EXISTENCE OF MULTIPLE SOLUTIONS FOR A HOLLING II TYPE FUNCTIONAL RESPONSE SYSTEM

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ABSTRACT. Applying the Mawhin continuation theorem, the homotopy invariance of topological degree and some construction skills, we obtain the existence of multiple positive periodic solutions for our new Holling II type functional response system. Moreover, an interesting result is discovered: when the male prey species population density is a constant, the existence of multiple positive periodic solutions is independent of \(\alpha(t)\) in some special domain \(\Omega_i\), \((i \in A, \text{in Section 2})\). At last, we take an example illustrate our interesting results.

1. INTRODUCTION

In recent years, the existence of periodic solutions for predatory-prey system has been widely investigated by many researchers. For example, in [1], Wu considers a prey-predator model with sex-structure:

\[
\begin{align*}
\dot{x}(t) &= \beta(t)x(t) - k(t)(x(t) + y(t))x(t) - \xi_1(t) x(t) z(t), \\
\dot{y}(t) &= \gamma_2(t)x(t) - d_1(t)y(t) - k(t)(x(t) + y(t))y(t) - \xi_1(t) y(t) z(t), \\
\dot{z}(t) &= z(t)(-d(t) - \gamma_2(t)z(t) + \xi_2(t)y(t) + \xi_2(t)x(t)),
\end{align*}
\]

(1.1)

where \(x(t)\) and \(y(t)\) represent female and male prey species population densities respectively, \(z(t)\) stands for predator species population density. All the parameters in system (1.1), are positive continuous \(\omega\)-periodic functions, with \(\omega > 0\). Then by virtue of coincidence degree theory, the sufficient conditions, which are depended on \(\xi_2(t)\), are gained for guaranteeing the existence of at least one positive periodic solution.

Besides, Zhang and Hou [2] discuss the following Holling II type functional response system with harvesting terms:

\[
\begin{align*}
\dot{s}(t) &= s(t)(a(t) - b(t)s(t) - \frac{s(t)y(t)}{m(t)z(t) + s(t)}) - h_1(t), \\
\dot{z}(t) &= z(t)(-d(t) + \frac{f(t)s(t)}{m(t)z(t) + s(t)}) - h_2(t),
\end{align*}
\]

where \(s(t)\) stands for prey species population density, \(z(t)\) is predator species population density, and \(h_i(t)\) \((i = 1, 2)\) is a harvesting term. Using coincidence degree
theory, the existence of four positive periodic solutions was obtained. For more
details in this direction, one can consult [3-7] and the references cited up there.

In this paper, we consider the following Holling II type response function model
with sex-structure and new multiple harvesting terms:

\[
\begin{align*}
\dot{x}(t) &= x(t)\left[\frac{\alpha(t)}{2} - b_1(t)x(t) - \frac{c(t)\alpha(t)}{m(t)\alpha(t)x(t)+(1-\alpha(t))y(t)}\right] - h_1, \\
\dot{y}(t) &= y(t)\left[-b_2(t)y(t) - \frac{m(t)\alpha(t)x(t)+(1-\alpha(t))y(t)}{m(t)x(t)+\alpha(t)x(t)+(1-\alpha(t))y(t)}\right] + \frac{\alpha(t)}{2} x - h_2 \equiv 0, \quad (1.2) \\
\dot{z}(t) &= z(t)\left[-d(t) + \frac{f(t)\alpha(t)x(t)+(1-\alpha(t))y(t)}{m(t)x(t)+\alpha(t)x(t)+(1-\alpha(t))y(t)}\right] - h_3,
\end{align*}
\]

where \( h_1 = h_1(t, x, \frac{z}{y}), h_2 = h_2(t, y, \frac{z}{y}), h_3 = h_3(t, z), \) and all the parameters in
(1.2) are positive continuous \( \omega \)-periodic functions.

Predator eats both male and female prey species, and at different times and/or
at different locations, the male and female prey species may be eaten differently. So
we investigate the new Holling II type functional response system (1.2) involving a
ratio parameter \( \alpha(t) \) with \( \alpha(t) \in [0, 1] \), and now the food of predator \( z(t) \) contains
two parts: \( \alpha(t)x(t) \) and \( (1 - \alpha(t))y(t) \).

Note that when \( x \) is very big, then \( h \) increases and vice versa. So the harvesting
term \( h \) should be \( x \)-dependent and it motivates us to substitute \( h(t, x) \) for \( h(t) \),
which has been studied by many scholars in the past few years. Moreover, we
discover that when sex-ratio \( \frac{x(t)}{y(t)} \) is large enough, one harvests female prey species
more and vice versa. This stipulates us to study the gender ratio of prey species
and replace \( h(t, x) \) by \( h(t, x, \frac{z}{y}) \).

In model (1.2), we assume that the males have the same birth rate as females,
and both of them are \( \frac{1}{2} \). By using the Mawhin continuation theorem ([8]) and
the homotopy invariance of topological degree, we discuss the existence of mul-
tiple positive periodic solutions for a improved Holling II type functional response
system. In this paper, the definitions of \( N(u, \lambda) \) and the operator \( G(u) \) are quite
different from the above mentioned papers, and an interesting result is revealed as
well: when \( x \) is a constant, no matter what \( \alpha(t) \) takes on, (1.2) always has at least
four positive periodic solutions in \( \Omega_i \) \((i \in A)\), with \( A = \{1, 2, 3, 4\} \).

