Electronic Journal of Mathematical Analysis and Applications,
Vol. 2(2) July 2014, pp. 1-11.
ISSN: 2090-792X (online)
http://fcag-egypt.com/Journals/EJMAA/

A NEW TECHNIQUE FOR SOLVING A CLASS OF NONLINEAR BOUNDARY VALUE PROBLEMS

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Abstract. In this paper, a new technique for solving a class of nonlinear Boundary Value Problems (BVPs) is introduced. The series solution obtained from this technique is satisfied by the given boundary condition at each partial sum. Based on the previous work of the author, convergence is discussed and the truncated error is estimated. The proposed technique is implemented to some problems that frequently appear in physics and engineering. The effectiveness of the proposed technique is verified through the comparison of the numerical results with those available in references.

1. Introduction

The BVPs play an important role in many fields especially in physics and engineering. The two-point boundary value problem occurs in a wide variety of problems, including the modeling of chemical reactions, heat transfer, diffusion, gas dynamics, and the solution of optimal control problems [1] and [2]. So, an accurate and fast techniques of solution are of great importance due to its wide application in scientific and engineering research [3] and [4]. Among these techniques, the series solution techniques such as Taylor method [5] and [6], homotopy analysis method (HAM) [7] and [8], homotopy perturbation method (HPM) [9] and [10] and Adomian decomposition method (ADM) [11], [12], [13] and [14]. Using these techniques, the solution is obtained by the analytic summation of the components of a convergent series. El-Kalla in [15] introduced a new expansion theorem by which the analytic summation constitutes the exact solution of some ordinary and partial differential equations. In this paper, this theorem is employed with ADM to introduce a new technique for solving the nonlinear BVP of the form

$$\frac{d^2y(x)}{dx^2} + p\left(x\right)\frac{dy\left(x\right)}{dx} + q\left(x\right)f(y) = r\left(x\right),\ x \in \left(a,b\right), \tag{1}$$

subjected to boundary conditions

$$y\left(a\right) = \alpha,\ y\left(b\right) = \beta, \tag{2}$$

2000 *Mathematics Subject Classification.* 34B15, 34L30, 47E05.
Key words and phrases. Nonlinear Boundary Value Problems; Contraction mapping; Series solution,
Submitted Oct. 7, 2013. Revised Nov. 19, 2013.

where a, b, α and β are finite constants. ADM has the advantage of dealing directly with the nonlinear problem avoiding any linearization, discretization or any unrealistic assumptions in which the solution is decomposed into a rapidly convergent series

$$y(x) = \sum_{i=0}^{\infty} y_i(x), \tag{3}$$

and the nonlinear term $f(y)$ is replaced by a series of the Adomian polynomials

$$f(y) = \sum_{m=0}^{\infty} A_m(y_0, y_1, ..., y_m), \tag{4}$$

where the traditional formula of A_m is

$$A_m = (\frac{1}{m!})(\frac{d^m}{d\lambda^m})[f(\sum_{i=0}^{\infty} \lambda^i y_i)]_{\lambda=0}. \tag{5}$$

In [16], the author deduced a new mathematical formula to the Adomian polynomials which can be written in the form

$$\tilde{A}_m = f(S_m) - \sum_{\upsilon=0}^{m-1} \tilde{A}_\upsilon, \ m \geq 1. \tag{6}$$

where the partial sum $S_m = \sum_{i=0}^{m} y_i(t)$ and $\tilde{A}_0 = f(y_0)$. Formula (6) has the advantage of absence of any derivative terms in the recursion, thereby allowing for ease of computations. Also, Formula (6) was used successfully in one dimensional problems [17] and [18] and in two dimensional problems [19] and [20] to study the convergence of ADM when applied to some classes of nonlinear equations. In this work, formula (6) is used in the convergence analysis and all the calculations of the numerical examples.

In direct application of ADM to problem (1), the inverse of the second-order differential operator is either two-fold definite or indefinite integral. If the inverse operator is defined by the indefinite integral the standard ADM yields the following recursive scheme [21]

$$\begin{aligned} y_0 &= c_1^0 + c_2^0 x + \int\int r(x)\,dxdx, \\ y_i &= c_1^i + c_2^i x - \int\int p(x)\frac{dy_{i-1}}{dx}dxdx - \int\int q(x)\tilde{A}_{i-1}dxdx, i \geq 1, \end{aligned} \tag{7}$$

