HARMONIC UNIVALENT FUNCTIONS WITH VARYING ARGUMENTS DEFINED BY USING INTEGRAL OPERATOR

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ABSTRACT. In this paper we define and investigate a new class of harmonic functions by using an integral operator with varying arguments. We obtain coefficients inequalities, extreme points and distortion theorem.

1. Introduction

A continuous complex-valued function $f = u + iv$ which is defined in a simply-connected complex domain $D$ is said to be harmonic in $D$ if both $u$ and $v$ are real harmonic in $D$. In any simply-connected domain we can write

$$f = h + g,$$

where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the co-analytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $D$ is that $|h'(z)| > |g'(z)|$, $z \in D$ (see [4]).

Denote by $S_H$ the class of functions $f$ of the form (1.1) that are harmonic univalent and sense-preserving in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) = 1 = 0$. Then for $f = h + g \in S_H$ we may express the analytic functions $h$ and $g$ as follows:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \quad (1.2)$$

In 1984 Clunie and Shell-Small [4] investigated the class $S_H$ as well as its geometric subclasses and obtained some coefficient bounds. Since then, there have been several related papers on $S_H$ and its subclasses (see [9], [11], [15] and [16]).

The integral operator $I_n^p$ is defined as follows (see [2], with $p = 1$, also see [7], with $\ell = 0$): (i) $I_0^0 f(z) = f(z)$; (ii) $I_1^1 f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} f(t) dt$; (iii) $I_n^\lambda f(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} I_n^{\lambda-1} f(t) dt$ ($\lambda \geq 0; n \in \mathbb{N} = \{1, 2, 3, \ldots\}$)

In this paper we now define the integral operator for harmonic univalent functions $f(z)$ such that $h(z)$ and $g(z)$ are given by (1.2) as follows:

$$I_n^\lambda f(z) = I_n^\lambda h(z) + (-1)^n I_n^\lambda g(z), \quad (1.3)$$

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where
\[ I^k_h(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)^{-n}a_kz^k] \quad \text{and} \quad I^k_g(z) = \sum_{k=1}^{\infty} [1 + \lambda(k-1)^{-n}b_kz^k] \quad (|b_1| < 1). \]

With the help of the modified integral operator \( I^k_g \) we let \( G_H(m, n, \gamma; \rho) \) be the family of harmonic functions \( f = h + g \), which satisfy the condition
\[
\text{Re} \left\{ \left( 1 + pe^{i\alpha} \right) \frac{I^k_h(z)}{I^k_g(z)} - pe^{i\alpha} \right\} \geq \gamma
\]
(A.4)

\((\alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; \lambda \geq 0; m \in \mathbb{N}, \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; m \geq n; z \in U)\),

where \( I^k_g \) is defined by (1.3).

we note that:
(i) Taking \( m = n + 1, \rho = 0, \lambda = 1, G_H(n + 1, n; 1, \gamma, 0) = H(n, \gamma) \) \((m \in \mathbb{N}; n \in \mathbb{N}_0; 0 \leq \gamma < 1)\) (see Cotirla [5]);
(ii) Taking \( \lambda = 1, G_H(m, n; 1, \gamma; \rho) = G_H(m, n; \gamma; \rho) \) (see El-Ashwah et al. [6]);
(iii) Taking \( m = n + 1, \rho = 0, G_H(m, n; \gamma; \rho) = H(n, \gamma); \quad (m \in \mathbb{N}; \lambda > 0; \rho \geq 0; 0 \leq \gamma < 1)\) (see Seoudy [13], with \( p = 1)\);
(iv) Taking \( \lambda = 1 \), then \( G_H(m, n, 1, \gamma, \rho) = H(m, \gamma; \rho) \) (see Güney and Sakar [5]).

