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ON APPROXIMATION OF CONJUGATE FUNCTIONS

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Abstract. In present paper, three quite new theorems on degree of approximation of a function $\tilde{f}$ belonging to the class $Lip(\alpha, r)$, class $Lip(\xi(t), r)$ and weighted class $W(L^r(f), \xi(t))$, $1 \leq r < \infty$ by (C,2)(E,1) product operators on its conjugate Fourier series have been established. Here, the function $\tilde{f}$ is conjugate to a 2π-periodic function f and $\xi(t)$ is non-negative and increasing function of t . The results obtained in this paper further extend several known results on linear operators.

1. Introduction

A number of researchers ([1], [2], [3], [4], [5], [7], [8], [9], [10], [12], [13], [16], [17], [18], [19], [21], [22], [23], [24], [25], [26], [27]) have studied error estimates $E_n(f)$ of a function belonging to different classes using different linear operators. A study on degree of approximation of conjugate of a function belonging to class $Lip(\xi(t), r)$ by product summability method on conjugate Fourier series has been made by Lal and Singh [14]. Recently, a study on $(C, 2)(E, 1)$ product summability of Fourier series and conjugate Fourier series has been made by Nigam [20].

Motivated by the work of earlier authors on degree of approximation of a function using linear operators, we, in present paper, use quite new product operators and obtain three quite new results. In fact, we establish three theorems on degree of approximation of a function $\tilde{f}$, conjugate to a 2π-periodic function f belonging to the class $Lip(\alpha, r)$, class $Lip(\xi(t), r)$ and weighted class $W(L^r(f), \xi(t))$, $1 \leq r < \infty$ by $(C, 2)(E, 1)$ product operators on its conjugate Fourier series.

2. Preliminaries

Let f be a 2π-periodic and Lebesgue integrable function. The Fourier series associated with f at a point x is defined by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x) \tag{1}$$

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with partial sum $s_n(f;x)$.

The conjugate series of the Fourier series (1) is given by

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx) \equiv \sum_{n=1}^{\infty} B_n(x) \tag{2}$$

with partial sum $\tilde{s}_n(f;x)$.

Throughout this paper, we shall call (2) as conjugate Fourier series of the funtion f.

The $L^{\infty}(f)$-norm of a function $f : R^n \to R$ is defined by

$$\|f\|_{\infty} = \sup\{| f(x) |: x \in R\} \tag{3}$$

The $L^r(f)$-norm of a function $f : R^n \to R$ is defined by

$$\|f\|_r = \left(\int_0^{2\pi} | f(x) |^r \, dx\right)^{\frac{1}{r}}, \ 1 \le r < \infty \tag{4}$$

The degree of approximation of a function $f : R^n \to R$ by a trigonometric polynomial $t_n(x)$ of order n under sup norm $\| \ \|_{\infty}$ is defined by

$$\| f(x) - t_n(x) \|_{\infty} = \sup\{| f(x) - t_n(x) |: x \in R\} \ [28] \tag{5}$$

and $E_n(f)$ of a function f belongs to $L^r(f)$ is given by

$$E_n(f) = \min_{t_n} \|f(x) - t_n(x)\|_r \tag{6}$$

This method of approximation is called trigonometric Fourier approximation (TFA).

Let $f(x)$ and $g(x)$ be two functions defined on some subset of the real numbers. One writes

$$f(x) = O(g(x))$$

if and only if there exists a positive real number M and a real number x_0 such that

$$|f(x)| \le M\,|g(x)| \text{ for all } x > x_0.$$

One says that a function f belongs to the class $Lip\alpha$ if

$$f(x+t) - f(x) = O(| t |^{\alpha}) \ for \ 0 < \alpha \le 1 \tag{7}$$

and that f belongs to the class $Lip(\alpha, r)$ if

$$\left(\int_0^{2\pi} | f(x+t) - f(x) |^r \, dx\right)^{\frac{1}{r}} = O(| t |^{\alpha}), \ 0 < \alpha \le 1 \text{ and } 1 \le r < \infty. \tag{8}$$

([15], definition 5.38)

Given a positive increasing function $\xi(t)$ and some r, $1 \le r < \infty$, one says that a function f belongs to the class $Lip((\xi(t), r)$ if

$$\left(\int_0^{2\pi} | f(x+t) - f(x) |^r \, dx\right)^{\frac{1}{r}} = O(\xi(t)) \tag{9}$$

and that f belongs to the class $W(L^r(f),\xi(t))$, $1 \leq r < \infty$ if

$$\left(\int_0^{2\pi} \mid \{f(x+t)-f(x)\}\ sin^{\beta}x \mid^r dx\right)^{\frac{1}{r}} = O(\xi(t)),\ \beta \geq 0. \tag{10}$$

If $\beta = 0$, our newly defined class $W(L^r(f),\xi(t))$ reduces to the class $Lip(\xi(t),r)$ and if $\xi(t) = t^{\alpha}$ then $Lip(\xi(t),r)$ class reduces to the class $Lip(\alpha,r)$.

Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with s_n for its n^{th} partial sum.

Let $\{t_n^{C_2}\}$ denote the sequence of $(C,2)$ means of the sequence $\{s_n\}$. If the $(C,2)$ transform of s_n is defined as

$$t_n^{C_2}(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\ s_k(f;x) \rightarrow s \text{ as } n \rightarrow \infty \tag{11}$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by $(C,2)$ method (Cesàro method).

