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ON A NEW CLASS OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

M.K. AOUF, A.O.MOSTAFA, A. Y. LASHIN, B.M.MUNASSAR

ABSTRACT. In this paper, we introduce a new class of analytic univalent functions defined in the open unit disc by using Dziok-Srivastava operator and obtain some results including coefficient inequality, distortion theorems, Hadamard products, radii of starlikeness and convexity and closure theorems of functions in this class.

1. INTRODUCTION

Let S denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \ , \tag{1}$$

which are analytic and univalent in the open unit disc $U = \{z : \mid z \mid < 1\}$ and normalized by $f(0) = 0 = f^{'}(0) - 1$. We denote by $S^* (\alpha)$ and $K(\alpha)$ the subclasses of S consisting of all functions which are, respectively, starlike and convex of order α $(0 \leq \alpha < 1)$. Thus,

$$S^*(\alpha) = \left\{ f \in S : \text{Re} \left(\frac{zf^{'}(z)}{f(z)} \right) > \alpha \, (0 \leq \alpha < 1; z \in U) \right\}$$

and

$$K(\alpha) = \left\{ f \in S : \text{Re} \left(1 + \frac{zf^{\ ''}(z)}{f^{\ '}(z)} \right) > \alpha \, (0 \leq \alpha < 1; z \in U) \right\}.$$

The classes $S^*(\alpha)$ and $K(\alpha)$ were introduced by Roberston [14].
Let $f \in S$ be given by (1) and $g \in S$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n. \tag{2}$$

We define the Hadmard product (or convolution) of f and g by

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$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n=(g*f)(z)\,,z\in U\ .$$

For positive real parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ $\left(\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-,\mathbb{Z}_0^-=0,-1,-2,...;j=1,2,...,s\right)$, the generalized hypergeometric function ${}_qF_s\left(\alpha_1,...,\alpha_q,\beta_1,...,\beta_s;z\right)$ is defined by

$${}_qF_s\left(\alpha_1,...,\alpha_q,\beta_1,...,\beta_s;z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\,...\,(\alpha_q)_n}{(\beta_1)_n\,...\,(\beta_s)_n}\frac{1}{n!}z^n,$$

$$\left(q\leq s+1;s,q\in\mathbb{N}_0=\mathbb{N}\cup\{0\}\,,\mathbb{N}=\{1,2,...\}\,;z\in U\right),$$

where $(\theta)_n$, is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$(\theta)_n=\frac{\Gamma(\theta+n)}{\Gamma(\theta)}=\begin{cases}1 & (n=0)\\ \theta(\theta+1)...(\theta+n-1) & (n\in\mathbb{N})\,.\end{cases}$$

For the function

$$h\left(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z\right)=z{}_qF_s\left(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z\right),$$

the Dziok-Srivastava linear operator(see [5]) $H_{q,s}\left(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s\right):S\longrightarrow S$, is defined by the Hadamard product as follows:

$$H_{q,s}\left(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s\right)f(z)=h\left(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z\right)*f(z)$$

$$=z+\sum_{n=2}^{\infty}\Gamma_n(\alpha_1)a_nz^n\quad(z\in U)\,,\tag{3}$$

where

$$\Gamma_n(\alpha_1)=\frac{(\alpha_1)_{n-1}\,...\,(\alpha_q)_{n-1}}{(\beta_1)_{n-1}...(\beta_s)_{n-1}}\frac{1}{(n-1)!}.\tag{4}$$

For brevity, we write

$$H_{q,s}\left(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z\right)f(z)=H_{q,s}(\alpha_1)\,f(z)\,.$$

For $0<\beta\leq1,\frac{1}{2}\leq\xi\leq1,0\leq\alpha<\frac{1}{2}$ and, $f\in S$ we define the class $S_{q,s}^*\left(\alpha_1,\alpha,\beta,\xi\right)$ by

$$\left|\frac{\dfrac{z\left(H_{q,s}(\alpha_1)f(z)\right)'}{H_{q,s}(\alpha_1)f(z)}-1}{2\xi\left[\dfrac{z\left(H_{q,s}(\alpha_1)f(z)\right)'}{H_{q,s}(\alpha_1)f(z)}-\alpha\right]-\left[\dfrac{z\left(H_{q,s}(\alpha_1)f(z)\right)'}{H_{q,s}(\alpha_1)f(z)}-1\right]}\right|<\beta.\tag{5}$$

We also let

$T_{q,s}^*\left(\alpha_1,\alpha,\beta,\xi\right)=S_{q,s}^*\left(\alpha_1,\alpha,\beta,\xi\right)\cap T$, where

$$T=\left\{f\in S:f(z)=z-\sum_{n=2}^{\infty}a_nz^n,a_n\geq0\ ;z\in U\right\}.\tag{6}$$

We note that:

(1) For $q = 2$ and $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in (5), the class $T^*_{q,s}(\alpha_1, \alpha, \beta, \xi)$ reduces to the class $T^*_{2,1}([1], \alpha, \beta, \xi) = T^*(\alpha, \beta, \xi)$,

$$T^*(\alpha,\beta,\xi) = \left\{ f \in T : \left| \frac{\frac{zf'(z)}{f(z)} - 1}{2\xi\left[\frac{zf'(z)}{f(z)} - \alpha\right] - \left[\frac{zf'(z)}{f(z)} - 1\right]} \right| < \beta, \right.$$

$$\left. 0 \le \alpha < \tfrac{1}{2},\ 0 < \beta \le 1, \tfrac{1}{2} \le \xi \le 1,\ z \in U \right\} \text{ (see Kulkarni [8])};$$