2. Main results

Firstly, let us review some notions and make some preparations.

Let \( X \) and \( Z \) be Banach spaces, that \( N : X \times [0, 1] \to Z \) be a continuous mapping,
that \( L : \text{Dom} L \subset X \to Z \) be a linear mapping, which is called to be a Fredholm
mapping of index zero if \( \dim \text{Ker} L = \text{codim} \text{Im} L < \infty \) and \( \text{Im} L \) is closed in \( Z \).

If \( L \) is a Fredholm mapping of index zero, then there exists continuous projectors
\( P : X \to X \) and \( Q : Z \to Z \) such that \( \text{Im} P = \text{Ker} L \) and \( \text{Ker} Q = \text{Im} L = \text{Im}(I - Q) \), and \( X = \text{Ker} L \oplus \text{Ker} P, Z = \text{Im} L \oplus \text{Im} Q \). Therefore, \( L|_{\text{Dom} L \cap \text{Ker} P} : (I - P)X \to \text{Im} L \) is invertible and its inverse is denoted by \( K_P \).

Let \( \Omega \subset X \) be bounded and open, then \( N \) is called \( L \)-compact on \( \bar{\Omega} \times [0, 1] \), if
\( QN(\bar{\Omega} \times [0, 1]) \) is bounded and \( K_P(I - Q)N : \bar{\Omega} \times [0, 1] \to X \) is compact.

For \( \text{Im} Q \) is isomorphic to \( \text{Ker} L \), there exists an isomorphism \( J : \text{Im} Q \to \text{Ker} L \).

Besides, we need Mawhin’s continuous theorem below.

**Lemma 2.1.** ([8, p40]) Let \( L \) be a Fredholm mapping of index zero, that \( N \) be
\( L \)-compact on \( \bar{\Omega} \times [0, 1] \). Assume
for each \( \lambda \in (0, 1) \), \( x \in \partial \Omega \cap \text{Dom} L \), \( Lx \neq \lambda Nx \);

(b) for each \( x \in \partial \Omega \cap \text{Ker} L \), \( QN(x, 0) \neq 0 \);

(c) \( \deg(JQN(x, 0), \Omega \cap \text{Ker} L, 0) \neq 0 \).

Then \( Lx = Nx \) has at least one solution in \( \overline{\Omega} \cap \text{Dom} L \).

For the sake of convenience, we introduce notations as follows:

\[
\begin{align*}
\alpha^L := \min_{t \in [0, \omega]} a(t), & \quad \alpha^M := \max_{t \in [0, \omega]} a(t), \\
h^L(t) := \min_{t \in [0, \omega]} h_1(t, x), & \quad h^M(t) := \max_{t \in [0, \omega]} h_1(t, x), \\
\lambda := \min \{\ln l^+, u_2(t)\}, & \quad \lambda := \min \{\ln l^+, u_2(t)\}, \\
S_1 := \max \{\ln l^+, u_2(t)\}, & \quad S_3 := \max \{\ln l^+, u_2(t)\},
\end{align*}
\]

Throughout this paper, we need the following assumptions:

\[
A_1 : \quad \frac{\alpha^L - \alpha^M}{mL} > 2b_1 h^1, \quad A_2 : \quad e^{S_1} L^L - e^{S_3} h^M m^M > 2m^M d^M h^M e^{S_3}.
\]

**Lemma 2.2.** \( l^- < H^- < H^+ < l^+ \).

**Proof.** In fact, from above assumptions, we have:

\[
l^- = 4b_1^L h^L \left[ 2b_1 \left\{ \frac{\alpha^M}{2} + \sqrt{\frac{(\alpha^M)^2}{4} - 4b_1^L h^L} \right\} \right]^{-1} < H^-,
\]

and

\[
H^+ < \left[ \frac{\alpha^M}{2} + \sqrt{\frac{(\alpha^M)^2}{4} - 4b_1^L h^L} \right] [2b_1^L]^{-1} = l^+.
\]

From (2.1), (2.2) and \( l^- < l^+ \), \( H^- < H^+ \), one verifies the conclusion.

**Theorem 2.1.** Let \( A_1 \) and \( A_2 \) hold. Then, (1.2) has at least four positive \( \omega \)-periodic solutions.

**Proof.** For \( \dot{y}(t) \equiv 0 \), \( y(t) \) is a constant. By making the change of variables

\[
x(t) = e^{u_1(t)}, \quad y(t) = e^{u_2(t)} \quad \text{and} \quad z(t) = e^{u_3(t)},
\]