where c_1^i and c_2^i are arbitrary constants that should be obtained using the boundary conditions for each partial sum S_m. It is difficult to obtain these arbitrary constants for each partial sum and more computational work is needed. Many authors, for example [22] and [23], have proposed modifications to ADM by defining the inverse operator to be a two-fold definite integral $\int_a^x dx_1 \int_a^{x_1} dx_2$ as in [22] or $\int_a^x dx_1 \int_b^{x_1} dx_2$ as in [23] but we still have difficulties in obtaining $\frac{dy}{dx} \downharpoonleft_{x=a}$ in [22] and $\frac{dy}{dx} \downharpoonleft_{x=b}$ in [23]. In the next section, a proposed technique is established to relax these difficulties and consequently to save the execution time on the processor. Each partial sum of the series solution obtained by this new technique is satisfied by the given boundary conditions without any additional work. In section three, convergence of the series solution is proved and the maximum absolute truncated error is estimated. In section four, some numerical examples, that frequently appear in physics and engineering, are introduced to verify the efficiency of the proposed technique.

2. The proposed technique

The proposed technique is based on the change of the canonical form (1) by introducing the differential operator

$$\pounds\left(.\right)=e^{-\int p(x)dx}\frac{d}{dx}\{e^{\int p(x)dx}\frac{d\left(.\right)}{dx}\}, \tag{8}$$

then (1) can be rewritten in the form

$$\pounds y\left(x\right)=r\left(x\right)-q\left(x\right)f(y). \tag{9}$$

Define the inverse of $\pounds$ to be

$$\pounds^{-1}(.)=\int_{a}^{x}e^{-\int p(\zeta)d\zeta}\int_{0}^{x}e^{\int p(\zeta)d\zeta}(.)d\zeta d\zeta, \tag{10}$$

and apply $\pounds^{-1}$ on both sides of (9) yields

$$y(x)-y\left(a\right)=\pounds^{-1}\left\{r\left(x\right)-q\left(x\right)f(y)\right\}. \tag{11}$$

Application of ADM to (11) yields the recursive relation

$$\begin{aligned} y_0\left(x\right) &= \alpha+\pounds^{-1}r\left(x\right), \\ y_i\left(x\right) &= -\pounds^{-1}q\left(x\right)\tilde{A}_{i-1},\ i\geq 1, \end{aligned} \tag{12}$$

and the series solution is denoted by $y_l\left(x\right)=\sum_{i=0}^{\infty}y_i\left(x\right)$ with $y_l\left(a\right)=\alpha$. Due to the nature of the Adomian series solution, $y_l\left(x\right)$ has a high accuracy near $x=a$ and some drawback near $x=b$.

If we define the inverse of the operator (8) to have the form

$$F^{-1}(.)=\int_{b}^{x}e^{-\int p(\zeta)d\zeta}\int_{0}^{x}e^{\int p(\zeta)d\zeta}(.)d\zeta d\zeta, \tag{13}$$

and then apply F^{-1} to both sides of (9), we get

$$y(x)-y\left(b\right)=F^{-1}\left\{r\left(x\right)-q\left(x\right)f(y)\right\}. \tag{14}$$

Application of ADM to (14) yields the recursive relation

$$\begin{aligned} y_0\left(x\right) &= \beta+F^{-1}r\left(x\right), \\ y_i\left(x\right) &= -F^{-1}q\left(x\right)\tilde{A}_{i-1},\ i\geq 1, \end{aligned} \tag{15}$$

and in this case the series solution is denoted by $y_r\left(x\right)=\sum_{i=0}^{\infty}y_i\left(x\right)$ with $y_r\left(b\right)=\beta$. Also, due to the nature of the Adomian series solution, $y_r\left(x\right)$ has a high accuracy near $x=b$ and some drawback near $x=a$.

In order to obtain solution with high accuracy in the whole interval $J=[a,b]$ and satisfied by both boundary conditions we set the solution in the form

$$y\left(x\right)=w_1\left(x\right)y_r\left(x\right)+w_2\left(x\right)y_l\left(x\right), a\leq x\leq b, \tag{16}$$

where $w_1\left(x\right)$ and $w_2\left(x\right)$ are a linear weighted functions satisfy the following:

i) $w_1\left(a\right)=0$ and linearly increasing to $w_1\left(b\right)=1$.

ii) $w_2\left(a\right)=1$ and linearly decreasing to $w_2\left(b\right)=0$.