Also we note that, by the special choices of \( \alpha, \gamma, \rho, m \) and \( n \), we obtain:

(i) Taking \( \rho = 0 \), then \( G_H(m, n, \gamma, 0) = H(m, n, \gamma) = \left\{ f \in S_H : \text{Re} \left\{ \left( 1 + pe^{i\alpha} \right) \frac{I^k_h(z)}{I^k_g(z)} - pe^{i\alpha} \right\} \geq \gamma \left( \alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; \lambda \geq 0; m \geq n; z \in U \right) \right\} \);

(ii) \( G_H(n + 1, n, \gamma, \rho) = G_H(n, \gamma; \rho) = \left\{ f \in S_H : \text{Re} \left\{ \left( 1 + pe^{i\alpha} \right) \frac{I^k_h(z)}{I^k_g(z)} - pe^{i\alpha} \right\} \geq \gamma \left( \alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; \lambda \geq 0; n \in \mathbb{N}_0; z \in U \right) \right\} \);

(iii) \( G_H(1, 0, \gamma, \rho) = G_H(\lambda, \gamma; \rho) = \left\{ f \in S_H : \text{Re} \left\{ \left( 1 + pe^{i\alpha} \right) \frac{f(z)}{I^k_g(z)} - pe^{i\alpha} \right\} \geq \gamma \left( \alpha \in \mathbb{R}; 0 \leq \gamma < 1; \rho \geq 0; \lambda \geq 0; z \in U \right) \right\} \).

Definition 1 [10]. Let \( V_H \) denoted the class of functions \( f = h + g \) for which \( h \) and \( g \) are of the form (1.2) and there exist a real number \( \phi \) so that, mod \( 2\pi \),

\[ \arg(a_k) + (k - 1)\phi \equiv \pi \quad \text{and} \quad \arg(b_k) + (k - 1)\phi \equiv 0, \quad k \geq 2. \]

We also let \( \overline{V_P} = \overline{H} \cap V_H \).

Some subclasses of harmonic univalent functions with varying arguments were introduced and studied by various authors (see [1] and [13]).

Also we now define the subclass \( \overline{V_P}(m, n, \lambda; \gamma, \rho) \) consists of harmonic functions \( f_n = h + g_n \) in \( G_H(m, n, \lambda; \gamma, \rho) \) such that \( h \) and \( g_n \) are of the form:
\[
h(z) = z + \sum_{k=2}^{\infty} a_kz^k, \quad g_n(z) = \sum_{k=1}^{\infty} b_kz^k \quad (|b_1| < 1). \quad (1.5)
\]

and there exist a real number \( \phi \) such that, mod \( 2\pi \),

\[ \arg(a_k) + (k - 1)\phi \equiv \pi, \quad k \geq 2 \quad \text{and} \quad \arg(b_k) + (k + 1)\phi \equiv (n - 1)\pi, \quad k \geq 1. \quad (1.6) \]

Also we note that, by the special choices of \( \alpha, \gamma, m \) and \( n \), we obtain:
(i) Taking \( \rho = 0 \), \( \overline{V_P}(m, n; \lambda; \gamma, 0) = \overline{V_P}(m, n; \lambda; \beta) \) \((m \in \mathbb{N}; n \in \mathbb{N}_0; \lambda \geq 0; 0 < \gamma \leq 1)\);
(ii) \( \overline{V_P}(n + 1, n; \lambda; \gamma, \rho) = \overline{V_P}(m, \lambda; \gamma, \rho) \) \((n \in \mathbb{N}_0; \lambda \geq 0)\);
(iii) \( \overline{V_P}(1, 0; \lambda; \gamma, \rho) = \overline{V_P}(\lambda; \gamma, \rho) \).
2. Geometric properties of the classes \( G_H(m, n, \lambda; \gamma, \rho) \) and \( V_{\Phi}(m, n, \lambda; \gamma, \rho) \)

Unless otherwise mentioned, we assume in the reminder of this paper that, \( \alpha \in \mathbb{R}, \ 0 \leq \gamma < 1, \ \rho \geq 0, \ \lambda \geq 0, \ m \in \mathbb{N}, \ n \in \mathbb{N}_0, \ m > n \) and \( z \in U \). We begin with a sufficient coefficients condition for functions in the class \( G_H(m, n, \lambda; \gamma, \rho) \).

**Theorem 1.** Let \( f = h + g \) be such that \( h \) and \( g \) are given by (1.3) and if

\[
\begin{align*}
\sum_{k=1}^{\infty} & \left[ \frac{(1+\rho)[1+\lambda(k-1)]^{-\gamma} - (\gamma + \rho)[1+\lambda(k-1)]^{-m}}{1 - \gamma} \right] |a_k| \\
+ & \left( \frac{(1+\rho)[1+\lambda(k-1)]^{-\gamma} - (1)^{m-n}(\gamma + \rho)[1+\lambda(k-1)]^{-m}}{1 - \gamma} \right] |b_k| \leq 2, \quad (2.1)
\end{align*}
\]

where \( a_1 = 1 \). Then \( f \in G_H(m, n, \lambda; \gamma, \rho) \).