(1.2) can be reformulated as:

\[
\begin{align*}
\dot{u}_1(t) &= \frac{a(t)}{t} - b_1(t) e^{u_1(t)} - \frac{m(t) e^{u_1(t)} + (1 - \alpha(t)) e^{u_1(t)} + (1 - \alpha(t)) e^{u_3(t)}}{m(t) e^{u_1(t)} + (1 - \alpha(t)) e^{u_1(t)} + (1 - \alpha(t)) e^{u_3(t)}}, \\
\dot{u}_3(t) &= -d(t) + \frac{m(t) e^{u_1(t)} + (1 - \alpha(t)) e^{u_1(t)} + (1 - \alpha(t)) e^{u_3(t)} - h_3(t, e^{u_3(t)}) e^{-u_3(t)}}{m(t) e^{u_1(t)} + (1 - \alpha(t)) e^{u_1(t)} + (1 - \alpha(t)) e^{u_3(t)}},
\end{align*}
\]

where \( u_2(t) \) is a constant. Let

\[
X = Z = \left\{ u = (u_1, u_3)^T \in C([0, \omega]; \mathbb{R}^2) : u(t + \omega) = u(t), \; t \in [0, \omega] \right\}
\]

and define

\[
\|u\| = \max_{t \in [0, \omega]} |u_1(t)| + \max_{t \in [0, \omega]} |u_3(t)|, \quad u \in X \text{ or } Z.
\]

Equipped with the above norm \( \| \cdot \| \), then \( X \) and \( Z \) are Banach spaces. Let the mappings be that

\[
N(u, \lambda) : X \times [0, 1] \to Z, \quad L : \text{Dom} L \subset X \to Z, \quad P : \text{Dom} L \cap X \to \text{Ker} L, \quad Q : Z \to Z/\text{Im} L,
\]
the generalized inverse (to $L$)

$$
N(u, \lambda) = \begin{pmatrix}
\frac{a(t)}{2} - b_1(t)e^{u_1(t)} - \frac{c(t)e^{u_3(t)}}{m(t)e^{u_3(t)} + \alpha(t)e^{u_1(t)} + (1-\alpha(t))e^{u_2(t)}} - h_1e^{-u_1(t)} \\
-d(t) + \frac{f(t)\alpha(t)e^{u_1(t)} + (1-\alpha(t))e^{u_2(t)}}{m(t)e^{u_3(t)} + \alpha(t)e^{u_1(t)} + (1-\alpha(t))e^{u_2(t)}} - h_3e^{-u_3(t)}
\end{pmatrix},
$$

$$Lu = \dot{u}(t), \quad Pu = \int_{0}^{\omega} u(t) dt, \quad u \in X, \quad Qz = \int_{0}^{\omega} z(t) dt, \quad z \in Z.$$ 

Then $\text{Ker} L = \mathbb{R}^2$, $\text{Im} L = \{ z \in Z : \int_{0}^{\omega} z(t) dt = 0 \}$ is closed in $Z, \text{dim Ker} L = 2 = \text{codim Im} L$, and $P, Q$ are continuous projectors such that $\text{Im} P = \text{Ker} L$ and $\text{Ker} Q = \text{Im} L = \text{Im}(I - Q)$. Hence, $L$ is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to $L$) $K_p : \text{Im} L \to \text{Ker} P \cap \text{Dom} L$ is given by

$$K_p(z) = \int_{0}^{s} z(t) dt - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{s} z(t) dt ds.$$ 

Thus

$$QN(u, \lambda) = \left( \frac{1}{\omega} \int_{0}^{\omega} F_1(s, \lambda) ds, \frac{1}{\omega} \int_{0}^{\omega} F_3(s, \lambda) ds \right)^T$$

and

$$K_p(I - Q)N(u, \lambda) = \left( \int_{0}^{\omega} F_1(s, \lambda) ds - \frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{s} F_1(s, \lambda) ds dt + \left( \frac{1}{\omega} - \frac{1}{\lambda} \right) \int_{0}^{\omega} F_1(s, \lambda) ds \right),$$

where

$$F_1(s, \lambda) = \frac{a(s)}{2} - b_1(s)e^{u_1(s)} - h_1(s, e^{u_1(s)}, e^{u_1(s)} - u_2(s))e^{-u_1(s)} - \frac{c(s)e^{u_3(s)}}{m(s)e^{u_3(s)} + \alpha(s)e^{u_1(s)} + (1-\alpha(s))e^{u_2(s)}},$$

$$F_3(s, \lambda) = -d(s) + \frac{f(s)\alpha(s)e^{u_1(s)} + (1-\alpha(s))e^{u_2(s)}f(s)}{m(s)e^{u_3(s)} + \alpha(s)e^{u_1(s)} + (1-\alpha(s))e^{u_2(s)}} - h_3(s, e^{u_3(s)})e^{-u_3(s)}.$$ 

Evidently, $QN$ and $K_p(I - Q)N$ are continuous and it is easy to derive the compactness of $K_p(I - Q)N(\Omega)$ for any open bounded set $\Omega \subset X$ by using the Arzela-Ascoli theorem. Moreover, $QN(\Omega)$ is bounded. Therefore, $N$ is $L$-compact on $\Omega$ for any open bounded set $\Omega$.