This means that $w_1\left(x\right)=\frac{x-a}{b-a}$ and $w_2\left(x\right)=\frac{x-b}{a-b}$ and consequently we have:

i) $w_1\left(x\right)+w_2\left(x\right)=1\ \forall\ a\leq x\leq b$.

ii) From (16) $y\left(a\right)=y_l\left(a\right)=\alpha$, since all integrals in the left series $y_l\left(x\right)$ are from a to x, and $y\left(b\right)=y_r\left(b\right)=\beta$, since all integrals in the right series $y_r\left(x\right)$ are from b to x. So, $y\left(x\right)$ in (16) satisfy both boundary conditions $\forall\ S_m$.

It is clear that at the mid-point $x = (a+b)/2$ we have $w_1 = w_2 = 1/2$ i.e. the solution $y(x)$, in (16), is the average of $y_r(x)$ and $y_l(x)$. But, in the interval $a < x < (a+b)/2$ we have $0 < w_1(x) < w_2(x) < 1$ so; $y_l(x)$ contributes the majority of the solution $y(x)$. Also, in the interval $(a+b)/2 < x < b$ we have $0 < w_2(x) < w_1(x) < 1$ so; $y_r(x)$ contributes the majority of $y(x)$. This implies that formula (16) represents a continuous solution to problem (1)-(2) with high accuracy in the whole interval $J = [a, b]$ and satisfied by both boundary conditions.

Fortunately, in cases when the integrals through the use of $\pounds^{-1}$ or F^{-1} are difficult, we can simplify the computations by means of El-Kalla theorem [15] in which the integral operator $\hat{I}(.) = e^{-\int p(x)dx}\left\{\int_0^x e^{\int p(x)dx}(.)d\zeta\right\}$ can be expanded to have the form

$$\hat{I}(.)=\sum_{k=0}^{\infty}(-1)_0^{k\ x}p(x)_{\ldots k-fold\ldots}\int_0^x p(x)\int_0^x (.)d\zeta_{\ldots k-fold\ldots}d\zeta d\zeta, \tag{17}$$

for example, the second order approximation of (17) is $\hat{I}_2(r(x)) = \int_0^x r(\zeta)d\zeta - \int_0^x p(x)\int_0^x r(\zeta)$ $d\zeta d\zeta$. In practice we use (17) in its truncated form

$$\hat{I}_N(.)=\sum_{k=0}^{N}(-1)_0^{k\ x}p(x)_{\ldots k-fold\ldots}\int_0^x p(x)\int_0^x (.)d\zeta_{\ldots k-fold\ldots}d\zeta d\zeta. \tag{18}$$

Using (18), recursive relations (12) and (15) take the new forms

$$\begin{aligned} y_0(x) &= \alpha + \int_a^x \hat{I}_N(r(\zeta))\,d\zeta, \\ y_i(x) &= -\int_a^x \hat{I}_N\left(q(\zeta)\tilde{A}_{i-1}\right)d\zeta,\ i \geq 1, \end{aligned} \tag{19}$$

and

$$\begin{aligned} y_0(x) &= \beta + \int_b^x \hat{I}_N(r(\zeta))\,d\zeta, \\ y_i(x) &= -\int_b^x \hat{I}_N\left(q(\zeta)\tilde{A}_{i-1}\right)d\zeta,\ i \geq 1, \end{aligned} \tag{20}$$

respectively.

3. Convergence analysis

Convergence of the Adomian series solution was studied for different problems and by many authors. In [24, 25], convergence was investigated when the method applied to a general functional equations and to specific type of equations in [26, 27]. In convergence analysis, Adomian's polynomials play a very important role which could affect directly on the accuracy as well as the convergence rate. In our analysis we suggest an alternative approach for proving the convergence. This approach mainly depends on the accelerated formula (6). As a direct result from this approach, the maximum absolute truncated error of the series solution is estimated. In our analysis we assume that the nonlinear term $f(y)$ is Lipschitzian with $|f(y) - f(z)| \leq L|y - z|$ and $|q(x)| \leq M$ where M and L are finite constants. Define a mapping $F : E \to E$ where, $E = (C[J], \|.\|)$ is the Banach space of all continuous functions on J with the norm $\|y(x)\| = \max_{\forall x \in J}|y(x)|$.

3.1. **Existence and Uniqueness Theorem.** Problem (1) has a unique solution whenever $0 < \phi < 1$, where $\phi = \frac{1}{2} LM (b-a)^2$.

