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INITIAL COEFFICIENT ESTIMATES FOR BI-λ-CONVEX AND BI-μ-STARLIKE FUNCTIONS CONNECTED WITH ARITHMETIC AND GEOMETRIC MEANS

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Abstract. In the present work, we propose to investigate the coefficient estimates for certain subclasses of bi-λ-convex and bi-μ-starlike functions of the Ma-Minda type in the open unit disk $\mathbb{U}$. Some interesting applications of the results presented here are also discussed

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < 1\}.$$

We shall denote by $\mathcal{S}$ the class of functions in $\mathcal{A}$ which are also univalent in $\mathbb{U}$.

For two functions f and g, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \qquad \text{and} \qquad |w(z)| < 1 \qquad (z \in \mathbb{U}),$$

such that

$$f(z) = g\big(w(z)\big) \qquad (z \in \mathbb{U}).$$

In particular, if the function g is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$f(0) = g(0) \qquad \text{and} \qquad f(\mathbb{U}) \subset g(\mathbb{U}).$$

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It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geqq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of all functions in $\mathcal{A}$, which are bi-univalent in $\mathbb{U}$ and given by (1). For a brief history of functions in the class Σ, see [24] (see also [3], [11] and [17]). In fact, judging by the remarkable flood of papers on the subject (see, for example, [2, 4, 5, 6, 7, 8, 9, 12, 14, 15, 19, 22, 23, 25, 26, 29, 27, 30] *as well as* the references to the related earlier works cited in each of these papers), the recent pioneering work of Srivastava *et al.* [24] appears to have revived the study of analytic and bi-univalent functions in recent years. Many interesting examples of functions which are in (or which are not in) the class Σ, together with various other properties and characteristics associated with the bi-univalent function class Σ (including also several open problems and conjectures involving estimates on the Taylor-Maclaurin coefficients of functions in Σ), can be found in the literature (see, for example, [1] and [30]; see also some of the aforecited papers).

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\alpha) := \left\{ f : f \in \mathcal{S} \,\text{and}\, \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\}$$

and

$$\mathcal{K}(\alpha) := \left\{ f : f \in \mathcal{S} \,\text{and}\, \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U};\ 0 \leqq \alpha < 1) \right\}.$$

For $0 \leqq \alpha < 1$, a function $f \in \Sigma$ is said to be in the class $\mathcal{S}^*_\Sigma(\alpha)$ of bi-starlike functions of order α in $\mathbb{U}$ or in the class $\mathcal{K}_\Sigma(\alpha)$ of bi-convex functions of order α in $\mathbb{U}$ if both f and f^{-1} are, respectively, starlike or convex functions of order α in $\mathbb{U}$. For $0 < \beta \leqq 1$, a function $f \in \Sigma$ is in the class $\mathcal{SS}^*_\Sigma(\beta)$ strongly bi-starlike functions of order β in $\mathbb{U}$ if both the functions f and f^{-1} are strongly starlike of order β in $\mathbb{U}$ (see [3]).

The arithmetic means of some functions and expressions is very frequently used in mathematics, especially in Geometric Function Theory. Making use of the arithmetic means, Mocanu [16] introduced the class $\mathcal{M}(\lambda)$ of λ-convex functions in $\mathbb{U}$ $(0 \leqq \lambda \leqq 1)$ (which are now referred to as Mocanu-convex functions) as follows:

$$\mathcal{M}(\lambda) := \left\{ f : f \in \mathcal{S} \,\text{and}\, \Re\left((1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left[1 + \frac{zf''(z)}{f'(z)}\right]\right) > 0 \qquad (z \in \mathbb{U}) \right\}.$$

In some case, the class $\mathcal{M}(\lambda)$ proclaims the class of starlike functions in $\mathbb{U}$. In some other case, the class $\mathcal{M}(\lambda)$ proclaims the class of convex functions in $\mathbb{U}$. In general, the class $\mathcal{M}(\lambda)$ of λ-convex functions in $\mathbb{U}$ determines the arithmetic bridge between starlikeness and convexity.

By using the geometric means, Lewandowski *et al.* [10] defined the class $\mathcal{L}(\mu)$ of μ-starlike functions in $\mathbb{U}$ $(0 \leqq \mu \leqq 1)$ as follows:

$$\mathcal{L}(\mu) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \Re\left(\left[\frac{zf'(z)}{f(z)} \right]^{\mu} \left[1 + \frac{zf''(z)}{f'(z)} \right]^{1-\mu} \right) > 0 \qquad (z \in \mathbb{U}) \right\}.$$

We note that the class $\mathcal{L}(\mu)$ of μ-starlike functions in $\mathbb{U}$ constitutes the geometric bridge between starlikeness and convexity.