(2) For $q = 2, s = 1$, $\alpha_1 = \lambda + 1\,(\lambda > -1)$ and $\alpha_2 = \beta_1 = 1$ in (5), the class $T^*_{q,s}(\alpha_1, \alpha, \beta, \xi)$ reduces to the class $T^*_{2,1}(\lambda + 1, \alpha, \beta, \xi) = T^*(\lambda, \alpha, \beta, \xi)$,

$$T^*(\lambda,\alpha,\beta,\xi) = \left\{ f \in T : \left| \frac{\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda f(z)} - 1}{2\xi\left[\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda f(z)} - \alpha\right] - \left[\frac{z\left(D^\lambda f(z)\right)'}{D^\lambda f(z)} - 1\right]} \right| < \beta, \right.$$

$$\left. 0 \le \alpha < \tfrac{1}{2},\ 0 < \beta \le 1, \tfrac{1}{2} \le \xi \le 1,\ z \in U \right\} \text{ (see Khairnar and Rajas [7])},$$

where $D^\lambda\,(\lambda > -1)$ is the Ruscheweyh derivative operator (see [15]).

We note also that:

(1) For $q = 2, s = 1, \alpha_1 = a\,(a > 0), \alpha_2 = 1$ and $\beta_1 = c\,(c > 0)$ in (5), the class $T^*_{q,s}(\alpha_1, \alpha, \beta, \xi)$ reduces to the class $T^*_{2,1}([a, 1; c], \alpha, \beta, \xi) = T^*(a, c, \alpha, \beta, \xi)$,

$$T^*(a,c,\alpha,\beta,\xi) = \left\{ f \in T : \left| \frac{\frac{z\left(L(a,c) f(z)\right)'}{L(a,c) f(z)} - 1}{2\xi\left[\frac{z\left(L(a,c) f(z)\right)'}{L(a,c) f(z)} - \alpha\right] - \left[\frac{z\left(L(a,c) f(z)\right)'}{L(a,c) f(z)} - 1\right]} \right| < \beta, \right.$$

$$\left. 0 \le \alpha < \tfrac{1}{2},\ 0 < \beta \le 1, \tfrac{1}{2} \le \xi \le 1,\ z \in U \right\},$$

where $L(a, c)$ is the Carlson - Shaffer operator(*see* [2]);

(2)For $q = 2, s = 1$, in (5), the class $T^*_{q,s}(\alpha_1, \alpha, \beta, \xi)$ reduces to the class $T^*_{2,1}([\alpha_1, \alpha_2; \beta_1], \alpha, \beta, \xi) = T^*(\alpha_1, \alpha_2, \beta_1, \alpha, \beta, \xi)$,

$$T^*(\alpha_1,\alpha_2,\beta_1,\alpha,\beta,\xi) = \left\{ f \in T : \left| \frac{\frac{z\left(I^{\alpha_1,\alpha_2}_{\beta_1} f(z)\right)'}{\left(I^{\alpha_1,\alpha_2}_{\beta_1} f(z)\right)} - 1}{2\xi\left[\frac{z\left(I^{\alpha_1,\alpha_2}_{\beta_1} f(z)\right)'}{\left(I^{\alpha_1,\alpha_2}_{\beta_1} f(z)\right)} - \alpha\right] - \left[\frac{z\left(I^{\alpha_1,\alpha_2}_{\beta_1} f(z)\right)'}{\left(I^{\alpha_1,\alpha_2}_{\beta_1} f(z)\right)} - 1\right]} \right| < \beta, \right.$$

$$\left. 0 \le \alpha < \tfrac{1}{2}, 0 < \beta \le 1, \tfrac{1}{2} \le \xi \le 1, \left(\alpha_1, \alpha_2 \in c, \beta_1 \notin Z_0^-\right), z \in U \right\},$$

where $I^{\alpha_1,\alpha_2}_{\beta_1}$ is the Hohlow operator(*see* [6]);

(3) For $q=2,\ s=1,\ \alpha_1=v+1\,(v>-1),\ \alpha_2=1$ and $\beta_1=v+2$ in (5), the class $T^*_{q,s}(\alpha_1,\alpha,\beta,\xi)$ reduces to the class $T^*_{2,1}([v+1,1;v+2],\alpha,\beta,\xi)=\ T^*(v,\alpha,\beta,\xi)$,

$$T^*(v,\alpha,\beta,\xi)=\left\{f\in T:\left|\frac{\frac{z(J_vf(z))'}{J_vf(z)}-1}{2\xi\left[\frac{z(J_vf(z))'}{J_vf(z)}-\alpha\right]-\left[\frac{z(J_vf(z))'}{J_vf(z)}-1\right]}\right|<\beta\right.$$

$$\left.0\le\alpha<\tfrac{1}{2},0<\beta\le1,\tfrac{1}{2}\le\xi\le1,z\in U\ \right\},$$

where J_v is the generalized Bernardi - Libera -Livingston operator see $([1],[9]$ and $[10])$;

(4) For $q=2,\ s=1,\ \alpha_1=2,\ \alpha_2=1$ and $\beta_1=2-\mu\,(\mu\neq2,3,...)$ in (5), the class $T^*_{q,s}(\alpha_1,\alpha,\beta,\xi)$ reduces to the class $T^*_{2,1}([2,1;2-\mu],\alpha,\beta,\xi)=T^*(\mu,\alpha,\beta,\xi)$,

$$T^*(\mu,\alpha,\beta,\xi)=\left\{f\in T:\left|\frac{\frac{z(\Omega^\mu_zf(z))'}{\Omega^\mu_zf(z)}-1}{2\xi\left[\frac{z(\Omega^\mu_zf(z))'}{\Omega^\mu_zf(z)}-\alpha\right]-\left[\frac{z(\Omega^\mu_zf(z))'}{\Omega^\mu_zf(z)}-1\right]}\right|<\beta,\right.$$

$$\left.0\le\alpha<\tfrac{1}{2},0<\beta\le1,\tfrac{1}{2}\le\xi\le1,z\in U\right\},$$

where Ω^μ_z is the Srivastava- Owa fractional derivative operator (see [12] and [13]);

