

SPECTRAL ANALYSIS OF ONE BOUNDARY VALUE-TRANSMISSION PROBLEM BY MEANS OF GREEN'S FUNCTION

F. S. MUHTAROV, K. AYDEMIR, O. SH. MUKHTAROV

ABSTRACT. The aim of this study is to investigate Sturm-Liouville equation together with eigenparameter dependent boundary conditions and two supplementary transmission conditions at one interior point. We develop Green's function method for spectral analysis of the considered problem in modified Hilbert space.

1. INTRODUCTION

Many physical phenomena, both classical mechanics and quantum mechanics, are described mathematically by Sturm-Liouville type problems. Particularly, these type problems arise directly as mathematical models of notation according to Newton's models of law, but more often as a result of using the method of separation of variables to solve various partial differential equations of physics, such as Laplace's equation, the heat equation and the wave equation. The Green's function is an important aid in solving of such problems. The formulae of Green's function for many problems with classical boundary conditions are presented in [4]. Chung and Yau [3] study discrete Green's function and their relationship with discrete Laplace equations. They discuss several methods for deriving Green's functions. Liu et al. [10] give an application of the estimate to discrete Green's function with a high accuracy analysis of the three-dimensional block finite element approximation. For multidimensional stationary problems and non-stationary problems the formulae for Green's function are more complicated and Green's functions are represented as functional series even for simple rectangular, spherical and cylindrical domains [12].

In this study we shall investigate one discontinuous eigenvalue problem which consists of Sturm-Liouville equation,

$$\tau u := -u''(x) + q(x)u(x) = \lambda u(x) \tag{1}$$

2000 *Mathematics Subject Classification.* 35J08, 47B25.

Key words and phrases. Sturm-Liouville problems, transmission conditions, Green's function, resolvent operator.

Submitted Aug. 3, 2014.

to hold in finite interval $(-1, 1)$ except at one inner point $0 \in (-1, 1)$, where discontinuity in u and u' are prescribed by transmission conditions

$$u(+0) = \alpha_0 u(-0), \quad (2)$$

$$u'(+0) = \beta_1 u'(-0) + \beta_0 u(-0), \quad (3)$$

together with the boundary conditions

$$\cos \alpha u(-1) + \sin \alpha u'(-1) = 0, \quad (4)$$

$$\cos \beta u(1) + \sin \beta u'(1) = 0, \quad (5)$$

where the potential $q(x)$ is real-valued, continuous in each interval $[-1, 0)$ and $(0, 1]$ and has a finite limits $q(\mp 0)$; $\alpha_0, \beta_0, \beta_1$ are real numbers; λ is a complex eigenparameter. This kind of problems may arise in spectral problems of the theory of heat and mass transfer [9], in diffraction problems [1] and varied physical transfer problems [7]. Note that, some spectral properties of such type problems are discussed in [2], [5], [6], [11] and [14]. The investigation of Green's function for problems with transmission conditions is quite a new area. In this paper, expression of Green's function for (1)-(5) have been obtained by modifying the classical method. The advantage of this method is that it is possible to construct the Green's function for BVP's under supplementary transmission conditions. In this study by using an own technique we introduce a new equivalent inner product in the Hilbert space $L_2(-1, 0) \oplus L_2(0, 1)$ and a linear operator in it such a way that the considered problem can be interpreted as eigenvalue problem for this operator. Moreover, we derive the resolvent operator by means of the Green's function and prove selfadjointness of the problem in modified Hilbert space.

2. THE CHARACTERISTIC FUNCTION AND EIGENVALUES

In this section we shall define two basic solutions $\phi(x, \lambda)$ and $\chi(x, \lambda)$ by own technique as follows. Let $\phi_1(x, \lambda)$ and $\chi_2(x, \lambda)$ be solutions of equation (1) on $[-1, 0)$ and $(0, 1]$ subject to initial conditions

$$\phi_1(-1, \lambda) = \sin \alpha, \quad \frac{\partial \phi_1(-1, \lambda)}{\partial x} = -\cos \alpha \quad (6)$$

and

$$\chi_2(1, \lambda) = -\sin \beta, \quad \frac{\partial \chi_2(1, \lambda)}{\partial x} = \cos \beta, \quad (7)$$