In order to use Lemma 2.1, we must find at least four appropriate open bounded subsets in $X$. Applying the operator equation $Lu = \lambda N(u, \lambda), \lambda \in (0, 1)$, one gains

$$\begin{cases}
\dot{u}_1(t) = \lambda \left[ \frac{a(t)}{2} - b_1(t)e^{u_1(t)} - h_1(t, e^{u_1(t)}, e^{u_1(t)} - u_2(t))e^{-u_1(t)} \right. \\
\left. - \frac{\lambda e^{u_3(t)}}{m(t)e^{u_3(t)} + \alpha(t)e^{u_1(t)} + (1-\alpha(t))e^{u_2(t)}} \right],
\end{cases}$$

$$\begin{cases}
\dot{u}_3(t) = \lambda \left[ \frac{f(t)\alpha(t)e^{u_1(t)} + (1-\alpha(t))e^{u_2(t)}}{m(t)e^{u_3(t)} - d(t) + \alpha(t)e^{u_1(t)} + (1-\alpha(t))e^{u_2(t)}} - h_3(t, e^{u_3(t)})e^{-u_3(t)} \right].
\end{cases}$$

Assume that $u = u(t) \in X$ is an $\omega$-periodic solution of (2.4) for some $\lambda \in (0, 1)$. Then for $i = 1, 3$, there exist $\xi_i, \eta_i \in [0, \omega]$ such that $u_i(\xi_i) = \max_{t \in [0, \omega]} u_i(t), u_i(\eta_i) = \min_{t \in [0, \omega]} u_i(t).$
Due to (2.1) and Lemma 2.1, then for any $t \in [0, \omega]$, one gets
\[
\ln l^- < u_1(\eta_i) < \ln l^+.
\]
A direct computation from (2.5) gives

\[ d^L \leq d(\xi_3) < \frac{f^M(\alpha(\xi_3)e^{u_1(\xi_3)} + (1 - \alpha(\xi_3))e^{u_2(\xi_3)}f^M)}{m^L e^{u_3(\xi_3)}} \]

\[ = \frac{f^M e^{S_3}}{m^L e^{u_3(\xi_3)}}. \]

It means that

\[ u_3(\xi_3) < \ln \frac{f^M e^{S_3}}{m^L d^L}. \] (2.11)

From (2.6) and by \( S_4 = \min\{\ln l^-, u_2(t)\} < \min\{u_1(t), u_2(t)\} \), it follows that

\[ h_3^{L_2} e^{-u_2(\eta_3)} \leq h_3(\xi_3, e^{u_3(\eta_3)})e^{-u_3(\eta_3)} < \frac{f^M e^{S_3}}{\alpha(\eta_3)e^{u_1(\eta_3)} + (1 - \alpha(\eta_3))e^{u_2(\eta_3)}}, \]

Then

\[ u_3(\eta_3) > \ln \frac{h_3^{L_2} e^{S_4}}{f^M e^{S_3}}. \] (2.12)

On account of (2.5), we have

\[ \frac{f(\xi_3)\alpha(\xi_3)e^{u_1(\xi_3)} + (1 - \alpha(\xi_3))e^{u_2(\xi_3)}f(\xi_3)}{m(\xi_3)e^{u_3(\xi_3)} + \alpha(\xi_3)e^{u_1(\xi_3)} + (1 - \alpha(\xi_3))e^{u_2(\xi_3)} = d(\xi_3) + h_3(\xi_3, e^{u_3(\xi_3)})e^{-u_3(\xi_3)}, \]

thus \( \frac{e^{S_4} f^L}{m^L e^{u_3(\xi_3)} + e^{S_3}} < d^M + h_3^M e^{-u_3(\xi_3)}. \) It hints that

\[ m^M d^L e^{2u_3(\xi_3)} + (e^{S_3} d^M + h_3^M m^M - e^{S_1} f^L)e^{u_3(\xi_3)} + h_3^M e^{S_3} > 0. \]

In view of \( A_2 \), one gains \( u^- > 0, u^+ > 0 \) and

\[ u_3(\xi_3) < \ln u^- \quad \text{or} \quad u_3(\xi_3) > \ln u^+. \] (2.13)

Analogously, by (2.6) 2, we get

\[ u_3(\eta_3) < \ln u^- \quad \text{or} \quad u_3(\eta_3) > \ln u^+. \] (2.14)

From (2.11) – (2.14), one concludes

\[ \ln \frac{h_3^{L_2} e^{S_4}}{f^M e^{S_3}} < u_3(t) < \ln u^- \quad \text{or} \quad \ln u^+ < u_3(t) < \ln \frac{f^M e^{S_3}}{m^L d^L}. \] (2.15)
Indeed, by $A_2$, then $e^{S_4} f^L - e^{S_3} d^M - h_3^M m^M > 0$, and

$$
u^- = \frac{e^{S_4} f^L - d^M e^{S_3} - h_3^M m^M - \sqrt{(d^M e^{S_3} + h_3^M m^M - e^{S_4} f^L)^2 - 4d^M m^M h_3^M e^{S_3}}}{2m^M d^M}$$

$$= (4d^M m^M h_3^M e^{S_3}) \left[ 2m^M d^M (e^{S_4} f^L - d^M e^{S_3} - h_3^M m^M) + \sqrt{(d^M e^{S_3} + h_3^M m^M - e^{S_4} f^L)^2 - 4d^M m^M h_3^M e^{S_3}} \right]^{-1}$$

$$= \frac{2h_3^M e^{S_3}}{e^{S_3} f^L}$$

Thus $\ln \frac{h_3^M e^{S_3}}{e^{S_3} f^L} < u_3(t) < \ln u^-$. Furthermore,

$$u^+ = \frac{e^{S_4} f^L - d^M e^{S_3} - h_3^M m^M + \sqrt{(d^M e^{S_3} + h_3^M m^M - e^{S_4} f^L)^2 - 4d^M m^M h_3^M e^{S_3}}}{2m^M d^M}$$