(5) For $q=2,\ s=1,\ \alpha_1=\mu\,(\mu>0),\ \alpha_2=1$ and $\beta_1=\lambda+1\,(\lambda>-1)$ in (5), the class $T^*_{q,s}(\alpha_1,\alpha,\beta,\xi)$ reduces to class $T^*_{2,1}([\mu,1;\lambda+1],\alpha,\beta,\xi)=T^*(\mu,\lambda,\alpha,\beta,\xi)$,

$$T^*(\mu,\lambda,\alpha,\beta,\xi)=\ \left\{f\in T:\left|\frac{\frac{z(I_{\lambda,\mu}f(z))'}{(I_{\lambda,\mu}f(z))}-1}{2\xi\left[\frac{z(I_{\lambda,\mu}f(z))'}{(I_{\lambda,\mu}f(z))}-\alpha\right]-\left[\frac{z(I_{\lambda,\mu}f(z))'}{(I_{\lambda,\mu}f(z))}-1\right]}\right|<\beta,\right.$$

$$\left.0\le\alpha<\tfrac{1}{2},0<\beta\le1,\tfrac{1}{2}\le\xi\le1,\mu>0,\lambda>0,z\in U\right\},$$

where $I_{\lambda,\mu}$ is the Choi - Saigo - Srivastava operator (see [4]);

(6) For $q=2,\ s=1,\ \alpha_1=2,\ \alpha_2=1$ and $\beta_1=k+1\,(k>-1)$ in (5), the class $T^*_{q,s}(\alpha_1,\alpha,\beta,\xi)$ reduces to the class $T^*_{2,1}([2,1;k+1],\alpha,\beta,\xi)=T^*(k,\alpha,\beta,\xi)$,

$$T^*(k,\alpha,\beta,\xi)=\left\{f\in T:\left|\frac{\frac{z(I_kf(z))'}{(I_kf(z))}-1}{2\xi\left[\frac{z(I_kf(z))'}{(I_kf(z))}-\alpha\right]-\left[\frac{z(I_kf(z))'}{(I_kf(z))}-1\right]}\right|<\beta,\right.$$

$$\left.0\le\alpha<\tfrac{1}{2},0<\beta\le1,\tfrac{1}{2}\le\xi\le1,k>-1,z\in U\right\},$$

where I_k is the Noor integral operator (see [11]);

(7) For $q=2,s=1,\ \alpha_1=c\,(c>0),\ \alpha_2=\lambda+1\,(\lambda>-1)$ and $\beta_1=a\,(a>0)$ in (5), the class $T^*_{q,s}(\alpha_1,\alpha,\beta,\xi)$ reduces to the class $T^*_{2,1}([c,\lambda+1;a],\alpha,\beta,\xi)=$

$T^{*}(c,a,\lambda,\alpha,\beta,\xi)$,

$$T^{*}(c,a,\lambda,\alpha,\beta,\xi)=\left\{f\in T:\left|\frac{\frac{z\left(I^{\lambda}(a,c)f(z)\right)'}{\left(I^{\lambda}(a,c)f(z)\right)}-1}{2\xi\left[\frac{z\left(I^{\lambda}(a,c)f(z)\right)'}{\left(I^{\lambda}(a,c)f(z)\right)}-\alpha\right]-\left[\frac{z\left(I^{\lambda}(a,c)f(z)\right)'}{\left(I^{\lambda}(a,c)f(z)\right)}-1\right]}\right|<\beta,\right.$$

$$\left.0\leq\alpha<\tfrac{1}{2},0<\beta\leq1,\tfrac{1}{2}\leq\xi\leq1,\lambda>-1,a>0,z\in U\right\},$$

where $I^{\lambda}(a,c)$ is the Cho - Kwon - Srivastava operator (see [3]).

2. Coefficient Inequality

Unless otherwise mentioed, we shall assume in the reminder of this paper that, the parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ are positive real numbers, $0<\beta\leq 1$, $\frac{1}{2}\leq\xi\leq 1$, $0\leq\alpha<\frac{1}{2}$, $n\geq 2$, $\Gamma_n(\alpha_1)$ is defined by (1.4) and $z\in U$.

Theorem 1. Let the function f be defined by (1.6). Then f is in the class $T^{*}_{q,s}(\alpha_1,\alpha,\beta,\xi)$ if and only if

$$\sum_{n=2}^{\infty}\left[(n-1)(1-\beta)+2\beta\xi(n-\alpha)\right]\Gamma_n(\alpha_1)a_n\leq 2\xi\beta(1-\alpha). \tag{7}$$