respectively. By virtue of [[13], Theorem 7] each of these solutions are entire functions of λ for fixed x . There is an unique solution $\phi_2(x, \lambda)$ and $\chi_1(x, \lambda)$ of equation (1) on $[-1, 0)$ and $(0, 1]$ satisfying the initial conditions

$$\phi_2(+0, \lambda) = \alpha_0 \phi_1(-0, \lambda), \quad \frac{\partial \phi_2(+0, \lambda)}{\partial x} = \beta_1 \frac{\partial \phi_1(-0, \lambda)}{\partial x} + \beta_0 \phi_1(-0, \lambda) \quad (8)$$

and

$$\chi_1(-0, \lambda) = \frac{1}{\alpha_0} \chi_2(+0, \lambda), \quad \frac{\partial \chi_1(-0, \lambda)}{\partial x} = \frac{1}{\beta_1} \frac{\partial \chi_2(+0, \lambda)}{\partial x} - \frac{\beta_0}{\alpha_0 \beta_1} \chi_2(+0, \lambda) \quad (9)$$

respectively. By applying the similar technique as in [5] we can prove that the solutions $\phi_2(x, \lambda)$ and $\chi_1(x, \lambda)$ are also an entire functions of parameter λ for each fixed x . Consequently, both of the functions $\phi(x, \lambda)$ and $\chi(x, \lambda)$ defined as

$$\phi(x, \lambda) = \begin{cases} \phi_1(x, \lambda), & x \in [-1, 0) \\ \phi_2(x, \lambda), & x \in (0, 1] \end{cases}$$

$$\chi(x, \lambda) = \begin{cases} \chi_1(x, \lambda), & x \in [-1, 0) \\ \chi_2(x, \lambda), & x \in (0, 1] \end{cases}$$

satisfies the equation (1) on whole $[-1, 0) \cup (0, 1]$ and the both transmission conditions (2) and (3). Moreover, $\phi(x, \lambda)$ satisfies one of boundary condition (namely the condition (4)) and $\chi(x, \lambda)$ satisfies the other boundary condition (5). Taking in view that the Wronskians $W(\phi_i, \chi_i; x) := \phi_i(x, \lambda)\chi_i'(x, \lambda) - \phi_i'(x, \lambda)\chi_i(x, \lambda)$ are independent of variable x we shall denote $w_i(\lambda) = W(\phi_i, \chi_i; x)$ ($i=1,2$). By using (8) and (9) we have

$$\begin{aligned} w_1(\lambda) &= \phi_1(-0, \lambda) \frac{\partial \chi_1(-0, \lambda)}{\partial x} - \chi_1(-0, \lambda) \frac{\partial \phi_1(-0, \lambda)}{\partial x} \\ &= \frac{1}{\alpha_0 \beta_1} (\phi_2(+0, \lambda) \frac{\partial \chi_2(+0, \lambda)}{\partial x} - \chi_2(+0, \lambda) \frac{\partial \phi_2(+0, \lambda)}{\partial x}) \\ &= \frac{1}{\alpha_0 \beta_1} w_2(\lambda) \end{aligned}$$

for each $\lambda \in \mathbb{C}$. It is convenient to introduce the notation

$$w(\lambda) := \alpha_0 \beta_1 w_1(\lambda) = w_2(\lambda). \quad (10)$$

Lemma 1 Let λ_0 be zero of $w(\lambda)$. Then the solutions $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent.

Proof. From $w_i(\lambda_0) = 0$ it follows that

$$\chi_i(x, \lambda_0) = k_i \phi_i(x, \lambda_0) \quad (11)$$

for some $k_i \neq 0$ ($i=1, 2$). Show that $k_1 = k_2$. Suppose, it possible, that $k_1 \neq k_2$. Making use (8) and (11) we have

$$\begin{aligned} \chi_2(+0, \lambda_0) &= \alpha_0 \chi_1(-0, \lambda_0) = k_1 \alpha_0 \phi_1(-0, \lambda_0) \\ &= k_1 \phi_2(+0, \lambda_0) = \frac{k_1}{k_2} \chi_2(+0, \lambda_0) \end{aligned} \quad (12)$$

Hence

$$\chi_2(+0, \lambda_0) = 0. \quad (13)$$

Similarly from (9) and (11) we derive that,

$$\frac{\partial \chi_2(+0, \lambda_0)}{\partial x} = 0. \quad (14)$$

Since the differential equation (1) considered only on $(0, 1]$ has a unique solution with given initial values u and u' at the point $x = 0$, we have $\chi_2(x, \lambda_0) = 0$ for any $x \in (0, 1]$ by (13) and (14). But this is contradict with (7). Consequently, $k_1 = k_2$, so $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent.