$$= \left[ e^{S_4} f^L - d^M e^{S_3} - h_3^M m^M + \sqrt{(e^{S_4} f^L - d^M e^{S_3} - h_3^M m^M)^2} \right] (2m^M d^M)^{-1}$$

$$< \left[ e^{S_4} f^M - d^M e^{S_3} - h_3^M m^M + \sqrt{(e^{S_4} f^M + d^M e^{S_3} + h_3^M m^M)^2} \right] (2m^L d^L)^{-1}$$

$$= \left[ e^{S_4} f^M - d^M e^{S_3} - h_3^M m^M + e^{S_3} f^M + d^M e^{S_3} + h_3^M m^M \right] (2m^L d^L)^{-1}$$

and $\ln u^+ < u_3(t) < \ln \frac{f^M e^{S_3}}{m^L d^L}$. Clearly, $\ln l^\pm$, $\ln H^\pm$, $u^\pm$, $S_3$, $S_4$ are independent of $\lambda$. Now, let $u = (u_1, u_2, u_3)$,

$$\Omega_1 = \{ u \in X : \ln l^- - C_1 < u_1(t) < \ln H^- + 2C_2, \ln \frac{h_3^M e^{S_3}}{f^M e^{S_3}} < u_3(t) < \ln u^- \},$$

$$\Omega_2 = \{ u \in X : \ln l^- - C_1 < u_1(t) < \ln H^- + 2C_2, \ln u^+ < u_3(t) < \ln \frac{f^M e^{S_3}}{m^L d^L} \},$$

$$\Omega_3 = \{ u \in X : \ln H^- - C_2 < u_1(t) < \ln l^+ + C_3, \ln \frac{h_3^M e^{S_3}}{f^M e^{S_3}} < u_3(t) < \ln u^- \},$$

$$\Omega_4 = \{ u \in X : \ln H^- - C_2 < u_1(t) < \ln l^+ + C_3, \ln u^+ < u_3(t) < \ln \frac{f^M e^{S_3}}{m^L d^L} \}.$$
Now we check that (b) in Lemma 2.1 holds, i.e., when \( u \in \partial \Omega_i \cap \text{Ker}L = \partial \Omega_i \cap \mathbb{R}^2, \) \( QN(u, 0) \neq (0, 0)^\top, \) (i \( \in A \)). If not, then when \( u \in \partial \Omega_i \cap \text{Ker}L = \partial \Omega_i \cap \mathbb{R}^2, \) is a constant vector \( u = (u_1, u_3)^\top \) with \( u \in \partial \Omega_i \) (i \( \in A \)) meets:

\[
\left\{ \begin{array}{l}
\int_0^\omega \left( \frac{a(t)}{2} - b_1(t)e^{u_1(t)} - h_1(t, e^{u_1(t)}, e^{u_1(t)} - e^{-u_1(t)}) \right) dt = 0, \\
\int_0^\omega \left( -d(t) + \frac{f(t)(1-\alpha(t))e^{u_1(t)} + f(t)(1-\alpha(t))e^{u_2(t)}}{m(t)e^{u_3(t)} + \alpha(t)e^{u_3(t)} + (1-\alpha(t))e^{u_2(t)}} - h_3(t, e^{u_3(t)})e^{-u_3(t)} \right) dt = 0.
\end{array} \right.
\]

Then there exist \( t_j \in [0, \omega] \) (j \( = 1, 3 \)) such that

\[
\left\{ \begin{array}{l}
\frac{a(t)}{2} - b_1(t)e^{u_1(t)} - h_1(t, e^{u_1(t)}, e^{u_1(t)} - e^{-u_1(t)}) = 0, \\
\frac{f(t)(1-\alpha(t))e^{u_1(t)} + f(t)(1-\alpha(t))e^{u_2(t)}}{m(t)e^{u_3(t)} + \alpha(t)e^{u_3(t)} + (1-\alpha(t))e^{u_2(t)}} - d(t) - h_3(t, e^{u_3(t)})e^{-u_3(t)} = 0.
\end{array} \right. \quad (2.16)
\]

Akin to obtain (2.10), (2.15), it is clear that the solution of (2.16) does not satisfy \( u \in \partial \Omega_i \cap \mathbb{R}^2, \) (i \( \in I \)). Hence, condition (b) in Lemma 2.1 holds.

Finally, let us show (c) in Lemma 2.1 is valid. Since \( \text{Ker}L = \text{Im}Q \), we can take \( J = I \). To this end, we define the homotopy mapping \( \phi : \text{Dom}L \times [0, 1] \to X \) by

\[
\phi(u_1, u_3, \mu) = \mu QN(u, 0) + (1 - \mu)G(u), \quad \mu \in [0, 1],
\]

where \( G(u) \) is

\[
G(u) = \left( \int_0^\omega \left[ \frac{a(t)}{2} - b_1(t)e^{u_1(t)} - h_1(t, e^{u_1(t)}, e^{u_1(t)} - e^{-u_1(t)}) \right] dt \right) + \left( \int_0^\omega \left[ -d(t) + \frac{f(t)(1-\alpha(t))e^{u_1(t)} + f(t)(1-\alpha(t))e^{u_2(t)}}{m(t)e^{u_3(t)} + \alpha(t)e^{u_3(t)} + (1-\alpha(t))e^{u_2(t)}} - h_3(t, e^{u_3(t)})e^{-u_3(t)} \right] dt \right).
\]