Proof. Assume that the inequality (7) holds true. We find from (6) that

$$\left|z\left[H_{q,s}(\alpha_1)\ f(z)\right]'-H_{q,s}(\alpha_1)f(z)\ \right|-\beta\left|2\xi\left\{z\left[H_{q,s}(\alpha_1)\ f(z)\right]'-\alpha H_{q,s}(\alpha_1)f(z)\right\}\right.$$

$$\left.-\left\{z\left[H_{q,s}(\alpha_1)\ f(z)\right]'-H_{q,s}(\alpha_1)f(z)\right\}\right|$$

$$=\left|\sum_{n=2}^{\infty}(n-1)\Gamma_n(\alpha_1)a_nz^n\right|-\beta\left|2\xi\left[(1-\alpha)z-\sum_{n=2}^{\infty}(n-\alpha)\Gamma_n(\alpha_1)a_nz^n\right]\right.$$

$$\left.+\sum_{n=2}^{\infty}(n-1)\Gamma_n(\alpha_1)a_nz^n\right|$$

$$\leq\sum_{n=2}^{\infty}\left[(n-1)+2\xi\beta(n-\alpha)-\beta(n-1)\right]\Gamma_n(\alpha_1)a_n-2\beta\xi(1-\alpha)$$

$$=\sum_{n=2}^{\infty}\left[(n-1)(1-\beta)+2\beta\xi(n-\alpha)\right]\Gamma_n(\alpha_1)a_n-2\beta\xi(1-\alpha)\leq 0.$$

Hence, by the maximum modulus theorem, we have $f\in T^{*}_{q,s}(\alpha_1,\alpha,\beta,\xi)$.
Conversely, let $f\in T^{*}_{q,s}(\alpha_1,\alpha,\beta,\xi)$. Then

$$\left|\frac{\frac{z\left(H_{q,s}(\alpha_1)f(z)\right)'}{H_{q,s}(\alpha_1)f(z)}-1}{2\xi\left[\frac{z\left(H_{q,s}(\alpha_1)f(z)\right)'}{H_{q,s}(\alpha_1)f(z)}-\alpha\right]-\left[\frac{z\left(H_{q,s}(\alpha_1)f(z)\right)'}{H_{q,s}(\alpha_1)f(z)}-1\right]}\right|<\beta$$

that is

$$\frac{\left|\sum\limits_{n=2}^{\infty}(n\ -1)\,\Gamma_n(\alpha_1)a_nz^n\right|}{\left|2\xi\left[(1-\alpha)\,z\ -\sum\limits_{n=2}^{\infty}(n\ -\alpha)\ \Gamma_n(\alpha_1)a_nz^n\right]+\sum\limits_{n=2}^{\infty}(n\ -1)\ \Gamma_n(\alpha_1)a_nz^n\right|}<\beta. \tag{8}$$

Now since $Re\ f\,(z)\leq|\ f\,(z)|$ for all z, we have

$$Re\left\{\frac{\sum\limits_{n=2}^{\infty}(n\ -1)\ \Gamma_n(\alpha_1)a_nz^n}{2\xi\left[(1-\alpha)\,z\ -\sum\limits_{n=2}^{\infty}(n\ -\alpha)\ \Gamma_n(\alpha_1)a_nz^n\right]+\sum\limits_{n=2}^{\infty}(n\ -1)\ \Gamma_n(\alpha_1)a_nz^n}\right\}<\beta. \tag{9}$$

Choose values of z on the real axis so that $\frac{z(H_{q,s}(\alpha_1)f(z))'}{H_{q,s}(\alpha_1)f(z)}$ is real. Then upon clearing the denominator in (9) and letting $z\to1^-$ through real values, we have

$$\frac{\sum\limits_{n=2}^{\infty}(n\ -1)\,\Gamma_n(\alpha_1)a_n}{2\xi\left[(1-\alpha)\ -\sum\limits_{n=2}^{\infty}(n\ -\alpha)\,\Gamma_n(\alpha_1)a_n\right]+\sum\limits_{n=2}^{\infty}(n\ -1)\,\Gamma_n(\alpha_1)a_n}\leq\beta.$$

That is

$$\sum_{n=2}^{\infty}\left[(n-1)\,(1-\beta)+2\beta\xi\,(n-\alpha)\right]\ \Gamma_n(\alpha_1)a_n\leq2\xi\beta\,(1-\alpha)\,. \tag{10}$$

This is the required condition, which compleles the proof of Theorem 1.
Setting $q=2,s=1,\alpha_1=\lambda+1\,(\lambda\geq-1)$ and $\alpha_2=\beta_1=1$ in Theorem 1, we get the following corollary which corrects Theorem 1 of [7].

Corollary 1. Let the function f defined by (6). Then f is in the class $T^*\,(\lambda,\alpha,\beta,\xi)$ if and only if

$$\sum_{n=2}^{\infty}\left[(n-1)\,(1-\beta)+2\beta\xi\,(n-\alpha)\right]\ \binom{\lambda+n-1}{\lambda}a_n\leq2\xi\beta\,(1-\alpha)\,.$$

Corollary 2. Let the function f defined by (6) be in the class $T^*_{q,s}\,(\alpha_1,\alpha,\beta,\xi)$ then we have

$$a_n\leq\frac{2\xi\beta\,(1-\alpha)}{\left[(n\ -1)\,(1-\beta)+2\beta\xi\,(n\ -\alpha)\right]\ \Gamma_n(\alpha_1)},(n\geq2)\,. \tag{11}$$

The result is sharp for the function f given by

$$f(z)=z-\frac{2\xi\beta\,(1-\alpha)}{\left[(n\ -1)\,(1-\beta)+2\beta\xi\,(n\ -\alpha)\right]\Gamma_n(\alpha_1)}z^n,n\geq2. \tag{12}$$

3. Growth and Distortion Theorems

Theorem 2. Let the function f defined by (6) be in the class $T^{*}_{q,s}(\alpha_1,\alpha,\beta,\xi)$. Then for $|z|=r<1$,we have

$$|f(z)| \geq r-\frac{2\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)}r^2 \tag{13}$$

and

$$|f(z)| \leq r+\frac{2\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)}r^2, \tag{14}$$

provided that $\Gamma_n(\alpha_1)\geq\Gamma_2(\alpha_1)\,(n\geq 2)$. The equalities in (13) and (14) are attained for the function f given by

$$f(z)=z-\frac{2\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)}z^2, \tag{15}$$

at $z=r$ and $z=re^{i(2n+1)\pi}\,(n\geq 2)$.