By the same technique as in [5] we can prove the following lemma.

Lemma 2 The set of eigenvalues the problem (1) – (5) coincide with the set of zeros of $w(\lambda)$.

Lemma 3 Each eigenvalue is the simple zero of $w(\lambda)$.

Proof. Let λ_0 be eigenvalue. We must show that $w'(\lambda_0) \neq 0$. Using the well-known Lagrange's formula we have

$$\begin{aligned} & \int_{-1}^0 \phi(x, \lambda) \phi(x, \lambda_0) dx + \frac{1}{\alpha_0 \beta_1} \int_0^1 \phi(x, \lambda) \phi(x, \lambda_0) dx \\ &= \frac{k_0}{\alpha_0 \beta_1 (\lambda - \lambda_0)} W(\phi(\cdot, \lambda), \phi(\cdot, \lambda_0); 1) \end{aligned} \quad (15)$$

Now in the right hand side of (15), letting $\lambda \rightarrow \lambda_0$ we derive that

$$\frac{k_0}{\alpha_0 \beta_1} \omega'(\lambda_0) = \int_{-1}^0 (\phi(x, \lambda_0))^2 dx + \frac{1}{\alpha_0 \beta_1} \int_0^1 (\phi(x, \lambda_0))^2 dx \quad (16)$$

so $w'(\lambda_0) \neq 0$.

Now by modifying the standard method (see, for example [13]) we prove the next theorem.

Theorem 1 The eigenvalues of the boundary-value-transmission problem (1) – (5) are real.

Proof. Let λ_0 be eigenvalue and y_0 be eigenfunction corresponding to this eigenvalue. By two partial integration we have

$$\begin{aligned} & \int_{-1}^0 (\lambda_0 y_0(x)) \overline{y_0(x)} dx + \frac{1}{\alpha_0 \beta_1} \int_0^1 (\lambda_0 y_0(x)) \overline{y_0(x)} dx \\ &= \int_{-1}^0 (\tau y_0)(x) \overline{y_0(x)} dx + \frac{1}{\alpha_0 \beta_1} \int_0^1 (\tau y_0)(x) \overline{y_0(x)} dx \\ &= \int_{-1}^0 y_0(x) \overline{(\tau y_0)(x)} dx + \frac{1}{\alpha_0 \beta_1} \int_0^1 y_0(x) \overline{(\tau y_0)(x)} dx + W[y_0, \bar{z}; -0] \\ &- W[y_0, \bar{y}_0; -1] + \frac{1}{\alpha_0 \beta_1} W[y_0, \bar{y}_0; 1] - \frac{1}{\alpha_0 \beta_1} W[y_0, \bar{y}_0; +0] \\ &= \int_{-1}^0 y_0(x) \overline{(\lambda_0 y_0)(x)} dx + \frac{1}{\alpha_0 \beta_1} \int_0^1 y_0(x) \overline{(\lambda_0 y_0)(x)} dx + W(y_0, \bar{z}; -0) \\ &- W[y_0, \bar{y}_0 - 1] + \frac{1}{\alpha_0 \beta_1} W[y_0, \bar{y}_0; 1] - \frac{1}{\alpha_0 \beta_1} W[y_0, \bar{y}_0; +0] \end{aligned} \quad (17)$$

From the boundary boundary conditions (2)-(3) it is follows obviously that

$$W(y_0, \bar{y}_0; -1) = 0 \text{ and } W(y_0, \bar{y}_0; 1) = 0. \quad (18)$$

The direct calculation gives

$$W(y_0, \bar{y}_0; -0) = \frac{1}{\alpha_0 \beta_1} W(y_0, \bar{y}_0; +0). \quad (19)$$

Substituting (18) and (19) in (17) we have the equality

$$(\lambda_0 - \bar{\lambda}_0) \left[\int_{-1}^0 (y_0(x))^2 dx + \frac{1}{\alpha_0 \beta_1} \int_0^1 (y_0(x))^2 dx \right] = 0$$

Thus, we get $\lambda_0 = \bar{\lambda}_0$ since $\alpha_0 \beta_1 > 0$. Consequently all eigenvalues of the problem (1) – (5) are real.