**Proof.** To prove that \( f \in G_H(m, n, \lambda; \gamma, \rho) \). We only need to show that if (2.1) holds, then the condition (1.4) is satisfied, then. Our aim is to show that

\[
\text{Re} \left\{ \frac{(1+pe^{i\alpha}) I^m_{\lambda} f(z) - pe^{i\alpha} I^m_{\lambda} f(z)}{I^m_{\lambda} f(z)} \right\} = \text{Re} \frac{A(z)}{B(z)} \geq \gamma. \quad (2.2)
\]

Using the fact that \( \text{Re} \{ w \} > \gamma \) if and only if \( |1 - \gamma + w| > |1 + \gamma - w| \), it suffices to show that

\[
|A(z) + (1-\gamma)B(z)| - |A(z) - (1+\gamma)B(z)| \geq 0, \quad (2.3)
\]

where \( A(z) = (1+pe^{i\alpha}) I^m_{\lambda} f(z) - pe^{i\alpha} I^m_{\lambda} f(z) \) and \( B(z) = I^m_{\lambda} f(z) \). Substituting for \( A(z) \) and \( B(z) \) in the left side of (2.3) we obtain

\[
\begin{align*}
&\left| (1+pe^{i\alpha}) I^m_{\lambda} f(z) - pe^{i\alpha} I^m_{\lambda} f(z) + (1-\gamma)I^m_{\lambda} f(z) \right| \\
&- \left| (1+pe^{i\alpha}) I^m_{\lambda} f(z) - pe^{i\alpha} I^m_{\lambda} f(z) - (1-\gamma)I^m_{\lambda} f(z) \right| \\
&= \left| (2-\gamma)z + \sum_{k=2}^{\infty} \left[ \frac{((1+\lambda(k-1))^{-\gamma} - (1\gamma) [1+\lambda(k-1)]^{-m})}{1 - \gamma} \right] |a_k| z^k \right| \\
&+ \left( \frac{((1+\lambda(k-1))^{-\gamma} - (1)^{m-n}[1+\lambda(k-1)]^{-m})}{1 - \gamma} \right] |b_k| z^k \right| \\
&\text{Substituting for } A(z) \text{ and } B(z) \text{ in the left side of (2.3) we obtain}
\end{align*}
\]
\[ (2 - \gamma) |z| - \sum_{k=2}^{\infty} \left[ (1 + \rho)[1 + \lambda(k - 1)]^{-n} - (\gamma + \rho - 1)[1 + \lambda(k - 1)]^{-m} \right] |a_k| |z|^k \]
\[- \sum_{k=1}^{\infty} \left[ (1 + \rho)[1 + \lambda(k - 1)]^{-n} - (-1)^{m-n} (\gamma + \rho - 1)[1 + \lambda(k - 1)]^{-m} \right] |b_k| |z|^k \]
\[- \gamma |z| - \sum_{k=2}^{\infty} \left[ (1 + \rho)[1 + \lambda(k - 1)]^{-n} - (\gamma + \rho + 1)[1 + \lambda(k - 1)]^{-m} \right] |a_k| |z|^k \]
\[ - \gamma |z| - \sum_{k=1}^{\infty} \left[ (1 + \rho)[1 + \lambda(k - 1)]^{-n} - (\gamma + \rho + 1)[1 + \lambda(k - 1)]^{-m} \right] |b_k| |z|^k \]
\[ \frac{\rho e^{i\alpha} \left[ (1 + \lambda(k - 1)]^{-m} - (1)^{m-n} (1 + \lambda(k - 1)]^{-n} \right] |b_k| |z|^k}{(1 + \lambda(k - 1)]^{-n} - (\gamma + \rho + 1)[1 + \lambda(k - 1)]^{-m}} \]
\[ > 2 \right\{ (1 - \gamma) - \sum_{k=2}^{\infty} \left[ (1 + \rho)[1 + \lambda(k - 1)]^{-n} - (\gamma + \rho + 1)[1 + \lambda(k - 1)]^{-m} \right] |a_k| \]
\[ + \sum_{k=1}^{\infty} \left[ (1 + \rho)[1 + \lambda(k - 1)]^{-n} - (1)^{m-n} (\gamma + \rho + 1)[1 + \lambda(k - 1)]^{-m} \right] |b_k| \right\} \]
\[ \geq 0, \text{ this by using } (2.1). \]