Let φ be an analytic and univalent function with positive real part in $\mathbb{U}$ such that $\varphi(0) = 1$, $\varphi'(0) > 0$, φ maps the unit disk $\mathbb{U}$ onto a region starlike with respect to 1, and is symmetric with respect to the real axis. The Taylor-Maclaurin series expansion of such functions is of the form given by

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \qquad (B_1 > 0). \tag{3}$$

Throughout this paper, we assume that the function φ satisfies the above conditions (including *also* the above-stated condition $B_1 > 0$ on the coefficient B_1) unless it is stated otherwise.

By $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ we denote the following classes of functions:

$$\mathcal{S}^*(\varphi) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad \frac{zf'(z)}{f(z)} \prec \varphi(z) \qquad (z \in \mathbb{U}) \right\}$$

and

$$\mathcal{K}(\varphi) := \left\{ f : f \in \mathcal{S} \quad \text{and} \quad 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \qquad (z \in \mathbb{U}) \right\}.$$

The classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are the extensions of the classical sets of starlike and convex functions in $\mathbb{U}$. In such a form, these classes were defined and studied by Ma and Minda [13]. A function f is bi-starlike of Ma-Minda type in $\mathbb{U}$ or bi-convex of Ma-Minda type in $\mathbb{U}$ if both f and f^{-1} are, respectively, Ma-Minda starlike in $\mathbb{U}$ or Ma-Minda convex in $\mathbb{U}$. These classes are denoted, respectively, by $\mathcal{S}^*_{\Sigma}(\varphi)$ and $\mathcal{K}_{\Sigma}(\varphi)$ (see [1]).

We now introduce the function classes $\mathcal{M}(\lambda, \varphi)$ and $\mathcal{L}(\mu, \varphi)$ as follows:

$$\mathcal{M}(\lambda, \varphi) := \left\{ f : f \in \mathcal{S} \qquad \text{and} \right.$$
$$\left. (1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \qquad (0 \leqq \lambda \leqq 1;\ z \in \mathbb{U}) \right\} \tag{4}$$

and

$$\mathcal{L}(\mu, \varphi) := \left\{ f : f \in \mathcal{S} \qquad \text{and} \right.$$
$$\left. \left(\frac{zf'(z)}{f(z)}\right)^{\mu} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\mu} \prec \varphi(z) \qquad (0 \leqq \mu \leqq 1;\ z \in \mathbb{U}) \right\}. \tag{5}$$

The classes $\mathcal{M}(\lambda, \varphi)$ and $\mathcal{L}(\mu, \varphi)$ are, respectively, the classes of Mocanu-convex functions in $\mathbb{U}$ and μ-starlike functions in $\mathbb{U}$ of Ma-Minda type. A function f is bi-Mocanu convex in $\mathbb{U}$ of Ma-Minda type or bi-μ-starlike in $\mathbb{U}$ of Ma-Minda type if both f and f^{-1} are, respectively, Mocanu-convex in $\mathbb{U}$ or μ-starlike in $\mathbb{U}$. These function classes are denoted by $\mathcal{M}_\Sigma(\lambda,\varphi)$ and $\mathcal{L}_\Sigma(\mu,\varphi)$, respectively (see [1]). Motivated by some of the above-mentioned works, we define the following subclass of the bi-univalent function class Σ.

Definition 1. Let $h : \mathbb{U} \to \mathbb{C}$ be a convex univalent function in $\mathbb{U}$ such that

$$h(0) = 1, \quad h(\bar{z}) = \overline{h(z)} \qquad \text{and} \qquad \Re\{h(z)\} > 0 \qquad (z \in \mathbb{U}).$$

Then a function $f(z)$ given by (1) is said to be in the class $\mathcal{M}_\Sigma^\beta(\delta,\mu,\lambda,h)$ if the following conditions are satisfied:

$$f \in \Sigma,\ e^{i\beta}\left((1-\lambda)\left[\frac{zf'(z)}{f(z)}\right]^\delta + \lambda\left[\frac{zf'(z)}{f(z)}\right]^\mu\left[1+\frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) \prec h(z)\cos\beta + i\sin\beta \tag{6}$$

$$\left(z \in \mathbb{U};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2};\ 0 \leqq \lambda \leqq 1;\ 0 \leqq \mu \leqq 1;\ 1 \leqq \delta \leqq 2\right)$$

and

$$e^{i\beta}\left((1-\lambda)\left[\frac{wg'(w)}{g(w)}\right]^\delta + \lambda\left[\frac{wg'(w)}{g(w)}\right]^\mu\left[1+\frac{wg''(w)}{g'(w)}\right]^{1-\mu}\right) \prec h(w)\cos\beta + i\sin\beta \tag{7}$$

$$\left(z \in \mathbb{U};\ -\frac{\pi}{2} < \beta < \frac{\pi}{2};\ 0 \leqq \lambda \leqq 1;\ 0 \leqq \mu \leqq 1;\ 1 \leqq \delta \leqq 2\right),$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots \tag{8}$$

as the extension of f^{-1} to $\mathbb{U}$.