Corollary 1 Let $u(x)$ and $v(x)$ be eigenfunctions corresponding to distinct eigenvalues. Then they are orthogonal in the sense of the following equality

$$\int_{-1}^0 u(x)v(x)dx + \frac{1}{\alpha_0\beta_1} \int_0^1 u(x)v(x)dx = 0. \quad (20)$$

3. THE GREEN'S FUNCTION

Let us consider the inhomogeneous differential equation

$$u'' + (\lambda - q(x))u = f(x), \quad x \in [-1, 0] \cup (0, 1] \quad (21)$$

subject to boundary conditions

$$\cos \alpha u(-1) + \sin \alpha u'(-1) = 0, \quad (22)$$

$$\cos \beta u(1) + \sin \beta u'(1) = 0, \quad (23)$$

and transmission conditions

$$u(+0) - \alpha_0 u(-0) = 0, \quad (24)$$

$$u'(+0) - \beta_1 u'(-0) - \beta_0 u(-0) = 0. \quad (25)$$

Making use of the definitions of the functions $\phi_i(x, \lambda)$, $\chi_i(x, \lambda)$ ($i = 1, 2$) we see that the general solution of the differential equation (21) can be represented in the form

$$u(x, \lambda) = \begin{cases} \frac{\chi_1(x, \lambda)}{\omega_1(\lambda)} \int_{-1}^x \phi_1(y, \lambda) f(y) dy + \frac{\phi_1(x, \lambda)}{\omega_1(\lambda)} \int_x^0 \chi_1(y, \lambda) f(y) dy \\ \quad + C_1 \phi_1(x, \lambda) + D_1 \chi_1(x, \lambda) & \text{for } x \in [-1, 0) \\ \frac{\chi_2(x, \lambda)}{\omega_2(\lambda)} \int_0^x \phi_2(y, \lambda) f(y) dy + \frac{\phi_2(x, \lambda)}{\omega_2(\lambda)} \int_x^1 \chi_2(y, \lambda) f(y) dy \\ \quad + C_2 \phi_2(x, \lambda) + D_2 \chi_2(x, \lambda) & \text{for } x \in (0, 1] \end{cases} \quad (26)$$

where C_i, D_i ($i = 1, 2$) are arbitrary constants. By substitution into the boundary conditions (22) and (23) we have at once that

$$D_1 = 0, \quad C_2 = 0. \quad (27)$$

Further, substitution (26) into transmission conditions (24) and (25) we have the inhomogeneous linear system of equations for C_1 and D_1 , the determinant of which is equal to $-\omega(\lambda)$ and therefore is not vanish by assumption. Solving that system we find

$$C_1 = \frac{1}{\omega_2(\lambda)} \int_0^1 \chi_2(y, \lambda) f(y) dy, \quad (28)$$

$$D_2 = \frac{1}{\omega_1(\lambda)} \int_{-1}^0 \phi_1(y, \lambda) f(y) dy \quad (29)$$

Putting (27), (28) and (29) in (26) we deduce that the problem (21)-(25) has an unique solution $u := u_f(x, \lambda)$ in the form

$$u_f(x, \lambda) = \begin{cases} \frac{\alpha_0 \beta_1 \chi_1(x, \lambda)}{\omega(\lambda)} \int_{-1}^x \phi_1(y, \lambda) f(y) dy + \frac{\alpha_0 \beta_1 \phi_1(x, \lambda)}{\omega(\lambda)} \int_x^0 \chi_1(y, \lambda) f(y) dy \\ \quad + \frac{\phi_1(x, \lambda)}{\omega_2(\lambda)} \int_0^1 \chi_2(y, \lambda) f(y) dy & \text{for } x \in [-1, 0) \\ \frac{\chi_2(x, \lambda)}{\omega(\lambda)} \int_0^x \phi_2(y, \lambda) f(y) dy + \frac{\phi_2(x, \lambda)}{\omega(\lambda)} \int_x^1 \chi_2(y, \lambda) f(y) dy \\ \quad + \frac{\alpha_0 \beta_1 \chi_2(x, \lambda)}{\omega(\lambda)} \int_{-1}^0 \phi_1(y, \lambda) f(y) dy & \text{for } x \in (0, 1] \end{cases} \quad (30)$$