Proof. Define the mapping $F : E \to E$ as, $Fy = y(a) + \pounds^{-1}\{r(x) - q(x) f(y)\}$ and let y and $z \in E$ we have

$$\begin{aligned}
\|Fy - Fz\| &= \max_{\forall x \in J} \left| \pounds^{-1} q(x) [f(y) - f(z)] \right| \\
&\leq \max_{\forall x \in J} \int_a^x \int_0^x |q(\zeta)| \, L \, |f(y) - f(z)| \, d\zeta d\zeta \\
&\leq LM \max_{\forall x \in J} |f(y) - f(z)| \int_a^x \int_0^x d\zeta d\zeta \\
&\leq \frac{1}{2} LM (b-a)^2 \|y - z\| \\
&\leq \phi \|y - z\|
\end{aligned}$$

Under the condition $0 < \phi < 1$ the mapping F is contraction therefore, by the Banach fixed-point theorem for contraction, there exist a unique solution to problem (1) and this completes the proof. □

3.2. **Convergence Theorem.** The Adomian series (3) of problem (1) converges whenever $0 < \phi < 1$ and $r(x)$ bounded.

Proof. Let, S_n and S_m be arbitrary partial sums with $n > m$. We are going to prove that $\{S_n\}$ is a Cauchy sequence in Banach space E

$$\begin{aligned}
\|S_n - S_m\| &= \max_{\forall x \in J} |S_n - S_m| = \max_{\forall x \in J} \left| \sum_{i=m+1}^{n} y_i(x) \right| \\
&= \max_{\forall x \in J} \left| \sum_{i=m+1}^{n} \pounds^{-1} q(x) \tilde{A}_{i-1} \right| = \max_{\forall x \in J} \left| \pounds^{-1} q(x) \sum_{i=m}^{n-1} \tilde{A}_i \right|.
\end{aligned}$$

From (6) we have $\sum_{i=m}^{n-1} \tilde{A}_i = f(S_{n-1}) - f(S_{m-1})$ so

$$\begin{aligned}
\|S_n - S_m\| &= \max_{\forall x \in J} \left| \pounds^{-1} q(x) [f(S_{n-1}) - f(S_{m-1})] \right| \\
&\leq LM \max_{\forall x \in J} |S_{n-1} - S_{m-1}| \int_a^x \int_0^x d\zeta d\zeta \\
&\leq \phi \|S_{n-1} - S_{m-1}\|.
\end{aligned}$$

Let, $n = m + 1$ we have

$$\|S_{m+1} - S_m\| \leq \phi \|S_m - S_{m-1}\| \leq \phi^2 \|S_{m-1} - S_{m-2}\| \leq \cdots \leq \phi^m \|S_1 - S_0\|.$$

Using the triangle inequality we have

$$\begin{aligned}
\|S_n - S_m\| &\leq \|S_{m+1} - S_m\| + \|S_{m+2} - S_{m+1}\| + \cdots + \|S_n - S_{n-1}\| \\
&\leq \left[\phi^m + \phi^{m+1} + \cdots + \phi^{n-1}\right] \|S_1 - S_0\| \\
&\leq \phi^m \left[1 + \phi + \cdots + \phi^{n-m-1}\right] \|S_1 - S_0\| \\
&\leq \phi^m \left[\frac{1 - \phi^{n-m}}{1 - \phi}\right] \|y_1(x)\|.
\end{aligned}$$

Since $0<\phi<1$ so, $(1-\phi^{n-m})<1$ and then we have

$$\|S_n - S_m\| \leq \frac{\phi^m}{1-\phi} \max_{\forall x\in J} |y_1(x)| \tag{21}$$

but $\max\limits_{\forall x\in J} |y_1(x)| < \infty$ (since $r(x)$ is bounded and α, β finite) and as $m \to \infty$ then $\|S_n - S_m\| \to 0$, from which we conclude that $\{S_n\}$ is a Cauchy sequence in E. The same result will be obtained if we use the mapping $Fy = y(b) + F^{-1}\{r(x) - q(x) f(y)\}$. Since $\max\limits_{\forall x\in J} |w_1(x)| = \max\limits_{\forall x\in J} |w_2(x)| = 1$ and $y(x)$ is a linear combination of two convergent series then, $y(x)$ is a convergent series and this completes the proof. □

3.3. **Error Estimate.** The maximum absolute truncation error of series (3) to problem (1) is estimated to be: $\max\limits_{\forall x\in J} |y(x) - \sum_{i=0}^{m} y_i(x)| \leq \frac{\psi\phi^{m+1}}{2L(1-\phi)}$ where, $\psi = \max\limits_{\forall x\in J} |f(y_0)|$.