Proof. Since for $n\geq 2$,

$$[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)\sum_{n=2}^{\infty}a_n$$

$$\leq\sum_{n=2}^{\infty}[(n-1)(1-\beta)+2\beta\xi(n-\alpha)]\,\Gamma_n(\alpha_1)a_n\leq 2\xi\beta(1-\alpha),$$

then

$$\sum_{n=2}^{\infty}a_n\leq\frac{2\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)}. \tag{16}$$

From (6) and (16), we have

$$|f(z)|\geq r-r^2\sum_{n=2}^{\infty}a_n\ \geq r-\frac{2\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)}r^2$$

and

$$|f(z)|\leq r+r^2\sum_{n=2}^{\infty}a_n\ \leq r+\frac{2\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)}r^2.$$

This completes the proof of Theorem 2.

Theorem 3. Let the function f defined by (6) be in the class $T^{*}_{q,s}(\alpha_1,\alpha,\beta,\xi)$. Then for $|z|=r<1$,

$$\left|f'(z)\right|\geq 1-\frac{4\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)}r \tag{17}$$

and

$$\left|f'(z)\right|\leq 1+\frac{4\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\ \Gamma_2(\alpha_1)}r. \tag{18}$$

The equalities in (17) and (18) are attained for the function f given by (15).

proof. For $n\geq 2$, we have

$$\left|f^{'}(z)\right| \leq 1-r\sum_{n=2}^{\infty}na_n, \tag{19}$$

and by Theorem 1, we have

$$\sum_{n=2}^{\infty}na_n \leq \frac{4\xi\beta\left(1-\alpha\right)}{\left[(1-\beta)+2\beta\xi\left(2-\alpha\right)\right]\Gamma_2(\alpha_1)}. \tag{20}$$

From (19) and (20), we have

$$\left|f^{'}(z)\right| \geq 1-r\sum_{n=2}^{\infty}na_n \ \geq 1-\frac{4\xi\beta\left(1-\alpha\right)}{\left[(1-\beta)+2\beta\xi\left(2-\alpha\right)\right]\ \Gamma_2(\alpha_1)}r$$

and

$$\left|f^{'}(z)\right| \leq 1+r\sum_{n=2}^{\infty}na_n \leq 1+\frac{4\xi\beta\left(1-\alpha\right)}{\left[(1-\beta)+2\beta\xi\left(2-\alpha\right)\right]\Gamma_2(\alpha_1)}r$$

This completes the proof of Theorem 3.

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In this section we obtain the radii of starlikeness, convexity and close-to-convexity for functions in the class $T^{*}_{q,s}\left(\alpha_1,\alpha,\beta,\xi\right)$.

Theorem 4. Let the function f defined by (6) be in the class $T^{*}_{q,s}\left(\alpha_1,\alpha,\beta,\xi\right)$. Then f is starlike of order $\delta, 0 \leq \delta < 1$ in disc $|z| < R_1$ where

$$R_1 = \inf_{n\geq 2}\left\{\frac{(1-\delta)\left[(n-1)\left(1-\beta\right)+2\beta\xi\left(n-\alpha\right)\right]\Gamma_n(\alpha_1)}{2\beta\xi\left(1-\alpha\right)\left(n-\delta\right)}\right\}^{\frac{1}{n-1}}. \tag{21}$$

The result is sharp, with the extermal function f given by (2.6).

Proof. It is sufficient to show that

$$\left|\frac{zf^{'}(z)}{f(z)}-1\right| \leq 1-\delta \text{ for } |z| < R_1, \tag{22}$$

where R_1 is given by (21). Indeed we find, again from (6) that

$$\left|\frac{zf^{'}(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{n=2}^{\infty}\left(n-1\right)a_n\left|z\right|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n\left|z\right|^{n-1}}.$$

Thus

$$\left|\frac{zf^{'}(z)}{f(z)}-1\right| \leq 1-\delta,$$

if

$$\sum_{n=2}^{\infty}\frac{(n-\delta)}{(1-\delta)}a_n\left|z\right|^{n-1} \leq 1. \tag{23}$$

But, by Theorem 1, (23) will be true if

$$\frac{(n-\delta)}{(1-\delta)}\left|z\right|^{n-1} \leq \frac{\left[(n\ -1)\left(1-\beta\right)+2\beta\xi\left(n\ -\alpha\right)\right]\ \Gamma_{n}(\alpha_{1})}{2\beta\xi\left(1-\alpha\right)}, \tag{24}$$

that is, if

$$\left|z\right| \leq \left\{\frac{(1-\delta)\left[(n-1)\left(1-\beta\right)+2\beta\xi\left(n-\alpha\right)\right]\Gamma_{n}(\alpha_{1})}{2\beta\xi\left(1-\alpha\right)\left(n-\delta\right)}\right\}^{\frac{1}{n-1}}, \quad (n \geq 2). \tag{25}$$

Theorem 4 follows easily from (25).