From this formula we derive that the Green's function of the problem (21)-(25) can be represented as

$$G_1(x, y; \lambda) = \begin{cases} \frac{\phi(y, \lambda)\chi(x, \lambda)}{\omega(\lambda)} & \text{for } -1 \leq y \leq x \leq 1, \quad x, y \neq 0 \\ \frac{\phi(x, \lambda)\chi(y, \lambda)}{\omega(\lambda)} & \text{for } -1 \leq x \leq y \leq 1, \quad x, y \neq 0 \end{cases} \quad (31)$$

and the formula (30) can be rewritten in terms of this Green's function as

$$u_f(x, \lambda) = \int_{-1}^0 G_1(x, y; \lambda) f(y) dy + \frac{1}{\alpha_0 \beta_1} \int_0^1 G_1(x, y; \lambda) f(y) dy \quad (32)$$

4. THE RESOLVENT OPERATOR AND SELFADJOINTNESS OF THE PROBLEM

In this section we shall introduce a special equivalent inner product in the Hilbert space $L_2[-1, 0] \oplus L_2(0, 1]$ and define a symmetric linear operator A in this space such a way that the considered problem can be interpreted as the eigenvalue problem of A . Namely, in the Hilbert Space $H = L_2[-1, 0] \oplus L_2(0, 1]$ of two-component vectors we define an equivalent inner product by

$$\langle F, G \rangle_H := \int_{-1}^0 F_1(x) \overline{G_1(x)} dx + \frac{1}{\alpha_0 \beta_1} \int_0^1 F_1(x) \overline{G_1(x)} dx$$

for $F = F_1(x)$, $G = G_1(x)$, $\in H$ and define a linear operator $A : H \rightarrow H$ with domain of definition

$$D(A) = \left\{ \begin{array}{l} F = F_1(x) : F_1(x) \text{ and } F_1'(x) \text{ are absolutely} \\ \text{continuous in each of intervals } [-1, 0) \text{ and } (0, 1], \\ \text{and has a finite limits } F_1(0\mp) \text{ and } F_1'(0\mp), \\ \tau F_1 \in L_2[-1, 0] \oplus L_2(0, 1], \quad \cos \alpha u(-1) + \sin \alpha u'(-1) = 0, \\ \cos \beta u(1) + \sin \beta u'(1) = 0, \quad u(+0) = \alpha_0 u(-0), \\ u'(+0) = \beta_1 u'(-0) + \beta_0 u(-0) \end{array} \right\}$$

and action law

$$Au := -u''(x) + q(x)u(x).$$

Then the problem (22) – (25) can be written in the operator form, as

$$(\lambda - A)u = f$$

in the Hilbert space H . If λ not an eigenvalue, we find that for $f \in H$ this operator equation has an unique solution

$$u_f(x, \lambda) := \langle G_1(x, \cdot, \lambda), \overline{f(\cdot)} \rangle_H. \quad (33)$$

Now, making use (31) and (33) we see that if $\text{Im} \lambda \neq 0$ then

$$u_f(\cdot, \lambda) \in D(A) \text{ for } f \in H, \quad (34)$$

and

$$\|u_f(\cdot, \lambda)\| \leq \frac{1}{|\text{Im} \lambda|} \|f\| \text{ for } f \in H. \quad (35)$$

Hence, each nonreal $\lambda \in \mathbb{C}$ is a regular point of an operator A and

$$R(\lambda, A)f = u_f(\cdot, \lambda) \text{ for } f \in H \quad (36)$$

Further, it is easy to see that $D(A)$ is dense in H , and because of (34) and (36)

$$(\lambda - A)D(A) = (\bar{\lambda} - A)D(A) = H \text{ for } \text{Im}\lambda \neq 0 \quad (37)$$

By virtue of [[8], Theorem 2.2.p.198], from this it follows that A is self-adjoint in the Hilbert Space H .