Let $\{t_n^{E_1}\}$ denote the sequence of $(E,1)$ means of the sequence $\{s_n\}$. If the $(E,1)$ transform of s_n is defined as

$$t_n^{E_1}(f;x) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k} s_k(f;x) \rightarrow s \text{ as } n \rightarrow \infty \tag{12}$$

the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by $(E,1)$ method (Euler method)([10]).

Thus if

$$t_n^{C_2E_1}(f;x) = \frac{2}{(n+1)(n+2)}\sum_{k=0}^{n}(n-k+1)\frac{1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu} s_{\nu}(f;x) \rightarrow s \text{ as } n \rightarrow \infty, \tag{13}$$

where $\{t_n^{C_2E_1}\}$ denote the sequence of $(C,2)(E,1)$ product means of the sequence s_n, the series $\sum_{n=0}^{\infty} u_n$ is said to be summable to s by $(C,2)(E,1)$ method.

Now, we mention Hölder's inequality, Minkowiski's inequality and second mean value theorem for integrals, which are used in the proofs of our main results.

Hölder's Inequality.
Supoose $f,g : R^n \rightarrow R$ are Lebesgue measurable. Then

$$\|fg\|_1 \leq \|f\|_r \|g\|_s \tag{14}$$

Minkowiski's Inequality.
Supoose $f,g : R^n \rightarrow R$ are Lebesgue measurable. Then

$$\|f+g\|_r \leq \|f\|_r + \|g\|_s \tag{15}$$

Second Mean Value Theorem for Integrals.
Let $f(x)$ and $g(x)$ be two functions, which are in (a,b). If $g(x)$ is a positive monotonic increasing function in (a,b) then there exists a value c, where $a \leq c \leq b$ such

that

$$\int_a^b f(x)g(x)dx = g(b)\int_c^d f(x)dx \tag{16}$$

We use the following notations.

$$\psi(t) = \psi(x,t) = f(x+t) - f(x-t)$$

$$\tilde{K}_n(t) = \frac{1}{\pi(n+1)(n+2)} \sum_{k=0}^{n} \left[\frac{n-k+1}{2^k} \sum_{\nu=0}^{k} \binom{k}{\nu} \frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}} \right]$$

$$\tilde{f}(x) = -\frac{1}{2\pi}\int_0^{2\pi} \psi(t)\cot\left(\frac{t}{2}\right) dt \text{ [29]}$$

3. The Main Results

We prove the following theorems:

3.1. **Theorem 1.** If a function $\tilde{f}(x)$, conjugate to a 2π-periodic function $f(x)$ belonging to the class $Lip(\alpha, r)$, $1 \le r < \infty$, then its degree of approximation by $(C,2)(E,1)$ means on its conjugate Fourier series (2) is given by

$$\| \tilde{f}(x) - t_n^{C_2E_1}(x) \|_r = O\left\{ \frac{1}{(n+1)^{\alpha-\frac{1}{r}}} \right\} \; for \; 0 < \alpha \le 1, \tag{17}$$

where $r^{-1} + s^{-1} = 1$, $1 \le r < \infty$ and $\alpha r > 1$, provided that

$$2^\tau \sum_{k=\tau}^{n} \left(\frac{n-k+1}{2^k} \right) = O(n+1)(n+2) \tag{18}$$

3.2. **Theorem 2.** If a function $\tilde{f}(x)$, conjugate to a 2π-periodic function $f(x)$ belonging to the class $Lip(\xi(t), r)$, $1 \le r < \infty$, then its degree of approximation by $(C,2)(E,1)$ means on its conjugate Fourier series (2) is given by

$$\| \tilde{f}(x) - t_n^{C_2E_1}(x) \|_r = O\left\{ (n+1)^{\frac{1}{r}} \; \xi\left(\frac{1}{n+1}\right) \right\}, \tag{19}$$

provided that $\xi(t)$ satisfies the conditions

$$\left\{ \int_0^{\frac{1}{n+1}} \left(\frac{t \mid \psi(t) \mid}{\xi(t)} \right)^r dt \right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{20}$$

and

$$\left\{ \int_{\frac{1}{n+1}}^{\pi} \left(\frac{t^{-\delta} \mid \psi(t) \mid}{\xi(t)} \right)^r dt \right\}^{\frac{1}{r}} = O\left\{(n+1)^\delta\right\} \tag{21}$$

uniformaly in x, in which δ is an arbitrary number with $s(1-\delta) - 1 > 0$, where $r^{-1} + s^{-1} = 1$, $1 \le r < \infty$ and (18) holds.

3.3. **Theorem 3.** If a function $\tilde{f}(x)$, conjugate to a 2π-periodic function $f(x)$ belonging to the class weighted $W(L_r(f), \xi(t))$, $1 \leq r < \infty$, then its degree of approximation by $(C,2)(E,1)$ means on its conjugate Fourier series (2) is given by

$$\| \tilde{f}(x) - t_n^{C_2E_1}(x) \|_r = O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\}, \tag{22}$$

provided that $\xi(t)$ satisfies the condition

$$\left\{\int_0^{\frac{1}{n+1}} \left(\frac{t \mid \psi(t) \mid}{\xi(t)}\right)^r \ \sin^{\beta r} t\ dt\right\}^{\frac{1}{r}} = O\left(\frac{1}{n+1}\right) \tag{23}$$

and (21) uniformaly in x, in which δ is an arbitrary number with $s(1-\delta)-1>0$, where $r^{-1}+s^{-1}=1$, $1 \leq r < \infty$ and

$$\frac{\xi(t)}{t} \text{ is non-increasing in t} \tag{24}$$

and (18) holds.