Theorem 5. Let the function f defined by (6) be in the class $T^{*}_{q,s}(\alpha_{1}, \alpha, \beta, \xi)$. Then f is convex of order $\delta, (0 \leq \delta < 1)$ in
disc $|z| < R_{2}$, where

$$R_{2} = \inf_{n\geq 2}\left\{\frac{(1-\delta)\left[(n-1)\left(1-\beta\right)+2\beta\xi\left(n-\alpha\right)\right]\Gamma_{n}(\alpha_{1})}{2\beta\xi n\left(1-\alpha\right)\left(n-\delta\right)}\right\}^{\frac{1}{n-1}}. \tag{26}$$

The result is sharp for the function f given by (12).
Proof. We must show that

$$\left|\frac{zf\,''(z)}{f\,'(z)}\right| \leq 1-\delta, \text{ for } |z| < R_{2},$$

where R_{2} is given by (26). Indeed we find from (6) that

$$\left|\frac{zf\,''(z)}{f\,'(z)}\right| \leq \frac{\sum\limits_{n=2}^{\infty} n\left(n-1\right)a_{n}\left|z\right|^{n-1}}{1-\sum\limits_{n=2}^{\infty} na_{n}\left|z\right|^{n-1}}.$$

Thus

$$\left|\frac{zf\,''(z)}{f\,'(z)}\right| \leq 1-\delta,$$

if

$$\sum_{n=2}^{\infty}\frac{n\left(n-\delta\right)}{(1-\delta)}a_{n}\left|z\right|^{n-1} \leq 1. \tag{27}$$

But, by Theorem 1, (27) will be true if

$$\frac{n\left(n-\delta\right)}{(1-\delta)}\left|z\right|^{n-1} \leq \frac{\left[(n\ -1)\left(1-\beta\right)+2\beta\xi\left(n\ -\alpha\right)\right]\ \Gamma_{n}(\alpha_{1})}{2\beta\xi\left(1-\alpha\right)},$$

that is, if

$$\left|z\right| \leq \left\{\frac{(1-\delta)\left[(n-1)\left(1-\beta\right)+2\beta\xi\left(n-\alpha\right)\right]\Gamma_{n}(\alpha_{1})}{2\beta\xi n\left(1-\alpha\right)\left(n-\delta\right)}\right\}^{\frac{1}{n-1}}, \ n \geq 2. \tag{28}$$

Theorem 5 follows easily from (28).

Corrolary 3. Let the function f defined by (6) be in the class $T_{q,s}^{*}(\alpha_1,\alpha,\beta,\xi)$. Then f is close-to-convex of order $\delta,(0\leq\delta<1)$ in the disc $|z|<R_3$, where

$$R_3=\inf_{n\geq 2}\left\{\frac{(1-\delta)\left[(n-1)(1-\beta)+2\beta\xi(n-\alpha)\right]\Gamma_n(\alpha_1)}{2\beta\xi n(1-\alpha)}\right\}^{\frac{1}{n-1}}. \tag{29}$$

The result is sharp, with the extermal function f given by (12).

5. Closure Theorems

Theorem 6. Let $\mu_j\geq 0$ for $j=1,2,...,m$, and $\sum\limits_{j=1}^{m}\mu_j\leq 1$. If the functions f_j defined by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,j}z^n\,(a_{n,j}\geq 0;j=1,2,...,m), \tag{30}$$

are in the class $T_{q,s}^{*}(\alpha_1,\alpha,\beta,\xi)$, for every $j=1,2,...,m$. Then the function $h(z)$ defined by

$$h(z)=z-\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m}\mu_j a_{n,j}\right)z^n,$$

is in the class $T_{q,s}^{*}(\alpha_1,\alpha,\beta,\xi)$.

Proof. Since $f_j\in T_{q,s}^{*}(\alpha_1,\alpha,\beta,\xi)$, it follows from Theorem 1, that

$$\sum_{n=2}^{\infty}\left[(n-1)(1-\beta)+2\beta\xi(n-\alpha)\right]\Gamma_n(\alpha_1)a_{n,j}\leq 2\xi\beta(1-\alpha),$$

for every $j=1,2,...,m$. Hence

$$\sum_{n=2}^{\infty}\left[(n-1)(1-\beta)+2\beta\xi(n-\alpha)\right]\left(\sum_{j=1}^{m}\mu_j a_{n,j}\right)\Gamma_n(\alpha_1)$$

$$=\sum_{j=1}^{m}\mu_j\left(\sum_{n=1}^{\infty}\left[(n-1)(1-\beta)+2\beta\xi(n-\alpha)\right]\Gamma_n(\alpha_1)a_{n,j}\right)$$

$$\leq 2\xi\beta(1-\alpha)\sum_{j=1}^{m}\mu_j\leq 2\beta\xi(1-\alpha).$$

By Theorem 1, it follows that $h(z)\in T_{q,s}^{*}(\alpha_1,\alpha,\beta;\xi)$, and so the proof of Theorem 6 is completed.

Theorem 7. Let $f_1(z)=z$ and

$$f_n(z)=z-\frac{2\xi\beta(1-\alpha)}{\left[(n-1)(1-\beta)+2\beta\xi(n-\alpha)\right]\Gamma_n(\alpha_1)}z^n\,(n\geq 2). \tag{31}$$

Then f is in the class $T_{q,s}^{*}(\alpha_1,\alpha,\beta;\xi)$, if and only if it can by expressed in the form

$$f(z)=\sum_{n=1}^{\infty}\mu_n f_n(z), \tag{32}$$

where $\mu_n \geq 0\,(n \geq 1)$ and $\sum\limits_{n=1}^{\infty} \mu_n = 1.$

Proof. Assume that

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z)$$

$$= z - \sum_{n=2}^{\infty} \frac{2\xi\beta(1-\alpha)}{[(n-1)(1-\beta) + 2\beta\xi(n-\alpha)]\ \Gamma_n(\alpha_1)} \mu_n z^n.$$

Then it follows that

$$\sum_{n=2}^{\infty} \frac{[(n-1)(1-\beta) + 2\beta\xi(n-\alpha)]\ \Gamma_n(\alpha_1)}{2\xi\beta(1-\alpha)} \cdot$$

$$\cdot \frac{2\xi\beta(1-\alpha)}{[(n-1)(1-\beta) + 2\beta\xi(n-\alpha)]\ \Gamma_n(\alpha_1)} \mu_n$$

$$= \sum_{n=2}^{\infty} \mu_n = 1 - \mu_1 \leq 1.$$

So, by Theorem 1, $f \in T^*_{q,s}(\alpha_1, \alpha, \beta, \xi)$.