5. COUNTEREXAMPLE

Recall that we had derived all results in this study under condition $\alpha_0\beta_1 > 0$. Let us show that this simple condition on the sign of the coefficients of the transmission conditions can not be omitted without putting any other condition on this coefficients. For this, consider the following special case of the problem (1)-(5) for which $\alpha_0\beta_1 < 0$

$$\begin{aligned} -u'' &= \lambda u, & x \in [-1, 0) \cup (0, 1] \\ u(-1) &= 0, & \lambda u(1) = u'(1) \\ u(-0) &= u(+0), & u'(-0) = -u'(+0) \end{aligned}$$

It is easy to verify that this problem has only the trivial solution for any $u = 0$ for any $\lambda \in \mathbb{C}$. Thus, if $\alpha_0\beta_1 < 0$ then the spectrum of the problem (1)-(5) may be empty.

REFERENCES

- [1] M. C. Agronovich , Spectral properties in problems of diffraction, in Generalized Method of Eigenoscillation in the Theory of Diffraction(Russian), (N.N. Voitovich, B.Z. Kazenelenbaum and A. N. Sivov), Nauka, Moscow, 1977.
- [2] B. Chanane, Eigenvalues of Sturm-Liouville problems with discontinuity conditions inside a finite interval, Appl. Math. Comput. 188, 1725-1732, 2007.
- [3] F. Chung and S.T. Yau, Discrete Greens functions, Journal of Combinatorial Theory, Series A, 91(1-2) pp. 191-214, 2000.
- [4] D. G. Duffy , em Green' s Functions with Applications, Studies in Advanced Mathematics, Chapman Hall/CRC, Boca Raton, Fla, USA, 2001.
- [5] M. Kadakal and O. Sh. Mukhtarov, Discontinuous Sturm-Liouville Problems Containing Eigenparameter in the Boundary Conditions, Acta Mathematica Sinica, English Series Sep., 22(5), pp. 1519-1528, 2006.
- [6] M. Kandemir, O. Sh. Mukhtarov and Y. Y. Yakubov, Irregular boundary value problems with discontinuous coefficients and the eigenvalue parameter, Mediterr, J. Math. 6, 317-338, 2009.
- [7] M. Kobayashi, Eigenvalues of discontinuous Sturm-Liouville problems with symmetric potentials, Computers. Math. Applic. 18(4), 357-364, 1989.
- [8] S. Lang, Real Analysis (Second edition) Addison-Wesley, Reading, Mass., 1983.
- [9] A. V. Likov and Yu. A. Mikhailov, The theory of Heat and Mass Transfer(Russian), Qosenergaizdat, 1963.
- [10] J. Liu, B. Jia and Q. Zhu, An estimate for the three-dimensional discrete Greens function and applications, J. Math. Anal. Appl., 370(2), pp. 350-363, 2010.
- [11] F. S. Muhtarov and K. Aydemir, Distributions of eigenvalues for Sturm-Liouville problem under jump conditions, Journal of New Results in Science 1 81-89, 2012.
- [12] A. D. Polyanin, Handbook of Linear Partial Differential Equations for Engineers and Scientists. Chapman Hall/CRC Press, Boca Raton, 2002.
- [13] E. C. Titchmarsh, Eigenfunctions Expansion Associated with Second Order Differential Equations I, second edn. Oxford Univ. Press, London, 1962.
- [14] E. Uğurlu and E. Bairamov, Dissipative operators with impulsive conditions, J. Math. Chem., 51, 1670-1680, DOI 10.1007/s10910-013-0172-5, 2013.

F. S. MUHTAROV
BAKU STATE UNIVERSITY, AZERBAIJAN
E-mail address: `ilgarm@ayhanm-xerox.com`

K. AYDEMIR
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, GAZIOSMANPAŞA UNIVERSITY, 60250 TOKAT,
TURKEY
E-mail address: `kadriye.aydemir@gop.edu.tr`

O. SH. MUKHTAROV
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, GAZIOSMANPAŞA UNIVERSITY, 60250 TOKAT,
TURKEY
E-mail address: `omukhtarov@yahoo.com`