4. Lemmas

Following lemmas are required for the proof of our theorems,

4.1. **Lemma 1.**

$$\left|\tilde{K}_n\left(t\right)\right| = O(n+1), \text{for } 0 \leq t \leq \frac{1}{n+1}$$

Proof. For $0 \leq t \leq \frac{1}{n+1}$, $\sin\frac{t}{2} \geq \frac{t}{\pi}$ and $|\cos nt| \leq 1$

$$\begin{aligned}
\left|\tilde{K}_n\left(t\right)\right| &= \frac{1}{\pi\left(n+1\right)\left(n+2\right)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right]\right| \\
&\leq \frac{1}{t\left(n+1\right)\left(n+2\right)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\right]\right| \\
&= \frac{1}{t\left(n+1\right)\left(n+2\right)}\left|\sum_{k=0}^{n}(n-k+1)\right| \\
&= O\left(\frac{1}{t}\right)
\end{aligned}$$

□

4.2. **Lemma 2.** For $0 \leq a \leq b \leq \infty$, $0 \leq t \leq \pi$ and for any n, we have

$$\left|\tilde{K}_n\left(t\right)\right| = O\left(\frac{1}{t}\right)$$

Proof. For $0 < \frac{1}{n+1} \leq t \leq \pi$, $\sin\left(\frac{t}{2}\right) \geq \frac{t}{\pi}$.

$$\begin{aligned}
\left|\tilde{K}_n\left(t\right)\right| &= \frac{1}{\pi\left(n+1\right)\left(n+2\right)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\sum_{\nu=0}^{k}\binom{k}{\nu}\frac{\cos\left(\nu+\frac{1}{2}\right)t}{\sin\left(\frac{t}{2}\right)}\right]\right| \\
&\leq \frac{1}{t\left(n+1\right)\left(n+2\right)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\binom{k}{\nu}e^{i\left(\nu+\frac{1}{2}\right)t}\right\}\right]\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)e^{i\nu t}\right\}\right]\right|\left|e^{i\frac{t}{2}}\right|\\
&\leq \frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=0}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)e^{i\nu t}\right\}\right]\right|\\
&\leq \frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=0}^{\tau-1}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)e^{i\nu t}\right\}\right]\right|\\
&+ \frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=\tau}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)e^{i\nu t}\right\}\right]\right|, \qquad (25)
\end{aligned}
$$

where τ denotes the integral part of $\frac{1}{t}$.

Now we consider first term of (25),

$$
\begin{aligned}
&\frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=0}^{\tau-1}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)e^{i\nu t}\right\}\right]\right|\\
&\leq \frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=0}^{\tau-1}\left[\frac{n-k+1}{2^k}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)\right\}\right]\right|\left|e^{i\nu t}\right|\\
&\leq \frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=0}^{\tau-1}\left[\frac{n-k+1}{2^k}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)\right\}\right]\right|\\
&= \frac{1}{t\,(n+1)\,(n+2)}\sum_{k=0}^{\tau-1}(n-k+1)\\
&= \frac{1}{t\,(n+1)\,(n+2)}\sum_{k=0}^{\tau-1}(n+1) - \frac{1}{t\,(n+1)\,(n+2)}\sum_{k=0}^{\tau-1}k\\
&= \frac{1}{t(n+2)}\sum_{k=0}^{\tau-1}1 - \frac{1}{t\,(n+1)\,(n+2)}\sum_{k=0}^{\tau-1}k\\
&= \frac{\tau-1}{t(n+2)} - \frac{\tau(\tau-1)}{t\,(n+1)\,(n+2)}\\
&\leq k\left(\frac{1}{t}\right) \qquad (26)
\end{aligned}
$$

Now considering second term of (25) and using Abel's lemma

$$
\begin{aligned}
&\frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=\tau}^{n}\left[\frac{n-k+1}{2^k}\mathrm{Re}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)e^{i\nu t}\right\}\right]\right|\\
&\leq \frac{1}{t\,(n+1)\,(n+2)}\left|\sum_{k=\tau}^{n}\left[\frac{n-k+1}{2^k}\max_{0\leq\ m\leq\ k}\left\{\sum_{\nu=0}^{k}\left(\begin{array}{c}k\\ \nu\end{array}\right)e^{i\nu t}\right\}\right]\right|\\
&\leq \frac{k}{t\,(n+1)\,(n+2)}2^{\tau}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right) \qquad (27)
\end{aligned}
$$

Combining (25) to (27),

$$\left|\tilde{K}_n(t)\right| \leq k\left(\frac{1}{t}\right) + k\left\{\frac{2^{\tau}}{t\,(n+1)\,(n+2)}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right)\right\} \tag{28}$$

□

4.3. **Lemma 3.** ([15], lemma 5.40) If $f(x)$ belongs to the class $Lip(\alpha, r)$ on $[0,\pi]$ then $\psi(t)$ belongs to the class $Lip(\alpha, r)$ on $[0,\pi]$.