Conversely, assume that the functions f defined by (6) belongs to the class $T^*_{q,s}(\alpha_1, \alpha, \beta, \xi)$. Then

$$a_n \leq \frac{2\xi\beta(1-\alpha)}{[(n-1)(1-\beta) + 2\beta\xi(n-\alpha)]\,\Gamma_n(\alpha_1)}\,(n \geq 2).$$

Setting

$$\mu_n = \frac{[(n-1)(1-\beta) + 2\beta\xi(n-\alpha)]\ \Gamma_n(\alpha_1)}{2\xi\beta(1-\alpha)} a_n\,(n \geq 2),$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

We can see that f can by expressed in the form (32). This completes the proof of Theorem 7.

Corrolary 4. The extreme points of the class $T^*_{q,s}(\alpha_1, \alpha, \beta, \xi)$ are the functions $f_1 = z$ and f_n given by (31).

6. Modified Hadamard Product

For the functions

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n \,(a_{n,j} \geq 0; j = 1, 2), \tag{33}$$

we denote by $(f_1 * f_2)$ the modified Hadamard product (or convolution) of the functions$f_1$andf_2,that is,

$$(f_1 * f_2)(z) = z - \sum_{n=2}^{\infty} a_{n,1} a_{n,2} z^n. \tag{34}$$

Theorem 8. Let the functions $f_j\,(j=1,2)$, defined by (33) be in the class $T^*_{q,s}(\alpha_1,\alpha,\beta,\xi)$. Then $(f_1 * f_2) \in T^*_{q,s}(\alpha_1,\mu,\beta,\xi)$, where

$$\mu = 1 - \frac{2\beta\xi\,(1-\alpha)^2\,[(1-\beta)+2\beta\xi]}{[(1-\beta)+2\beta\xi\,(2-\alpha)]^2\,\Gamma_2(\alpha_1) - 4\beta^2\xi^2\,(1-\alpha)^2}. \tag{35}$$

The result is sharp.

Proof. Employing the technique used ealier by Schild and Silverman [16], we need to find the largest μ such that

$$\sum_{n=2}^{\infty} \frac{[(n-1)(1-\beta)+2\beta\xi\,(n-\mu)]\;\Gamma_n(\alpha_1)}{2\xi\beta\,(1-\mu)} a_{n,1}a_{n,2} \le 1. \tag{36}$$

Since $f_j \in T^*_{q,s}(\alpha_1,\alpha,\beta,\xi)\,(j=1,2)$, we readily see that

$$\sum_{n=2}^{\infty} \frac{[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]\,\Gamma_n(\alpha_1)}{2\xi\beta\,(1-\alpha)} a_{n,1} \le 1, \tag{37}$$

and

$$\sum_{n=2}^{\infty} \frac{[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]\;\Gamma_n(\alpha_1)}{2\xi\beta\,(1-\alpha)} a_{n,2} \le 1. \tag{38}$$

By the Cauchy-Schwarz inequality, we have

$$\sum_{n=2}^{\infty} \frac{[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]\;\Gamma_n(\alpha_1)}{2\xi\beta\,(1-\alpha)} \sqrt{a_{n,1}a_{n,2}} \le 1. \tag{39}$$

Thus it is sufficient to show that

$$\frac{[(n-1)(1-\beta)+2\beta\xi\,(n-\mu)]\;\Gamma_n(\alpha_1)}{2\xi\beta\,(1-\mu)} a_{n,1}a_{n,2} \le \frac{[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]\;\Gamma_n(\alpha_1)}{2\xi\beta\,(1-\alpha)} \sqrt{a_{n,1}a_{n,2}}\;\;(n\ge 2), \tag{40}$$

that is, that

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{(1-\mu)\,[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]}{(1-\alpha)\,[(n-1)(1-\beta)+2\beta\xi\,(n-\mu)]}\quad (n\ge 2). \tag{41}$$

Note that

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{2\xi\beta\,(1-\alpha)}{[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]\;\Gamma_n(\alpha_1)}\;\;(n\ge 2). \tag{42}$$

Consequently, we need only to prove that

$$\frac{2\xi\beta\,(1-\alpha)}{[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]\;\Gamma_n(\alpha_1)} \le \frac{(1-\mu)\,[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]}{(1-\alpha)\,[(n-1)(1-\beta)+2\beta\xi\,(n-\mu)]}, \tag{43}$$

or, equivalently, that

$$\mu \le 1 - \frac{2\beta\xi\,(1-\alpha)^2\,(n-1)\,[(1-\beta)+2\beta\xi]}{[(n-1)(1-\beta)+2\beta\xi\,(n-\alpha)]^2\,\Gamma_n(\alpha_1) - 4\beta^2\xi^2\,(1-\alpha)^2}\;\;(n\ge 2). \tag{44}$$