Proof. From (21) in Theorem 2 we have $\|S_n - S_m\| \leq \frac{\phi^m}{1-\phi} \max\limits_{\forall x\in J} |y_1(x)|$. As $n \to \infty$ then $S_n \to y(x)$ so, $\|y(x) - S_m\| \leq \frac{\phi^m}{1-\phi} \max\limits_{\forall x\in J} |y_1(x)|$ i.e.

$$\max_{\forall x\in J} \left| y(x) - \sum_{i=0}^{m} y_i(x) \right| \leq \frac{\phi^m}{1-\phi} \max_{\forall x\in J} |y_1(x)|. \tag{22}$$

But, $y_1 = -\pounds^{-1} q(x) \tilde{A}_0$, $\tilde{A}_0 = f(y_0)$, then, $\max\limits_{\forall x\in J} |y_1(x)| \leq \max\limits_{\forall x\in J} |f(y_0)| \frac{1}{2} M (b-a)^2 \leq \frac{\phi}{L}\psi$ so, (22) will be

$$\max_{\forall x\in J} \left| y(x) - \sum_{i=0}^{m} y_i(x) \right| \leq \frac{\psi\phi^{m+1}}{L(1-\phi)}. \tag{23}$$

which is the maximum absolute truncation error of the Adomian series solution of (1) and this completes the proof. □

4. Applications and numerical results

In this section, we implemented the proposed technique to some singular linear and nonlinear problems that frequently appear in physics and engineering. The effectiveness of the proposed technique is verified through the comparison of the numerical results with those available in references.

Example 1 Consider the Bessel equation of order zero [28]:

$$y'' + \frac{1}{x} y' + y = 0, 0 < x < 1,\ y(0) = \frac{1}{J_0(1)}, y(1) = 1,$$

with exact solution $y(x) = \frac{J_0(x)}{J_0(1)}$. In this example, $p(x) = \frac{1}{x}$, $q(x) = 1$, $r(x) = 0$, $\alpha = \frac{1}{J_0(1)}, \beta = 1$ and since $f(y) = y$ then $\tilde{A}_i = y_i$. In this case $y_l(x) = \sum_{i=0}^{\infty} y_i$ such that:

$y_0(x) = \frac{1}{J_0(1)} + \pounds^{-1}(0) = \frac{1}{J_0(1)}$,
$y_i(x) = -\pounds^{-1}(y_{i-1})$, $i \geq 1$, where $\pounds^{-1}(.) = \int_0^x \frac{1}{x} \int_0^x x(.)\, dx dx$,
and $y_r(x) = \sum_{i=0}^{\infty} y_i$ such that:
$y_0(x) = 1 + F^{-1}(0) = 1$,
$y_i(x) = -F^{-1}(y_{i-1})$ $i \geq 1$, where $F^{-1}(.) = \int_1^x \frac{1}{x} \int_0^x x(.)\, dx dx$.

This problem was solved in [28] by an improved Adomian method (IADM). Using the MATHEMATICA package we compare the absolute errors $|y(x) - S_5(x)|$ using the new technique with $|y(x) - S_{12}(x)|$ using IADM in [28]. Table 1 shows that, 6-term approximation using the new technique is more accurate than 13- term approximation using IADM.

Table 1 Numerical results for Example 1

x	$\lvert y(x) - S_{12}(x)\rvert$ using IADM in [28]	$\lvert y(x) - S_5(x)\rvert$ using new technique
0.0	0.0	0.0
0.1	$1.6729E-06$	$8.89010E-10$
0.2	$1.8402E-06$	$1.0011E-09$
0.3	$2.2581E-06$	$5.6011E-09$
0.4	$1.7706E-06$	$8.2106E-09$
0.5	$1.2296E-06$	$6.2034E-08$
0.6	$8.1541E-07$	$8.3121E-08$
0.7	$5.1621E-07$	$6.8087E-08$
0.8	$2.9587E-07$	$3.1115E-09$
0.9	$1.2888E-07$	$1.0120E-09$
1.0	0.0	0.0

Example 2 Consider the inhomogeneous Bessel equation [28]

$$y'' + \frac{1}{x}y' + y = 4 - 9x + x^2 - x^3, 0 < x < 1,\ y(0) = y(1) = 0,$$

with exact solution $y(x) = x^2 - x^3$. In this example, $p(x) = \frac{1}{x}$, $q(x) = 1$, $r(x) = 4 - 9x + x^2 - x^3$, $\alpha = \beta = 0$ and since $f(y) = y$ then $\tilde{A}_i = y_i$. In this case $y_l(x) = \sum_{i=0}^{\infty} y_i$ such that:

$y_0(x) = 0 + \pounds^{-1}\left(4 - 9x + x^2 - x^3\right) = x^2 - x^3 + \frac{x^4}{16} - \frac{x^5}{25}$,