5. Proof of the main results

We obtain our main theorems in the following way.

5.1. **Proof of theorem 1.** It is well known that $s_n(f;x)$ of the series (2) is given by

$$\tilde{s}_n(f;x) - \tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\psi(t)\frac{\cos(n+\frac{1}{2})t}{\sin\frac{t}{2}}dt$$

Using (2), the $(E,1)$ transform of $s_n(f;x)$ is given by

$$\tilde{f}(x) - t_n^{E_1}(x) = \frac{1}{2\pi\ 2^n}\int_0^{\pi}\psi(t)\left\{\sum_{k=0}^{n}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{\cos(k+\frac{1}{2})t}{\sin\frac{t}{2}}\right\}dt$$

The $(C,2)(E,1)$ transform of $s_n(f;x)$ is given by

$$\begin{aligned}
\tilde{f}(x) - t_n^{C_2E_1}(x) &= \frac{1}{\pi\ (n+1)(n+2)}\sum_{k=0}^{n}\left[\frac{(n-k+1)}{2^k}\int_0^{\pi}\frac{\psi(t)}{\sin\frac{t}{2}}\ \left\{\sum_{\upsilon=0}^{k}\left(\begin{array}{c} k \\ \upsilon \end{array}\right)\cos\left(\upsilon+\frac{1}{2}\right)t\right\}dt\ \right] \\
&= \int_0^{\pi}\psi(t)\ \tilde{K}_n(t)\ dt \\
&= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\psi(t)\tilde{K}_n(t)\ dt \\
&= I_{1.1} + I_{1.2}\ \text{(say)}
\end{aligned} \tag{29}$$

We consider,

$$\mid I_{1.1} \mid \leq \int_0^{\frac{1}{n+1}} \mid \psi(t) \mid\mid \tilde{K}_n(t) \mid dt$$

Using Hölder's inequality (14) and lemma 3,

$$\begin{aligned}
\mid I_{1.1} \mid &\leq \left[\int_0^{\frac{1}{n+1}}\left\{\frac{t \mid \psi(t) \mid}{t^{\alpha}}\right\}^r dt\right]^{\frac{1}{r}}\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\mid \tilde{K}_n(t) \mid}{t^{1-\alpha}}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\mid \tilde{K}_n(t) \mid}{t^{1-\alpha}}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{1}{t^{2-\alpha}}\right\}^s dt\right]^{\frac{1}{s}} \text{ by lemma 1} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} t^{\alpha s - 2s} dt\right]^{\frac{1}{s}}
\end{aligned}$$

$$
\begin{aligned}
&=O\left(\frac{1}{n+1}\right)\left[\left(\frac{1}{n+1}\right)^{\frac{\alpha s-2s+1}{s}}\right]\\
&=O\left[\left(\frac{1}{n+1}\right)^{\alpha-1+\frac{1}{s}}\right]\\
&=O\left[\left(\frac{1}{n+1}\right)^{\alpha-\left(1-\frac{1}{s}\right)}\right]\\
I_{1.1} &=O\left[\left(\frac{1}{n+1}\right)^{\alpha-\frac{1}{r}}\right] \qquad (30)
\end{aligned}
$$

Similarly as above, we have

$$
\begin{aligned}
\mid I_{1.2}\mid \leq &\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\mid\psi(t)\mid}{t^{\alpha}}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\mid\tilde{K}_n(t)\mid}{t^{-\delta-\alpha}}\right\}^{s}dt\right]^{\frac{1}{s}}\\
=&O\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}t^{\alpha-\frac{1}{r}}}{t^{\alpha}}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\mid\tilde{K}_n(t)\mid}{t^{-\delta-\alpha}}\right\}^{s}dt\right]^{\frac{1}{s}}\\
=&O\left[\int_{\frac{1}{n+1}}^{\pi}\left\{t^{-\frac{1}{r}-\delta}\right\}^{r}dt\right]^{\frac{1}{r}} \text{ using lemma 2}\\
&\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{1}{t^{-\delta-\alpha}}\left\{\left(\frac{1}{t}\right)+\left\{\left(\frac{1}{t\,(n+1)\,(n+2)}\right)2^{\tau}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right)\right\}\right\}\right\}^{s}dt\right]^{\frac{1}{s}}\\
=&O\left[\int_{\frac{1}{n+1}}^{\pi}t^{-1-\delta r}dt\right]^{\frac{1}{r}}\\
&\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{1}{t^{-\delta-\alpha}}\left\{\left(\frac{1}{t}\right)+\left\{\left(\frac{1}{t\,(n+1)\,(n+2)}\right)2^{\tau}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right)\right\}\right\}\right\}^{s}dt\right]^{\frac{1}{s}}\\
=&O(n+1)^{\delta}\\
&\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{1}{t^{-\delta-\alpha}}\left\{\left(\frac{1}{t}\right)+\left\{\left(\frac{1}{t\,(n+1)\,(n+2)}\right)2^{\tau}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right)\right\}\right\}\right\}^{s}dt\right]^{\frac{1}{s}}\\
=&\ O(n+1)^{\delta}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{1}{t^{-\delta-\alpha}}\left\{\frac{1}{t}+\frac{1}{t}\right\}\right\}^{s}dt\right]^{\frac{1}{s}}\\
=&O(n+1)^{\delta}\left[\int_{\frac{1}{n+1}}^{\pi}t^{s\alpha+s\delta-s}dt\right]^{\frac{1}{s}}\\
=&O\left[(n+1)^{\delta}\left\{(n+1)^{-s\alpha-s\delta+s-1}\right\}^{\frac{1}{s}}\right]\\
=&O\left[(n+1)^{\delta}(n+1)^{-\alpha-\delta+1-\frac{1}{s}}\right]\\
I_{1.2} =&O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right] \qquad (31)
\end{aligned}
$$