The harmonic univalent functions
\[ f(z) = z + \sum_{k=2}^{\infty} \frac{(1 + \rho)[1 + \lambda(k - 1)]^{-n} - (\gamma + \rho + 1)[1 + \lambda(k - 1)]^{-m} x_k z^k}{\sum_{k=2}^{\infty} (1 + \rho)[1 + \lambda(k - 1)]^{-n} - (\gamma + \rho + 1)[1 + \lambda(k - 1)]^{-m} y_k z^k}, \quad (2.4) \]

where \( \sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1 \), shows that the coefficient bound given by (2.1) is sharp. This completes the proof of Theorem 1.

In the following theorem, it is shown that the condition (2.1) is also necessary for function \( f_n = h + g_n \), where \( h \) and \( g_n \) are of the form (1.5).

**Theorem 2.** Let \( f_n = h + g_n \), where \( h \) and \( g_n \) are given by (1.5). Then \( f_n \in \mathcal{PH}(m, n; \lambda; \gamma, \rho) \), if and only if the coefficient condition (2.1) holds.

**Proof.** Since \( \mathcal{PH}(m, n; \lambda; \gamma, \rho) \subseteq \mathcal{G}_H(m, n; \lambda; \gamma, \rho) \), we only need to prove the “only if” part of the theorem. For functions \( f_n = h + g_n \), where \( h \) and \( g_n \) are given by (1.5), the inequality (1.4) with \( f = f_n \) is equivalent to
\[ \text{Re} \left\{ \frac{(1 + \rho e^{i\alpha})[z + \sum_{k=2}^{\infty} (1 + \lambda(k - 1)]^{-n} a_k z^k + (-1)^n \sum_{k=1}^{\infty} (1 + \lambda(k - 1)]^{-m} b_k \bar{z}^k]}{z + \sum_{k=2}^{\infty} (1 + \lambda(k - 1)]^{-m} a_k z^k + (-1)^n \sum_{k=1}^{\infty} (1 + \lambda(k - 1)]^{-m} b_k \bar{z}^k]} \right\} \]
\[ - \text{Re} \left\{ \frac{(\gamma + \rho e^{i\alpha})[z + \sum_{k=2}^{\infty} (1 + \lambda(k - 1)]^{-m} a_k z^k + (-1)^n \sum_{k=1}^{\infty} (1 + \lambda(k - 1)]^{-m} b_k \bar{z}^k]}{z + \sum_{k=2}^{\infty} (1 + \lambda(k - 1)]^{-m} a_k z^k + (-1)^n \sum_{k=1}^{\infty} (1 + \lambda(k - 1)]^{-m} b_k \bar{z}^k]} \right\} > 0. \]

The above condition holds for all values of \( \alpha \in \mathbb{R} \) and \( z \in U \). Upon choosing \( \phi \) according to (1.6) and substituting \( \alpha = 0 \) and \( z = re^{i\phi} (0 < r < 1) \), we must have
\[ E \left\{ 1 - \left[ \sum_{k=2}^{\infty} (1 + \lambda(k - 1)]^{-m} |a_k| - (-1)^{m+n-1} \sum_{k=1}^{\infty} (1 + \lambda(k - 1)]^{-m} |b_k| \right] r^{k-1} > 0, \right\} \]
\[ (2.5) \]
where

\[
E = (1 - \gamma) - \left( \sum_{k=2}^{\infty} \left[ (1 + \rho) [1 + \lambda (k-1)]^{-n} - (\gamma + \rho) [1 + \lambda (k-1)]^{-m} \right] |a_k| \right) r^{k-1}
\]

\[
- \left( \sum_{k=1}^{\infty} \left[ (1 + \rho) [1 + \lambda (k-1)]^{-n} - (-1)^{m-n} (\gamma + \rho) [1 + \lambda (k-1)]^{-m} \right] |b_k| \right) r^{k-1}
\]

If the inequality (2.1) does not hold, then \( E \) is negative for \( r \) sufficiently close to 1. Thus there exists \( z_0 = r_0 \) in \( (0, 1) \) for which the quotient in (2.5) is negative. But this is a contradiction, then the proof of Theorem 2 is completed. \( \square \)

We now obtain the distortion theorem for functions in the class \( V_{\mathcal{F}}(m, n; \lambda; \gamma, \rho) \).