$y_i(x) = -\pounds^{-1}(y_{i-1})$, $i \geq 1$, where $\pounds^{-1}(.) = \int_0^x \frac{1}{x} \int_0^x x(.)\,dxdx$,

and $y_r(x) = \sum_{i=0}^{\infty} y_i$ such that:

$y_0(x) = 0 + F^{-1}\left(4 - 9x + x^2 - x^3\right) = x^2 - x^3 + \frac{x^4}{16} - \frac{x^5}{25} - \frac{9}{400}$,

$y_i(x) = -F^{-1}(y_{i-1})$ $i \geq 1$, where $F^{-1}(.) = \int_1^x \frac{1}{x} \int_0^x x(.)\,dxdx$.

This problem was solved in [28] by IADM. Also, we verify the high accuracy of the new technique by comparing the absolute errors $|y(x) - S_3(x)|$ using the new technique with $|y(x) - S_9(x)|$ using IADM in [28]. Table 2 shows that, 4-term approximation using the new technique is more accurate than 10- term approximation using IADM.

Table 2 Numerical results for Example 2

x	$\lvert y(x)-S_9(x)\rvert$ using IADM in [28]	$\lvert y(x)-S_3(x)\rvert$ using new technique
0.0	0.0	0.0
0.1	$2.2418E-05$	$1.0110E-08$
0.2	$2.5259E-05$	$3.3010E-08$
0.3	$3.0855E-05$	$7.4051E-08$
0.4	$2.4171E-05$	$1.0019E-07$
0.5	$1.6781E-05$	$2.2001E-07$
0.6	$1.1127E-05$	$5.0110E-07$
0.7	$7.0442E-06$	$6.1097E-07$
0.8	$4.0377E-06$	$3.0912E-08$
0.9	$1.7590E-06$	$1.0120E-08$
1.0	0.0	0.0

Example 3 Consider the non-linear BVP [28]

$$y'' + \frac{0.5}{x}y' = e^y\left(0.5 - e^y\right), 0 < x < 1,\ y(0) = \ln(2), y(1) = 0,$$

with exact solution $y(x) = \ln\left(\frac{2}{1+x^2}\right)$. In this example, $p(x) = \frac{0.5}{x}$, $q(x) = 1$, $r(x) = 0$, $\alpha = \ln(2), \beta = 0$ and $f(y) = e^y(0.5 - e^y)$. In this case $y_l(x) = \sum_{i=0}^{\infty} y_i$ such that:

$y_0(x) = \ln(2) + \pounds^{-1}(0) = \ln(2)$,

$y_i(x) = \pounds^{-1}\left(\tilde{A}_{i-1}\right)$, $i \geq 1$, where $\pounds^{-1}(.) = \int_0^x \frac{1}{\sqrt{x}} \int_0^x \sqrt{x}\,(.)\,dxdx$,

and $y_r(x) = \sum_{i=0}^{\infty} y_i$ such that:

$y_0(x) = 0 + F^{-1}(0) = 0$,

$y_i(x) = F^{-1}\left(\tilde{A}_{i-1}\right)$ $i \geq 1$, where $F^{-1}(.) = \int_1^x \frac{1}{\sqrt{x}} \int_0^x \sqrt{x}\,(.)\,dxdx$.

In table 3, MATHEMATICA is used to compare the absolute errors $|y(x) - S_5(x)|$ with those of Ref. [28].

Table 3 Numerical results for Example 3

x	$\lvert y(x)-S_5(x)\rvert$ using IADM in [28]	$\lvert y(x)-S_5(x)\rvert$ using new technique
0.0	0.0	0.0
0.1	$1.8523E-07$	$1.9010E-13$
0.2	$3.0918E-07$	$4.2110E-13$
0.3	$5.0820E-07$	$7.9959E-13$
0.4	$7.5002E-07$	$9.8919E-13$
0.5	$3.6001E-06$	$5.6103E-12$
0.6	$3.1127E-06$	$7.9121E-12$
0.7	$7.0412E-07$	$7.5040E-12$
0.8	$4.0349E-07$	$6.9840E-13$
0.9	$1.7097E-07$	$1.7121E-13$
1.0	0.0	0.0

Example 4 Consider the problem arising in astronomy [29]

$$y'' + \frac{2}{x}y' + y^5 = 0, 0 < x < 1,\ y(0) = 1, y(1) = \frac{\sqrt{3}}{2},$$

with exact solution $y(x) = \frac{1}{\sqrt{1+\frac{x^2}{3}}}$. In this example, $p(x) = \frac{2}{x}$, $q(x) = 1$, $r(x) = 0$, $\alpha = 1, \beta = \frac{\sqrt{3}}{2}$ and $f(y) = y^5$. In this case $y_l(x) = \sum_{i=0}^{\infty} y_i$ such that:

$y_0(x) = 1 + \pounds^{-1}(0) = 1$,

$y_i(x) = \pounds^{-1}\left(\tilde{A}_{i-1}\right)$, $i \geq 1$, where $\pounds^{-1}(.) = \int_0^x \frac{1}{x^2}\int_0^x x^2 (.)\, dxdx$,

and $y_r(x) = \sum_{i=0}^{\infty} y_i$ such that:

$y_0(x) = \frac{\sqrt{3}}{2} + F^{-1}(0) = \frac{\sqrt{3}}{2}$,

$y_i(x) = F^{-1}\left(\tilde{A}_{i-1}\right)$ $i \geq 1$, where $F^{-1}(.) = \int_1^x \frac{1}{x^2}\int_0^x x^2 (.)\, dxdx$.

This problem was solved in [29] using a combination of the ADM and the reproducing kernel method (RKM). In table 4, MATHEMATICA is used to compare the absolute errors $|y(x) - S_5(x)|$ with those of Ref. [29].

Table 4 Numerical results for Example 4

x	$\lvert y(x) - S_5(x)\rvert$ in [29]	$\lvert y(x) - S_5(x)\rvert$ using new technique
0.0	0.0	0.0
0.1	$1.5560E-09$	$7.7206E-11$
0.2	$1.1101E-09$	$9.9033E-11$
0.3	$4.2003E-08$	$4.1303E-10$
0.4	$6.5002E-08$	$7.0511E-10$
0.5	$8.3011E-08$	$9.7490E-10$
0.6	$7.0120E-08$	$8.9181E-10$
0.7	$3.9492E-08$	$3.8900E-10$
0.8	$1.0301E-09$	$9.3010E-11$
0.9	$1.7123E-09$	$6.6908E-11$
1.0	0.0	0.0

Also, example 4 is used to verify the validity of the convergence analysis by comparing the exact absolute error (EAE) $\Delta = \left|y(x) - \sum_{i=0}^{m} y_i(x)\right|$ and the maximum absolute error (MAE) $\Delta^* = \frac{\psi\phi^{m+1}}{L(1-\phi)}$, using formula (23), for different values of m in table 5.

Table 5: EAE and MAE for Example 4

m	EAE (Δ)	MAE(Δ^*)
5	$9.7490E-10$	$3.9042E-8$
10	$2.0032E-14$	$1.5002E-12$
15	$7.2090E-18$	$8.0160E-16$
20	$3.2030E-21$	$3.6609E-19$

The proposed technique is not only more accurate but also faster than the traditional techniques and consequently save the execution time on the processor. Table 6 shows the comparison between the execution time, on the same processor, of the proposed technique and those in references.

Table 6: Comparison between the execution time on the same processor

Execution time using:	Example 1	Example 2	Example 3	Example 4
The proposed technique	34.9 sec	26.3 sec	44.7 sec	38.4 sec
Techniques in references	78.8 sec	65.8 sec	72.5 sec	66.2 sec

5. Conclusion

ADM has difficulties when applied to nonlinear BVPs and many approaches have been presented to overcome these difficulties. The proposed technique gives an accurate and faster solution when applied to singular linear and nonlinear problems without additional computational work and this solution is satisfied by the given boundary conditions. Convergence of the series solution obtained from this technique is proved. Convergence analysis is reliable enough to obtain an explicit formula (23) to the maximum absolute truncated error of the Adomian series solution.