Combining (29) to (31),

$$\left|\tilde{f}(x) - t_n^{C_2E_1}(x)\right| = O\left[\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right]$$

Using $L^r(f)$-norm, we get

$$\begin{aligned}\parallel \tilde{f}(x) - t_n^{C_2E_1}(x) \parallel_r &= \left\{\int_0^{2\pi} \mid \tilde{f}(x) - t_n^{C_2E_1}(x) \mid^r dx\right\}^{\frac{1}{r}} \\ &= O\left[\left\{\int_0^{2\pi} \left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}^r dx\right\}^{\frac{1}{r}}\right] \\ &= O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right] \\ \parallel \tilde{f}(x) - t_n^{C_2E_1}(x) \parallel_r &= O\left\{\frac{1}{(n+1)^{\alpha-\frac{1}{r}}}\right\}\end{aligned}$$

This completes the proof of theorem 1.

5.2. **Proof of theorem 2.** Following the proof of theorem 1

$$\begin{aligned}\tilde{f}(x) - t_n^{C_2E_1}(x) &= \left[\int_0^{\frac{1}{n+1}} + \int_{\frac{1}{n+1}}^{\pi}\right]\psi(t)\tilde{K}_n(t)\ dt \\ &= I_{2.1} + I_{2.2} \text{ (say)}\end{aligned} \tag{32}$$

Now we consider,

$$\mid I_{2.1} \mid \leq \int_0^{\frac{1}{n+1}} \mid \psi(t) \mid \mid \tilde{K}_n(t) \mid dt$$

Using Hölder's inequality (14) and the fact that $\psi(t) \in Lip(\xi(t), r)$,

$$\begin{aligned}\mid I_{2.1} \leq &\left[\int_0^{\frac{1}{n+1}} \left\{\frac{t \mid \psi(t) \mid}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t) \mid \tilde{K}_n(t) \mid}{t}\right\}^s dt\right]^{\frac{1}{s}} \\ = &\ O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t) \mid \tilde{K}_n(t) \mid}{t}\right\}^s dt\right]^{\frac{1}{s}} \text{ by (20)} \\ = &\ O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}} \left\{\frac{\xi(t)}{t^2}\right\}^s dt\right]^{\frac{1}{s}} \text{ by lemma 1}\end{aligned}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals (16)

$$\begin{aligned}I_{2.1} &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}} \frac{dt}{t^{2s}}\right]^{\frac{1}{s}} \text{ for some } 0 \leq \in \leq \frac{1}{n+1} \\ &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-2s+1}}{-2s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}} \\ &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left\{(n+1)^{2-\frac{1}{s}}\right\}\end{aligned}$$

$$= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \tag{33}$$

Using Hölder's inequality (14),

$$\begin{aligned}
|I_{2.2}| &\leq \int_{\frac{1}{n+1}}^{\pi} |\psi(t)|\,\left|\tilde{K}_n(t)\right| dt \\
&\leq \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)\left|\tilde{K}_n(t)\right|}{t^{-\delta}}\right\}^s dt\right]^{\frac{1}{s}} \\
&\leq k\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{t^{-\delta}|\psi(t)|}{\xi(t)}\right\}^r dt\right]^{\frac{1}{r}} \text{ using lemma 2} \\
&\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\left(\frac{\xi(t)}{t^{1-\delta}}\right) + \left\{2^{\tau}\,\frac{\xi(t)}{t^{1-\delta}(n+1)(n+2)}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^k}\right)\right\}\right\}^s dt\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{n+1}}^{\pi} \left\{\frac{\xi(t)}{t^{1-\delta}}\right\}^s dt\right]^{\frac{1}{s}} \text{ by (18) and (21)}
\end{aligned}$$

Now putting $t = \frac{1}{y}$,

$$I_{2.2} = O\left\{(n+1)^{\delta}\right\}\left[\int_{\frac{1}{\pi}}^{n+1} \left\{\frac{\xi\left(\frac{1}{y}\right)}{(y)^{\delta-1}}\right\}\frac{dy}{y^2}\right]^{\frac{1}{s}}$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals (16),

$$\begin{aligned}
I_{2.2} &= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1} \frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq \eta \leq n+1 \\
&= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1} \frac{dy}{y^{s(\delta-1)+2}}\right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq 1 \leq n+1 \\
&= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{y^{s(1-\delta)-1}}{s(1-\delta)-1}\right\}_1^{n+1}\right]^{\frac{1}{s}} \\
&= O\left\{(n+1)^{\delta}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1-\delta)-\frac{1}{s}}\right] \\
&= O\left\{\xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{1-\frac{1}{s}}\right] \\
&= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}
\end{aligned} \tag{34}$$