For \( u \in \partial \Omega_i \cap \text{Ker}L = \partial \Omega_i \cap \mathbb{R}^2 \) and \( \mu \in [0, 1], \) then they fulfill

\[
\phi(u_1, u_3, \mu) = \mu QN(u, 0) + (1 - \mu)G(u) \neq (0, 0)^\top.
\]

Otherwise, there are \( \mu \) and constant vector \( u = (u_1, u_3)^\top \) such that \( \phi(u_1, u_3, \mu) = (0, 0)^\top. \) Thanks to the mean value theorem, there exist \( \bar{t}_4, \bar{t}_6 \in [0, \omega], \) so that

\[
\left\{ \begin{array}{l}
\mu \left[ \frac{a(t_4)}{2} - b_1(t_4)e^{u_1(t_4)} - h_1(t_4, e^{u_1(t_4)}, e^{u_1(t_4)} - e^{-u_1(t_4)}) \right] + (1 - \mu) \left[ \frac{a(t_6)}{2} - b_1(t_4)e^{u_1(t_6)} - h_1(t_6, e^{u_1(t_6)}, e^{u_1(t_6)} - e^{-u_1(t_6)}) \right] = 0, \\
\mu \left[ \frac{f(t_4)(1-\alpha(t_4))e^{u_1(t_4)} + f(t_6)(1-\alpha(t_6))e^{u_2(t_6)}}{m(t_4)e^{u_3(t_4)} + \alpha(t_4)e^{u_3(t_6)} + (1-\alpha(t_6))e^{u_2(t_6)}} - h_3(t_4, u_1(t_4), e^{u_3(t_4)} - e^{-u_3(t_4)}) \right] \left[ \frac{f(t_4)e^{u_2(t_4)}}{m(t_4)e^{u_3(t_4)} + \alpha(t_4)e^{u_3(t_6)} + (1-\alpha(t_6))e^{u_2(t_6)}} - h_3(t_6, u_3(t_6), e^{u_3(t_6)} - d(t_6)) \right] = 0.
\end{array} \right. \quad (2.17)
\]

We make the following Claims.

**Claim 1.** \( \ln l^- \leq u_1(\bar{t}_4) < \ln H^- \) or \( \ln H^+ < u_1(\bar{t}_4) \leq \ln l^+. \) (2.18)

From (2.17)_1, we obtain

\[
\left\{ \begin{array}{l}
\frac{a(t_4)}{2} - b_1(t_4)e^{u_1(t_4)} - h_1(t_4, e^{u_1(t_4)}, e^{u_1(t_4)} - e^{-u_1(t_4)}) \leq \\
\frac{a(t_6)}{2} - b_1(t_4)e^{u_1(t_6)} - h_1(t_6, e^{u_1(t_6)}, e^{u_1(t_6)} - e^{-u_1(t_6)}) \leq (1 - \mu)h_1(t_4)e^{u_1(t_4)} - h_1(t_6, e^{u_1(t_6)}, e^{u_1(t_6)} - e^{-u_1(t_6)})
\end{array} \right.
\]

\[
\leq 0.
\]
It means that
\[ \frac{a^L}{2} - b_1^M e^{u_1(\bar{t}_4)} - \frac{c^M}{mL} - h_1^M e^{-u_1(\bar{t}_4)} < 0. \]
So \( u_1(\bar{t}_4) < \ln H^- \) or \( u_1(\bar{t}_4) > \ln H^+ \). In like manner, from (2.17)_1, one achieves
\[
0 = \frac{a(\bar{t}_4)}{2} - b_1^M e^{u_1(\bar{t}_4)} - \mu h_1 (\bar{t}_4, e^{u_1(\bar{t}_4)}; e^{u_1(\bar{t}_4)}, \frac{e^{u_1(\bar{t}_4)}}{e^{u_1(\bar{t}_4)}}) e^{-u_1(\bar{t}_4)} - (1 - \mu) h_1^L e^{-u_1(\bar{t}_4)}
+ \mu (b_1^M - b_1(\bar{t}_4)) e^{u_1(\bar{t}_4)}
\leq \frac{a(\bar{t}_4)}{2} - b_1^M e^{u_1(\bar{t}_4)} - h_1^L e^{-u_1(\bar{t}_4)} + \mu (b_1^M - b_1(\bar{t}_4)) e^{u_1(\bar{t}_4)}
\leq \frac{a(\bar{t}_4)}{2} - (1 - \mu) b_1^M e^{u_1(\bar{t}_4)} - \mu b_1(\bar{t}_4) e^{u_1(\bar{t}_4)} - h_1^L e^{-u_1(\bar{t}_4)}
\leq \frac{a(\bar{t}_4)}{2} - b_1(\bar{t}_4) e^{u_1(\bar{t}_4)} - h_1^L e^{-u_1(\bar{t}_4)}
\leq \frac{a^M}{2} - b_1^L e^{u_1(\bar{t}_4)} - h_1^L e^{-u_1(\bar{t}_4)},
\]
which suggests
\[ b_1^L e^{2u_1(\bar{t}_4)} - \frac{a^M}{2} e^{u_1(\bar{t}_4)} + h_1^L < 0. \]
Thus \( \ln l^- \leq u_1(\bar{t}_4) \leq \ln l^+ \), and according to \( l^- < H^- < H^+ < l^+ \), the desired result follows.