Since

$$\Phi(n)=1-\frac{2\beta\xi(1-\alpha)^2(n-1)[(1-\beta)+2\beta\xi]}{[(n-1)(1-\beta)+2\beta\xi(n-\alpha)]^2\Gamma_n(\alpha_1)-4\beta^2\xi^2(1-\alpha)^2}, \quad (45)$$

is an increasing function of n $(n\geq 2)$, letting $n=2$ in (45), we obtain

$$\mu\leq\Phi(2)=1-\frac{2\beta\xi(1-\alpha)^2[(1-\beta)+2\beta\xi]}{[(1-\beta)+2\beta\xi(2-\alpha)]^2\Gamma_2(\alpha_1)-4\beta^2\xi^2(1-\alpha)^2}, \quad (46)$$

which proves the main assertion of Theorem 8.
Finally, by taking the functions $f_j(z)\,(j=1,2)$ given by

$$f_j(z)=z-\frac{2\xi\beta(1-\alpha)}{[(1-\beta)+2\beta\xi(2-\alpha)]\;\Gamma_2(\alpha_1)}z^2\;\;(j=1,2), \quad (47)$$

we can see that the result is sharp.
Theorem 9. Let the functions $f_j\,(j=1,2)$ defined by (33) be in the class $T^*_{q,s}(\alpha_1,\alpha,\beta,\xi)$. Then the function

$$h(z)=z-\sum_{n=2}^{\infty}\left(a_{n,1}^2+a_{n,2}^2\right)z^n \quad (48)$$

belongs to the class $T^*_{q,s}(\alpha_1,\tau,\beta,\xi)$, where

$$\tau=1-\frac{4\beta\xi(1-\alpha)^2[1-\beta+2\beta\xi]}{[(1-\beta)+2\beta\xi(2-\alpha)]^2\Gamma_2(\alpha_1)-8\beta^2\xi^2(1-\alpha)^2}. \quad (49)$$

The result is sharp for the functions $f_j\,(j=1,2)$ defined by (47).
Proof. By virtue of Theorem 1, we obtain

$$\sum_{n=2}^{\infty}\left[\frac{[(n-1)(1-\beta)+2\beta\xi(n-\alpha)]\;\Gamma_n(\alpha_1)}{2\xi\beta(1-\alpha)}\right]^2a_{n,1}^2\leq$$

$$\left[\sum_{n=2}^{\infty}\frac{[(n-1)(1-\beta)+2\beta\xi(n-\alpha)]\;\Gamma_n(\alpha_1)}{2\xi\beta(1-\alpha)}a_{n,1}\right]^2\leq 1 \quad (50)$$

and

$$\sum_{n=2}^{\infty}\left[\frac{[(n-1)(1-\beta)+2\beta\xi(n-\alpha)]\;\Gamma_n(\alpha_1)}{2\xi\beta(1-\alpha)}\right]^2a_{n,2}^2\leq$$

$$\left[\sum_{n=2}^{\infty}\frac{[(n-1)(1-\beta)+2\beta\xi(n-\alpha)]\;\Gamma_n(\alpha_1)}{2\xi\beta(1-\alpha)}a_{n,2}\right]^2\leq 1. \quad (51)$$

It follows from (50) and (51) that

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{[(n-1)(1-\beta)+2\beta\xi(n-\alpha)]\Gamma_n(\alpha_1)}{2\xi\beta(1-\alpha)}\right]^2\left(a_{n,1}^2+a_{n,2}^2\right)\leq 1. \quad (52)$$

Therefore, we need to find the largest τ such that

$$\frac{[(n-1)(1-\beta)+2\beta\xi(n-\tau)]\;\Gamma_n(\alpha_1)}{2\xi\beta(1-\tau)}\leq$$

$$\frac{1}{2}\left[\frac{[(n-1)(1-\beta)+2\beta\xi(n-\alpha)]\Gamma_n(\alpha_1)}{2\xi\beta(1-\alpha)}\right]^2\;(n\geq 2), \quad (53)$$

that is, that

$$\tau \leq 1 - \frac{4\beta\xi (1-\alpha)^2 (n-1) [1-\beta+2\beta\xi]}{[(n-1)(1-\beta)+2\beta\xi (n-\alpha)]^2 \Gamma_n(\alpha_1) - 8\beta^2\xi^2 (1-\alpha)^2}. \tag{54}$$

Since

$$D(n) = 1 - \frac{4\beta\xi (1-\alpha)^2 (n-1) [1-\beta+2\beta\xi]}{[(n-1)(1-\beta)+2\beta\xi (n-\alpha)]^2 \Gamma_n(\alpha_1) - 8\beta^2\xi^2 (1-\alpha)^2},$$

is an increasing function of n $(n \geq 2)$, we readily have

$$\tau \leq D(2) = 1 - \frac{4\beta\xi (1-\alpha)^2 [1-\beta+2\beta\xi]}{[(1-\beta)+2\beta\xi (2-\alpha)]^2 \Gamma_2(\alpha_1) - 8\beta^2\xi^2 (1-\alpha)^2}$$

and Theorem 9 follows at once.

Remark. Putting $q = 2, s = 1, \alpha_1 = \lambda+1\,(\lambda > -1)$ and $\alpha_2 = \beta_1 = 1$ in this paper, we correct the results obtained by Khairnar and Rajas [7].

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M.K. Aouf, Department of Mathematics, Faculty of Science,, University of Mansoura, Mansoura, Egypt

E-mail address: mkaouf127@yahoo.com

A.O.Mostafa, Department of Mathematics, Faculty of Science,, University of Mansoura, Mansoura, Egypt
E-mail address: adelaeg254@yahoo.com

Y. Lashin, Department of Mathematics, Faculty of Science,, University of Mansoura, Mansoura, Egypt
E-mail address: aylashin@yahoo.com

B.M.Munassar, Department of Mathematics, Faculty of Science,, University of Mansoura, Mansoura, Egypt
E-mail address: ß