Combining (32) to (34),

$$\left|\tilde{f}(x) - t_n^{C_2E_1}(x)\right| = O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\} \tag{35}$$

Using $L^r(f)$-norm, we get

$$
\begin{aligned}
\left\|\tilde{f}(x)-t_n^{C_2E_1}(x)\right\|_r &= \left\{\int_0^{2\pi}\left|\tilde{f}(x)-t_n^{C_2E_1}(x)\right|^r dx\ \right\}^{\frac{1}{r}} \\
&= O\left[\left\{\int_0^{2\pi}\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}^r dx\right\}^{\frac{1}{r}}\right] \\
&= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\int_0^{2\pi} dx\right\}^{\frac{1}{r}}\right] \\
\left\|\tilde{f}(x)-t_n^{C_2E_1}(x)\right\|_r &= O\left\{(n+1)^{\frac{1}{r}}\,\xi\left(\frac{1}{n+1}\right)\right\}
\end{aligned}
$$

This completes the proof of theorem 2.

5.3. **Proof of theorem 3.** Following the proof of theorem 1

$$
\begin{aligned}
\left|\tilde{f}(x)-t_n^{C_2E_1}(x)\right| &= \left[\int_0^{\frac{1}{n+1}}+\int_{\frac{1}{n+1}}^{\pi}\right]\psi(t)\tilde{K}_n(t)\ dt \\
&= I_{3.1}+I_{3.2}\ \text{(say)}
\end{aligned}
\tag{36}
$$

we have
$|\ \psi(x+t)-\psi(x)\ |\leq|\ f(u+x+t)-f(u+x)\ |+|\ f(u-x-t)-f(u-x)\ |$

Hence, by Minkowiski's inequality (15),

$$
\begin{aligned}
&\left[\int_0^{2\pi}|\ \{\psi(x+t)-\psi(x)\}\sin^{\beta}x\ |^r\ dx\right]^{\frac{1}{r}} \\
&\leq\left[\int_0^{2\pi}|\ \{f(u+x+t)-f(u+x)\}\sin^{\beta}x\ |^r\ dx\right]^{\frac{1}{r}} \\
&+\left[\int_0^{2\pi}|\ \{f(u-x-t)-f(u-x)\}\sin^{\beta}x\ |^r\ dx\right]^{\frac{1}{r}}=O\{\xi(t)\}.
\end{aligned}
$$

Then f belongs to $W(L^r(f),\xi(t))\Rightarrow\psi$ belongs to $W(L^r(f),\xi(t))$
Now consider,

$$|\ I_{3.1}\ |\leq\int_0^{\frac{1}{n+1}}|\ \psi(t)\ |\ |\ \tilde{K}_n(t)\ |\ dt$$

Using Hölder's inequality (14) and the fact that $\psi(t)\in W(L_r,\xi(t))$,

$$
\begin{aligned}
|\ I_{3.1}\ |\ &\leq\left[\int_0^{\frac{1}{n+1}}\left\{\frac{t\ |\ \psi(t)\ |\ \sin^{\beta}t}{\xi(t)}\right\}^r\ dt\right]^{\frac{1}{r}}\ \left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)\ |\ \tilde{K}_n(t)\ |}{t\ \sin^{\beta}t}\right\}^s\ dt\right]^{\frac{1}{s}} \\
&= O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)\ |\ \tilde{K}_n(t)\ |}{t\ \sin^{\beta}t}\right\}^s\ dt\right]^{\frac{1}{s}}\quad\text{by (23)}
\end{aligned}
$$

Since $\sin t\geq\frac{2t}{\pi}$ and using lemma 1,

$$|\ I_{3.1}\ |=O\left(\frac{1}{n+1}\right)\left[\int_0^{\frac{1}{n+1}}\left\{\frac{\xi(t)}{t^{2+\beta}}\right\}^s\ dt\right]$$

Since $\xi(t)$ is a positive increasing function and using second mean value theorem for integrals (16),

$$\begin{aligned}
\mid I_{3.1}\mid &= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\in}^{\frac{1}{n+1}}\frac{dt}{t^{(2+\beta)s}}\right]^{\frac{1}{s}} \quad \text{for some } 0\leq\in\leq\frac{1}{n+1}\\
&= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)\right\}\left[\left\{\frac{t^{-(2+\beta)s+1}}{-(2+\beta)s+1}\right\}_{\in}^{\frac{1}{n+1}}\right]^{\frac{1}{s}}\\
&= O\left\{\left(\frac{1}{n+1}\right)\xi\left(\frac{1}{n+1}\right)(n+1)^{2+\beta-\frac{1}{s}}\right\}\\
I_{3.1} &= O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\} \qquad (37)
\end{aligned}$$

Now using lemma 2,

$$\begin{aligned}
I_{3.2} &\leq k\left[\int_{\frac{1}{n+1}}^{\pi}\frac{|\psi(t)|}{t}dt\right]+k\left[\int_{\frac{1}{n+1}}^{\pi}2^{\tau}\frac{|\psi(t)|}{t\,(n+1)\,(n+2)}\sum_{k=\tau}^{n}\left(\frac{n-k+1}{2^{k}}\right)dt\right]\\
&= I_{3.2.1}+I_{3.2.2}\ (\text{say}) \qquad (38)
\end{aligned}$$