**Claim 2.** \( \ln \frac{h_1^L e^{S_4}}{f^M e^{S_3}} < u_3(\bar{t}_6) < \ln u^- \) or \( \ln u^+ < u_3(\bar{t}_6) < \ln \frac{f^M e^{S_3}}{mL e^{L^2}} \).

Employing \( S_3 = \max\{\ln l^+, u_2(t)\} \), \( S_4 = \min\{\ln l^-, u_2(t)\} \), and (2.17)_2, we fulfill the conclusion easily so we omit the details.

From Claim 1 and Claim 2, when \( \mu \in [0, 1] \) and \( \phi(u_1, u_3, \mu) = 0 \), then \( u \in \partial \Omega_1 \cap \text{Ker } L(\partial \Omega_1 \cap \mathbb{R}^2) \), so when \( u \in \partial \Omega_1 \cap \text{Ker } L = \partial \Omega_1 \cap \mathbb{R}^2 \) and \( \mu \in [0, 1] \), \( \phi(u_1, u_3, \mu) \neq 0 \).

By the homotopy invariance of topological degree and note the system below
\[
\begin{cases}
2 \frac{a(\bar{t}_4)}{2} - b_1^M e^{u_1(\bar{t}_4)} - h_1^L e^{-u_1(\bar{t}_4)} = 0, \\
-d(\bar{t}_4) + \frac{f^L e^{S_4}}{(m+2e^{3})e^{u_3(\bar{t}_6)}} - h_3^M e^{S_4} e^{-u_3(\bar{t}_6)} = 0,
\end{cases}
\]
one gains
\[
\text{deg}(JQN(u, 0), \Omega_1 \cap \text{Ker } L, (0, 0)^T) = \text{deg}(\phi(u, 1), \Omega_1 \cap \text{Ker } L, (0, 0)^T) = \text{deg}(\phi(u, 0), \Omega_1 \cap \text{Ker } L, (0, 0)^T) = \text{deg}(JG(u), \Omega_1 \cap \text{Ker } L, (0, 0)^T) = \text{sign} \left[ -b_1^M e^{u_1(\bar{t}_4)} + h_1^L e^{-u_1(\bar{t}_4)} \right] \times \text{sign} \left[ \frac{-f^L e^{S_4}}{(m^M + 2e^{3})(e^{u_3(\bar{t}_6)})} + h_3^M e^{S_4} e^{-u_3(\bar{t}_6)} \right].
\]

For \( \text{sign} \left[ -b_1^M e^{u_1(\bar{t}_4)} + h_1^L e^{-u_1(\bar{t}_4)} \right] \), from \( a(\bar{t}_4) - b_1^M e^{u_1(\bar{t}_4)} - h_1^L e^{-u_1(\bar{t}_4)} = 0 \), then
\[
-b_1^M e^{u_1(\bar{t}_4)} + h_1^L e^{-u_1(\bar{t}_4)} = (a(\bar{t}_4)) - 2b_1^M e^{u_1(\bar{t}_4)}.
\]
Employing
\[
e^{u_1(\bar{t}_4)} = \ln \frac{a(\bar{t}_4) \pm \sqrt{(a(\bar{t}_4))^2 - 4b_1^M h_1^L}}{2b_1^M},
\]
one ascertains

\[
\text{sign}\left[ -b^M_1 e^{u_1(\bar{t}_4)} + h^1_1 e^{-u_1(\bar{t}_4)} \right] = \text{sign}\left[ a(\bar{t}_4) \right] - 2b^M_1 e^{u_1(\bar{t}_4)} = \pm 1.
\]

For \( \text{sign}\left[ \frac{-f_L e^{S_4}}{(m^M + 2e^{S_3})e^{u_3(\bar{t}_6)}} + h^M_3 e^{-u_3(\bar{t}_6)} \right] \), according to

\[
-d(\bar{t}_6) + \frac{f_L e^{S_4}}{(m^M + 2e^{S_3})e^{u_3(\bar{t}_6)}} - h^M_3 e^{S_4} e^{-u_3(\bar{t}_6)} = 0,
\]

it yields

\[
-\frac{f_L e^{S_4}}{(m^M + 2e^{S_3})e^{u_3(\bar{t}_6)}} = -d(\bar{t}_6) - h^M_3 e^{S_4} e^{-u_3(\bar{t}_6)}.
\]

Thus

\[
\text{sign}\left[ \frac{-f_L e^{S_4}}{(m^M + 2e^{S_3})e^{u_3(\bar{t}_6)}} + h^M_3 e^{S_4} e^{-u_3(\bar{t}_6)} \right] = \text{sign}\left[ -d(\bar{t}_6) \right] = -1.
\]

Therefore,

\[
\begin{align*}
\text{deg}(JQN(u, 0), \Omega_i \cap \text{Ker } L, (0, 0)^	op) \\
= \text{sign}\left[ -b^M_1 e^{u_1(\bar{t}_4)} + h^1_1 e^{-u_1(\bar{t}_4)} \right] \times \text{sign}\left[ \frac{-f_L e^{S_4}}{(m^M + 2e^{S_3})e^{u_3(\bar{t}_6)}} + h^M_3 e^{S_4} e^{-u_3(\bar{t}_6)} \right] \\
= \pm 1 \times (-1) \\
= \pm 1 \neq 0.
\end{align*}
\]