**Theorem 3.** Let \( f_n = h + g_n \), where \( h \) and \( g_n \) are given by (ref.1.5) and \( f_n \in V_{\mathcal{F}}(m, n; \lambda; \gamma, \rho) \). Then for \( |z| = r < 1 \), we have

\[
|f_n(z)| \leq (1 + |b_1|) r + \left[ \frac{1 - \gamma}{(1 + \rho) (1 + \lambda)^{-n} - (\gamma + \rho) (1 + \lambda)^{-m}} - \frac{(1 + \rho) - (-1)^{m-n} (\gamma + \rho)}{(1 + \rho) (1 + \lambda)^{-n} - (\gamma + \rho) (1 + \lambda)^{-m}} |b_1| \right] r^2,
\]

(2.6)

and

\[
|f_n(z)| \geq (1 - |b_1|) r - \left[ \frac{1 - \gamma}{(1 + \rho) (1 + \lambda)^{-n} - (\gamma + \rho) (1 + \lambda)^{-m}} - \frac{(1 + \rho) - (-1)^{m-n} (\gamma + \rho)}{(1 + \rho) (1 + \lambda)^{-n} - (\gamma + \rho) (1 + \lambda)^{-m}} |b_1| \right] r^2.
\]

(2.7)

The equalities in (2.6) and (2.7) are attained for the functions \( f \) given by

\[
f_{1, \phi}(z) = (1 + |b_1|) \varpi + \left[ \frac{1 - \gamma}{(1 + \rho) (1 + \lambda)^{-n} - (\gamma + \rho) (1 + \lambda)^{-m}} - \frac{(1 + \rho) - (-1)^{m-n} (\gamma + \rho)}{(1 + \rho) (1 + \lambda)^{-n} - (\gamma + \rho) (1 + \lambda)^{-m}} |b_1| \right] \varpi^2 e^{-i\phi},
\]

(2.8)

and

\[
f_{2, \phi}(z) = (1 - |b_1|) \varpi - \left[ \frac{1 - \gamma}{(1 + \rho) (1 + \lambda)^{-n} - (\gamma + \rho) (1 + \lambda)^{-m}} - \frac{(1 + \rho) - (-1)^{m-n} (\gamma + \rho)}{(1 + \rho) (1 + \lambda)^{-n} - (\gamma + \rho) (1 + \lambda)^{-m}} |b_1| \right] \varpi^2 e^{-i\phi},
\]

(2.9)

where \( |b_1| \leq \frac{1 - \gamma}{(1 + \rho) - (-1)^{m-n} (\gamma + \rho)} \).

Proof. We prove the first inequality.
Let \( f_n \in V_{\mathcal{W}}(m, n, \lambda; \gamma, \rho) \), we have
\[
|f_n(z)| \leq (1 + |b_1|) + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k \leq (1 + |b_1|) r + r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|)
\]
\[
\leq (1 + |b_1|) r + r^2 \sum_{k=2}^{\infty} \left[ \frac{1}{(1 + \rho)(1 + \lambda)^{-n} - (\gamma + \rho)(1 + \lambda)^{-m}} \sum_{j=2}^{\infty} \frac{(1 + \rho)(1 + \lambda)^{-n} - (\gamma + \rho)(1 + \lambda)^{-m}}{1 - \gamma} |a_k| + |b_k| \right]
\]
\[
\leq (1 + |b_1|) r + \frac{1}{(1 + \rho)(1 + \lambda)^{-n} - (\gamma + \rho)(1 + \lambda)^{-m}} \left[ 1 - \frac{(1 + \rho) - (-1)^{m-n}(\gamma + \rho)}{1 - \gamma} |b_1| \right] r^2
\]
\[
\leq (1 + |b_1|) r + \left[ 1 - \frac{1}{(1 + \rho)(1 + \lambda)^{-n} - (\gamma + \rho)(1 + \lambda)^{-m}} \left[ 1 - \frac{(1 + \rho) - (-1)^{m-n}(\gamma + \rho)}{1 - \gamma} |b_1| \right] r^2 \right].
\]
The proof of the second inequality is similar, thus it is left.

\[\square\]

The bounds given in Theorem 3 for functions \( f_n = h + g_n \) such that \( h \) and \( g_n \) are given by (1.3) also hold for functions \( f = h + g \) such that \( h \) and \( g \) are given by (1.2) if the coefficient condition (2.1) is satisfied.