References

[1] V. Hlavacek, M. Marek, and M. Kubicek, Modeling of chemical reactors-X multiple solutions of enthalpy and mass balances for a catalytic reaction within a porous catalyst particle. Chemical Engineering Science, 23, (1968), 1083–1097.
[2] J. Cisternas, P. Holmes and I. G. Kevrekidis, Buckling in response to applied heat sources, Physica D, 177, (2003),71–100.
[3] T. Estrin, T. Higgins, The solution of boundary value problems by multiple Laplace transformations, J. of Franklin Institute, 252, (2), (1951), 153-167.
[4] J. Chen, Fast multilevel augmentation methods for nonlinear boundary value problems, Comput. Math. Appl, 61, (2011), 612–619.
[5] G. Jacobsohn, A discrete Taylor series method for the solution of two-point boundary-value problems, J. of Franklin Institute, 338, (1), (2001), 61-68.
[6] M. Razzaghi, M. Razzaghi, Solution of linear two-point boundary value problems via Taylor series, J. of Franklin Institute, 326, (4), (1989), 511-521.
[7] S. Abbasbandy, E. Magyari, E. Shivanian, The homotopy analysis method for multiple solutions of nonlinear boundary value problems, Commun Nonlinear Sci Numer Simulat, 14, (9-10), (2009), 3530-3536.
[8] Si Xinhui, Z. Liancun, Z. Xinxin, Si Xinyi, Homotopy analysis method for the asymmetric laminar flow and heat transfer of viscous fluid between contracting rotating disks, Appl. Math. Modelling, 36, (4), (2012), 1806-1820.
[9] A.M.A. El-Sayed, A. Elsaid, I.L. El-Kalla, D. Hammad, A homotopy perturbation technique for solving partial differential equations of fractional order in finite domains, Appl. Math. Comp., 218, (2012), 8329–8340.
[10] Zhi-Yong Zhang, Xue-Lin Yong, Yu-Fu Chen, Comparison of approximate symmetry and approximate homotopy symmetry to the Cahn–Hilliard equation, J. of Computational and Applied Mathematics, 237, (2013) 197–207.
[11] G. Adomian, Solving Frontier problems of Physics: The Decomposition method, Kluwer, 1995.
[12] A. Wazwaz, Laplace transform-ADM for handling nonlinear Volterra integro-differential equations, Appl. Math. and Comp., 216, (2010), 1304-1309.
[13] A. El-Sayed, I. L. El-Kalla, E. Ziada, Analytical and numerical solutions of nonlinear fractional differential equations, Appl. Num. Math., 60, (2010), 788–797.
[14] I. L. El-Kalla, A new approach for solving a class of nonlinear integro-differential equations, Commun Nonlinear Sci Numer Simulat, 17, (2012), 4634–4641.
[15] I. L. El-Kalla, New results on the analytic summation of Adomian series for some classes of differential and integral equations, Appl. Math. Comp., 217, (2010), 3756–3763.
[16] I. L. El-kalla, Error analysis of Adomian series solution to a class of nonlinear differential equations, Applied math. E-Notes, 7(2007), 214-221.
[17] I. L. El-Kalla, Convergence of the Adomian method applied to a class of nonlinear integral equations, Appl. Math. Lett., 21, (2008), 372-376.
[18] I. L. El-Kalla, Error estimate of the series solution to a class of fractional differential equations, Commun Nonlinear Sci Numer Simulat, 16, (2011), 1408–1413.

[19] E. A. Az-Zo'bi and K. Al-Khaled, A new convergence proof of the Adomian decomposition method for a mixed hyperbolic elliptic system of conservation laws, Appl. Math. and Comp., 217, (2010), 4248–4256.
[20] I. L. El-Kalla, Error estimates for series solutions to a class of nonlinear integral equations of mixed type, J Appl. Math. Comput., 38, (2012), 341-351.
[21] G. Adomian, R. Rach, Modified decomposition solution of linear and nonlinear boundary-value problems, Nonlinear Anal., 23, (1994), 615–619.
[22] A. M. Wazwaz, A reliable algorithm for obtaining positive solutions for nonlinear boundary value problems, Comput. Math. Appl., 41, (2001), 1237–1244.
[23] M. Benabidallah, Y. Cherruault, Application of the Adomain method for solving a class of boundary problems, Kybernetes, 33, (1) (2004), 118–132.
[24] Y. Cherruault, G.adomian, Decomposition method: new ideas for proving convergence of decomposition methods, Comput. Math. Appl., 29, (7), (1995), 103-108.
[25] M. M. Hosseini, H. Nasabzadeh, On the convergence of Adomian decomposition method, Appl. Math. Comp., 182, (2006), 536–543.
[26] K.Abbaoui and Y.Cherruault, Convergence of Adomian's method applied to differential equations, Comput. Math. Appl., Vol. 28, No. 5, (1994), 103-109.
[27] R. Rajaram, M. Najafi, Analytical treatment and convergence of the Adomian decomposition method for a system of coupled damped wave equations, Appl. Math. Comp., 212, (2009), 72-81.
[28] A. Ebaid, A new analytical and numerical treatment for singular two-point boundary value problems via the Adomian decomposition method, J. of Computational and Applied Mathematics, 235, (2011), 1914-1924.
[29] F. Geng, M. Cui, A novel method for nonlinear two-point boundary value problems: Combination of ADM and RKM, Appl. Math. Comp., 217, (2011), 4676–4681. differential equations

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