Using Hölder's inequality (14), $|\sin t|\leq 1$, $\sin t\geq\frac{2t}{\pi}$, (21), (24) and second mean value theorem for integrals (16),

$$\begin{aligned}
\mid I_{3.2.1}\mid &\leq k\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\mid\psi(t)\mid\sin^{\beta}t}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{-\delta+1}\ \sin^{\beta}t}\right\}^{s}dt\right]^{\frac{1}{s}}\\
&\leq k\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{t^{-\delta}\mid\psi(t)\mid}{\xi(t)}\right\}^{r}dt\right]^{\frac{1}{r}}\left[\int_{\frac{1}{n+1}}^{\pi}\left\{\frac{\xi(t)}{t^{\beta+1-\delta}}\right\}^{s}dt\right]^{\frac{1}{s}}\\
&= O\{(n+1)^{\delta}\}\left[\int_{\frac{1}{\pi}}^{n+1}\left\{\frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}}\right\}^{s}\frac{dy}{y^{2}}\right]^{\frac{1}{s}} \quad \text{by putting } t=\frac{1}{y}\\
&= O\left\{(n+1)^{\delta}\ \xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{\eta}^{n+1}\frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi}\leq\eta\leq n+1\\
&= O\left\{(n+1)^{\delta}\ \xi\left(\frac{1}{n+1}\right)\right\}\left[\int_{1}^{n+1}\frac{dy}{y^{s(\delta-1-\beta)+2}}\right]^{\frac{1}{s}} \quad \text{for some } \frac{1}{\pi}\leq 1\leq n+1\\
&= O\left\{(n+1)^{\delta}\ \xi\left(\frac{1}{n+1}\right)\right\}\left[(n+1)^{(1+\beta-\delta)-\frac{1}{s}}\right]\\
I_{3.2.1} &= O\left\{(n+1)^{\beta+\frac{1}{r}}\ \xi\left(\frac{1}{n+1}\right)\right\} \qquad (39)
\end{aligned}$$

Similarly using (21) and (24), $|\sin t| \leq 1$, $\sin t \geq \frac{2t}{\pi}$ and second mean value theorem for integrals (16),

$$
\begin{aligned}
| I_{3.2.2} | &\leq k \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} \mid \psi(t) \mid \sin^{\beta} t}{\xi(t)} \right\}^{r} dt \right]^{\frac{1}{r}} \\
&\left[\int_{\frac{1}{n+1}}^{\pi} \left\{ 2^{\tau} \frac{|\xi(t)|}{t^{1-\delta} \sin^{\beta} t \, (n+1)\,(n+2)} \sum_{k=\tau}^{n} \left(\frac{n-k+1}{2^{k}} \right) \right\}^{s} dt \right]^{\frac{1}{s}} \\
&\leq k \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{t^{-\delta} \mid \psi(t) \mid}{\xi(t)} \right\}^{r} dt \right]^{\frac{1}{r}} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{\beta+1-\delta}} \right\}^{s} dt \right]^{\frac{1}{s}} \text{ by (18)} \\
&= O\{(n+1)^{\delta}\} \left[\int_{\frac{1}{n+1}}^{\pi} \left\{ \frac{\xi(t)}{t^{\beta+1-\delta}} \right\}^{s} dt \right]^{\frac{1}{s}} \\
&= O\{(n+1)^{\delta}\} \left[\int_{\frac{1}{\pi}}^{n+1} \left\{ \frac{\xi\left(\frac{1}{y}\right)}{y^{\delta-1-\beta}} \right\}^{s} \frac{dy}{y^{2}} \right]^{\frac{1}{s}} \text{ by putting } t = \frac{1}{y} \\
&= O\left\{ (n+1)^{\delta} \, \xi\left(\frac{1}{n+1}\right) \right\} \left[\int_{\eta}^{n+1} \frac{dy}{y^{s(\delta-1-\beta)+2}} \right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq \eta \leq n+1 \\
&= O\left\{ (n+1)^{\delta} \, \xi\left(\frac{1}{n+1}\right) \right\} \left[\int_{1}^{n+1} \frac{dy}{y^{s(\delta-1-\beta)+2}} \right]^{\frac{1}{s}} \text{ for some } \frac{1}{\pi} \leq 1 \leq n+1 \\
&= O\left\{ (n+1)^{\delta} \, \xi\left(\frac{1}{n+1}\right) \right\} \left[(n+1)^{(1+\beta-\delta)-\frac{1}{s}} \right] \\
I_{3.2.1} &= O\left\{ (n+1)^{\beta+\frac{1}{r}} \, \xi\left(\frac{1}{n+1}\right) \right\} \qquad (40)
\end{aligned}
$$

Combining (36) to (40),

$$
| \tilde{f}(x) - t_{n}^{C_2 E_1}(x) | = O\left\{ (n+1)^{\beta+\frac{1}{r}} \, \xi\left(\frac{1}{n+1}\right) \right\}.
$$