Next using the same technique used earlier by Aghalary [1] we obtain the extreme points of the class \( V_{\mathcal{W}}(m, n, \lambda; \gamma, \rho) \).

**Theorem 4.** The closed convex hull of the class \( V_{\mathcal{W}}(m, n, \lambda; \gamma, \rho) \) denoted by \( e\text{clco} V_{\mathcal{W}}(m, n; \gamma, \rho) \) is
\[
\left\{ f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} b_k z^k \in G_{\mathcal{W}}(m, n, \lambda; \gamma, \rho) : \sum_{k=1}^{\infty} \frac{(1 + \rho)(1 + \lambda(k - 1))^{-n} - (\gamma + \rho)(1 + \lambda(k - 1))^{-m}}{1 - \gamma} |a_k| + (1 + \rho)(1 + \lambda(k - 1))^{-n} - (\gamma + \rho)(1 + \lambda(k - 1))^{-m} |b_k| \leq 2 \right\},
\]
where \( a_1 = 1 \). Set \( \lambda_k = \frac{1}{1 - \gamma} (1 + \rho)(1 + \lambda(k - 1))^{-n} - (\gamma + \rho)(1 + \lambda(k - 1))^{-m} \) and \( \mu_k = \frac{1}{1 - \gamma} (1 + \rho)(1 + \lambda(k - 1))^{-n} - (\gamma + \rho)(1 + \lambda(k - 1))^{-m} \). For \( b_1 \) fixed, \( |b_1| \leq \frac{1 - \gamma}{(1 + \rho) - (-1)^{m-n}(\gamma + \rho)} \), the extreme points of the class \( V_{\mathcal{W}}(m, n, \lambda; \gamma, \rho) \) are
\[
\left\{ z + \lambda_k x z^k + b_1 z \right\} \cup \left\{ z + \mu_k x z^k + b_1 z \right\},
\]
where \( k \geq 2 \) and \( |x| = 1 - \frac{(1 + \rho) - (-1)^{m-n}(\gamma + \rho)}{1 - \gamma} \).

**Proof.** Any function \( f \in V_{\mathcal{W}}(m, n, \lambda; \gamma, \rho) \) may be expressed as
\[
f(z) = z + \sum_{k=2}^{\infty} |a_k| e^{i\theta_k} z^k + b_1 z + \sum_{k=2}^{\infty} |b_k| e^{i\theta_k} z^k,
\]
where the coefficients satisfy the inequality \( \text{[2.1]} \). Set
\[
h_k(z) = z^k, \quad g_k(z) = z^k + \alpha_k e^{i\theta_k} z^k, \quad h_k(z) = z^k + \mu_k e^{i\theta_k} z^k, \quad k = 2, 3, \ldots,
\]
writing \( X_k = \frac{|a_k|}{\lambda_k}, \quad Y_k = \frac{|b_k|}{\mu_k}, \quad k = 2, 3, \ldots \) and \( X_1 = 1 - \sum_{k=2}^{\infty} X_k \).

Then
\[
y_1 = 1 - \sum_{k=2}^{\infty} Y_k,
\]
we have
\[
f(z) = \sum_{k=1}^{\infty} \left( X_k h_k(z) + Y_k g_k(z) \right).
\]