The argument appeals above for (2.19), adapted to \( G(u) \) now yields that the topological degree \( \text{deg}(JQN(x, 0), \Omega_i \cap \text{Ker } L, (0, 0)^	op) \) should be independent of \( \Omega_i \). Consequently, for \( i = 2, 3, 4, \)

\[
\text{deg}(JQN(u, 0), \Omega_2 \cap \text{Ker } L, (0, 0)^	op) = \pm 1.
\]

So far, we have confirmed that \( \Omega_i \) (\( i \in A \)) satisfies all the assumptions in Lemma 2.1. So, (2.3) has at least four different \( \omega \)-periodic solutions. For this reason, (1.2) has at least four different positive \( \omega \)-periodic solutions. This completes the proof.

**Remark 2.1.**

(i) The definition for \( G(u) \) is quite different from the previous papers, especially in \( G(u)_2 \). And in \( G(u)_1 \), we omit a complex term and make some special terms.

(ii) In the definition of the operator \( N(u, \lambda) \), the first equation involves \( \lambda \), but the second equation is independent of \( \lambda \).

(iii) When defining \( \Omega_i (i \in A) \), we use a priori bounds for \( u_1(t) \), so that we can supersede the particular \( G(u) \) for \( QN(u, 0) \) to compute \( \text{deg}(JQN(u, 0), \Omega_i \cap \text{Ker } L, (0, 0)^	op) \).
3. An Example

Consider the following Holling II type functional response system with a ratio parameter \( \alpha(t) \) with \( \alpha(t) \in [0, 1] \), and new harvesting terms:

\[
\begin{align*}
\dot{x}(t) &= x(t) \left[ \frac{1}{2} (30 + 2 \cos t) - \frac{2 \cos t x(t)}{2 + \sin t + \alpha(t) x(t) + (1 - \alpha(t)) y(t)} \right] - (5 + \cos t) x(t) - \frac{17}{4} - \sin t - \frac{\cos^2 t}{1 + 2 \sin t + 3 - x(t)}, \\
y(t) &= 2,
\end{align*}
\]

with

\[
\dot{z}(t) = z(t) \left[ \frac{(61 + \sin t) \alpha(t) x(t) + (1 - \alpha(t)) y(t)(11 + \sin t)}{2 + \sin t + \alpha(t) x(t) + (1 - \alpha(t)) y(t) - \frac{4 + \sin t}{1 + 2 - \sin(t)}} \right] - 2 - \frac{\sin t}{1 + 2 - \sin(t)}.
\]

Now, \( \alpha(t) = 30 + 2 \cos t, b_1(t) = 5 + \cos t, c(t) = \frac{2 + \cos t}{3}, m(t) = 2 + \sin t, h_1(t, x, \frac{y}{y}) = \frac{17}{4} + \sin t + \frac{\cos^2 t}{1 + 2 \sin t + 3 - x(t)} \), \( d(t) = \frac{3 + \cos t}{2}, f(t) = 61 + \sin t, h_3(t, z) = 2 + \frac{\sin t}{1 + 2 - \sin(t)} \).

\[
a_{L} - c_{M} = 14 - 1 = 13 > 2 \sqrt{h_{1}^{M} h_{1}^{M}}.
\]

This proves \( A_1 \). Next, we check that \( A_2 \) is valid and by virtue of \( f^L = 60, b_1^L = 4, h_3^M < 3, a^M = 32, d^M = 2, h_1^1 = \frac{13}{4} \),

\[
l^+ = \frac{a^M}{2} + \sqrt{\frac{(a^M)^2}{4} - 4b_1^L h_1^L} = 16 + \sqrt{196 - 4 \times 4 \times \frac{13}{4}} = \frac{7}{2},
\]

\[
l^- = \frac{a^M}{2} - \sqrt{\frac{(a^M)^2}{4} - 4b_1^L h_1^L} = 16 - \sqrt{196 - 4 \times 4 \times \frac{13}{4}} = \frac{1}{2},
\]

one deduces \( S_3 = \max \{\ln l^+, \ln 2\} = \ln \frac{7}{2}, S_4 = \min \{\ln l^-, \ln 2\} = \ln \frac{1}{2} \) and

\[
e^{S_4} f^L - e^{S_4} d^M - h_3^M m^M > \frac{1}{2} \times 60 - \frac{7}{2} \times 2 - 3 \times 2 = 17,
\]

\[
2 \sqrt{m^M d^M h_3^M e^{S_3} < 2 \times \sqrt{3 \times 2 \times 3 \times \frac{7}{2}} < 17 < e^{S_4} f^L - e^{S_4} d^M - h_3^M m^M}.
\]

Hence \( A_2 \) is legitimate. By Theorem 2.1, (3.1) possesses at least four positive \( \omega \)-periodic solutions.

**Remark 3.1.** In the example, from the expression of the harvesting term \( h_1(t, x, \frac{y}{y}) \), it is easy to find that when prey species population density \( x(t) \) is very large, \( h_1 \) increases. In addition, in the first equation, we also discover that when the sex-ratio \( \frac{z(t)}{y(t)} \) is large enough, \( h_1 \) increases as well and vice versa.

**References**


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