Using $L^{r}(f)$-norm, we get

$$
\begin{aligned}
\| \tilde{f}(x) - t_{n}^{C_2 E_1}(x) \|_{r} &= \left\{ \int_{0}^{2\pi} | \, C_{n}^{2} E_{n}^{1} - f(x) \, |^{r} \, dx \right\}^{\frac{1}{r}} \\
&= O\left[\left\{ \int_{0}^{2\pi} \left\{ (n+1)^{\beta+\frac{1}{r}} \, \xi\left(\frac{1}{n+1}\right) \right\}^{r} dx \right\}^{\frac{1}{r}} \right] \\
&= O\left\{ (n+1)^{\beta+\frac{1}{r}} \, \xi\left(\frac{1}{n+1}\right) \right\} \left[\left\{ \int_{0}^{2\pi} dx \right\}^{\frac{1}{r}} \right] \\
\| \tilde{f}(x) - t_{n}^{C_2 E_1}(x) \|_{r} &= O\left\{ (n+1)^{\beta+\frac{1}{r}} \, \xi\left(\frac{1}{n+1}\right) \right\}
\end{aligned}
$$

This completes the proof of theorem 3.

6. Corollaries

Following corollaries can be derived from our main theorems.

6.1. **Corollary 1.** If $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$, then the weighted class $W(L^r(f), \xi(t))$, $1 \leq r < \infty$ reduces to the class $Lip(\alpha, r)$, and then the degree of approximation of a function $\tilde{f}$, conjugate to a 2π-periodic function f belonging to the class $Lip(\alpha, r)$, $r^{-1} \leq \alpha \leq 1$ is given by

$$\| \tilde{f}(x) - t_n^{C_2E_1}(x) \|_r = O\left\{\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right\}.$$

Proof. The result follows by setting $\beta = 0$ in (22). □

6.2. **Corollary 2.** If $r \to \infty$ in corollary 1, then the class $Lip(\alpha, r)$ reduces to the class $Lip\alpha$ and then the degree of approximation of a function $\tilde{f}$, conjugate to a 2π-periodic function f belonging to the class $Lip\alpha$, $0 < \alpha < 1$ is given by

$$\| \tilde{f}(x) - t_n^{C_2E_1}(x) \|_r = O\left\{\frac{1}{(n+1)^{\alpha}}\right\}$$

6.3. **Corollary 3.** If $\xi(t) = t^\alpha$, $0 < \alpha \leq 1$, then the class $Lip(\xi(t), r)$, $1 \leq r < \infty$ reduces to the class $Lip(\alpha, r)$ and then the degree of approximation of a function $\tilde{f}$, conjugate to a 2π-periodic function f belonging to the class $Lip(\alpha, r)$, $r^{-1} \leq \alpha \leq 1$ is given by

$$\| \tilde{f}(x) - t_n^{C_2E_1}(x) \|_r = O\left\{\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right\}.$$

Proof. We have

$$\left\|\tilde{f}(x) - t_n^{C_2E_1}(x)\right\|_r = O\left\{\int_0^{2\pi} \left|\tilde{f}(x) - t_n^{C_2E_1}(x)\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\} = O\left\{\int_0^{2\pi} \left|\tilde{f}(x) - t_n^{C_2E_1}(x)\right|^r dx\right\}^{\frac{1}{r}}$$

or

$$O(1) = O\left\{\int_0^{2\pi} \left|\tilde{f}(x) - t_n^{C_2E_1}(x)\right|^r dx\right\}^{\frac{1}{r}} O\left\{\frac{1}{O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}}\right\}$$

Hence,

$$\left|\tilde{f}(x) - t_n^{C_2E_1}(x)\right| = O\left\{(n+1)^{\frac{1}{r}} \xi\left(\frac{1}{n+1}\right)\right\}$$

for if not the right-hand side will be $O(1)$, therefore

$$\begin{aligned}\left|\tilde{f}(x) - t_n^{C_2E_1}(x)\right| &= O\left\{\left(\frac{1}{n+1}\right)^\alpha (n+1)^{\frac{1}{r}}\right\} \\ &= O\left(\frac{1}{(n+1)^{\alpha - \frac{1}{r}}}\right)\end{aligned}$$

□

6.4. **Corollary 4.** If $r \to \infty$ in corollary 3, then the class $Lip(\alpha, r)$ reduces to the class $Lip\alpha$ and then the degree of approximation of a function $\tilde{f}$, conjugate to a 2π-periodic function f belonging to the class $Lip\alpha$, $0 < \alpha < 1$ is given by

$$\| \tilde{f}(x) - t_n^{C_2 E_1}(x) \|_r = O\left\{\frac{1}{(n+1)^\alpha}\right\}$$

7. Conclusion and Research Perspective:

Motivated by the work of earlier authors on degree of approximation of function using linear operators, we, in the present work, for the first time, studied the degree of approximation of a function $\tilde{f}$, conjugate to a periodic function f belonging to different classes by $(C, 2)(E, 1)$ product operators. The advantage of considering product operators over linear operators can be understood with the observation that the infinite series, which is neither summable by the left linear operators nor by the right linear operators individually, is summable to some number by the product operators obtained from the same linear operators placed in the same sequential order. Thus, the method of product operators is more powerful than the methods of individual linear operators. In studies of error estimates through trigonometric Fourier approximation (TFA), product operators give better approximation than individual linear operators. Further results can be obtained using other suitable product operators that may allow someone to get better error estimates through trigonometric Fourier approximation (TFA).

8. application

Theory of approximation of functions has wide applications in signal processing and image processing. Currently, this theory is also being used to study the convergence of wavelet packet expansions of functions.

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