In particular, setting
\[
f_1(z) = z + \overline{b_1},
\]
and
\[
f_k(z) = z + \frac{\lambda_k x z^k + \overline{b_1} z + \mu_k y z^k}{1 - \gamma},
\]
we see that extreme points of the class \( V_{\Pi}(m, n, \lambda, \gamma, \rho) \) are contained in \( \{f_k(z)\} \). To see that \( f_1(z) \) is not an extreme point, note that \( f_1(z) \) may be written as
\[
f_1(z) = \frac{1}{2} \left\{ f_1(z) + \lambda \left( 1 - \frac{(1 + \rho[1 + \lambda(k - 1)]^{-n} - (-1)^{m-n}(\gamma + \rho)[1 + \lambda(k - 1)]^{-m} |b_1|}{1 - \gamma} \right) z^k \right\}
\]
\[
+ \frac{1}{2} \left\{ f_1(z) - \lambda \left( 1 - \frac{(1 + \rho[1 + \lambda(k - 1)]^{-n} - (-1)^{m-n}(\gamma + \rho)[1 + \lambda(k - 1)]^{-m} |b_1|}{1 - \gamma} \right) z^k \right\},
\]
a convex linear combination of functions in the class \( V_{\Pi}(m, n, \lambda, \gamma, \rho) \). Next we will show if both \( |x| \neq 0 \) and \( |y| \neq 0 \), then \( f_k \) is not an extreme point. Without loss of generality, assume \( |x| \geq |y| \). Choose \( \epsilon > 0 \) small enough so that \( \epsilon < \frac{|y|}{|x|} \). Set \( A = 1 + \epsilon \) and \( B = 1 - \frac{|\epsilon x|}{|y|} \), we then see that both
\[
t_1(z) = z + \lambda_k x A z^k + \overline{b_1} z + \mu_k y B z^k
\]
and
\[
t_2(z) = z + \lambda_k x (2 - A) z^k + \overline{b_1} z + \mu_k y (2 - B) z^k,
\]
are in the class \( V_{\Pi}(m, n, \lambda, \gamma, \rho) \) and note that
\[
f_k(z) = \frac{1}{2} (t_1(z) + t_2(z)).
\]
The extremal coefficient bounds shows that functions of the form \( \text{[2.10]} \) are the extreme points for the class \( V_{\Pi}(m, n, \lambda, \gamma, \rho) \), then the proof of Theorem 4 is completed.

Finally we will examine the closure properties of the class \( V_{\Pi}(m, n, \lambda, \gamma, \rho) \) under the generalized Bernardi-Libera-Livingston integral operator (see \( \text{[3, 12]} \)) \( L_c(f) \) which is defined by
\[
L_c(f(z)) = \frac{c+1}{z} \int_0^z t^{c-1} f(t) \, dt \quad (c > -1).
\]

**Theorem 5.** Let \( f_0 = h + g_0 \in V_{\Pi}(m, n, \lambda, \gamma, \rho) \), where \( h \) and \( g_0 \) are given by \( \text{[1, 9]} \). Then \( L_c(f_0(z)) \) belongs to the class \( V_{\Pi}(m, n, \lambda, \gamma, \rho) \).
Proof. From the representation of $L_n(f_a(z))$, it follows that

$$L_n(f_a(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} (h(t) + \mathcal{F}_n(t)) \, dt$$

$$= \frac{c+1}{z^c} \int_0^z t^{c-1} \left\{ h + \sum_{k=2}^{\infty} a_k t^k + \sum_{k=1}^{\infty} b_k t^k \right\} \, dt$$

$$= z + \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k,$$

where $A_k = \frac{c+k}{c+1} a_k$, $B_k = \frac{c+k}{c+1} b_k$. Therefore, we have,

$$\sum_{k=1}^{\infty} \left[ \frac{(1 + \rho)[1 + \lambda(k-1)]^{-n} - (\gamma + \rho)[1 + \lambda(k-1)]^{-m} c + 1}{1 - \gamma} a_k + \frac{(1 + \rho)[1 + \lambda(k-1)]^{-n} - (-1)^{m-n}(\gamma + \rho)[1 + \lambda(k-1)]^{-m} c + 1}{1 - \gamma} b_k \right]$$

$$\leq \sum_{k=1}^{\infty} \left[ \frac{(1 + \rho)[1 + \lambda(k-1)]^{-n} - (\gamma + \rho)[1 + \lambda(k-1)]^{-m}}{1 - \gamma} a_k + \frac{(1 + \rho)[1 + \lambda(k-1)]^{-n} - (-1)^{m-n}(\gamma + \rho)[1 + \lambda(k-1)]^{-m}}{1 - \gamma} b_k \right] \leq 2,$$

and the proof of Theorem 5 is completed. \(\Box\)

Remark 1. (i) Putting $m = n + 1$ ($n \in \mathbb{N}_0$), $\rho = 0$ and $\lambda = 1$ in the above results, we obtain the corresponding results obtained by Cotiria [5];

(ii) Putting $\lambda = 1$ in the above results, we obtain the corresponding results obtained by El-Ashwah et al. [8];

(iii) Putting $m = n + 1$ ($n \in \mathbb{N}_0$) and $\rho = 0$ in the above results, we obtain the corresponding results obtained by Seoudy [13], with $p = 1$;

(iv) Putting $\lambda = 1$ in the above results, we obtain the corresponding results obtained by Güney and Sakar [8].

References


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