Remark 1. If we set

$$h(z) = h_{A,B}(z) := \frac{1+Az}{1+Bz} \qquad (-1 \leqq B < A \leqq 1;\ z \in \mathbb{U}) \tag{9}$$

in the class $\mathcal{M}_\Sigma^\beta(\delta,\mu,\lambda,h)$, we have the class which we denote, for convenience, by $\mathcal{M}_\Sigma^\beta(\delta,\mu,\lambda,h_{A,B})$.

Remark 2. Setting $\delta = 1$ and $\mu = 0$ in the above Definition, we have

$$\mathcal{M}_\Sigma^\beta(1,0,\lambda,h) =: \mathcal{M}_\Sigma^\beta(\lambda,h).$$

In particular, for $\beta = 0$, the class

$$\mathcal{M}_\Sigma^0(\lambda,h) =: \mathcal{M}_\Sigma(\lambda,h)$$

was introduced and studied by Ali *et al.* [1].

Remark 3. By setting $\delta = 1$ and $\lambda = 1$ in the above Definition, we have

$$\mathcal{M}_\Sigma^\beta(1,\mu,1,h) =: \mathcal{L}_\Sigma^\beta(\mu,h).$$

In particular, for $\beta = 0$, the class

$$\mathcal{L}_\Sigma^0(\mu,h) =: \mathcal{L}_\Sigma(\mu,h)$$

was introduced and studied by Ali *et al.* [1]. **Remark 4.** If we take

$$h(z) = h_\alpha(z) := \frac{1+(1-2\alpha)z}{1-z} \qquad (0 \leqq \alpha < 1;\ z \in \mathbb{U}) \tag{10}$$

in the class $\mathcal{M}_{\Sigma}^{\beta}(\delta,\mu,\lambda,h)$, we are led to the class which we denote, for convenience, by $\mathcal{M}_{\Sigma}^{\beta}(\delta,\mu,\lambda,h_{\alpha})$. In particular, for $\delta=1$, $\mu=0$ and $\beta=0$, the class

$$\mathcal{M}_{\Sigma}^{0}(1,0,\lambda,h_{\alpha}) =: \mathcal{M}_{\Sigma}(\lambda,h_{\alpha})$$

was introduced and studied by Li and Wang [12].

Remark 5. Upon replacing $h(z)$ by $h_{\alpha}(z)$ as given by (10) in Remark 3, we have

$$\mathcal{L}_{\Sigma}^{\beta}(\mu,h) =: \mathcal{L}_{\Sigma}^{\beta}(\mu,h_{\alpha}).$$

Remark 6. Putting $\mu=1$ in the class

$$\mathcal{M}_{\Sigma}^{\beta}(1,\mu,1,h) =: \mathcal{L}_{\Sigma}^{\beta}(\mu,h),$$

we have

$$\mathcal{M}_{\Sigma}^{\beta}(1,1,1,h) = \mathcal{L}_{\Sigma}^{\beta}(1,h) =: \mathcal{S}_{\Sigma}^{*}(\beta,h).$$

The class $\mathcal{S}_{\Sigma}^{*}(\beta,h)$ was introduced by Orhan *et al.* [18]. In particular, for $\beta=0$, the class $\mathcal{S}_{\Sigma}^{*}(0,h) =: \mathcal{S}_{\Sigma}^{*}(h)$ was considered by Ali *et al.* [1] and Srivastava *et al.* [23]. Moreover, in the special case when $\beta=0$ and $h(z)$ is replaced by $h_{\alpha}(z)$ given by (10), the class

$$\mathcal{S}_{\Sigma}^{*}(0,h_{\alpha}) =: \mathcal{S}_{\Sigma}^{*}(h_{\alpha})$$

was introduced by Brannan and Taha [3] and studied by Bulut [2], Caglar *et al.* [4], Li and Wang [12], Magesh and Yamini [15], Magesh *et al.* [14], and others.

Remark 7. Taking $\mu=0$ in the class

$$\mathcal{M}_{\Sigma}^{\beta}(1,\mu,1,h) =: \mathcal{L}_{\Sigma}^{\beta}(\mu,h),$$

we have

$$\mathcal{M}_{\Sigma}^{\beta}(1,0,1,h) = \mathcal{L}_{\Sigma}^{\beta}(0,h) =: \mathcal{K}_{\Sigma}(\beta,h).$$

The class $\mathcal{K}_{\Sigma}^{*}(\beta,h)$ was introduced by Orhan *et al.* [18]. In its particular case when $\beta=0$, the class $\mathcal{K}_{\Sigma}(0,h) =: \mathcal{K}_{\Sigma}(h)$ was considered by Ali *et al.* [1]. Furthermore, in the particular case when $\beta=0$ and $h(z)$ is replaced by $h_{\alpha}(z)$ given by (10), the class $\mathcal{K}_{\Sigma}(0,h_{\alpha}) =: \mathcal{K}_{\Sigma}(h_{\alpha})$ was introduced by Brannan and Taha [3] and studied by Li and Wang [12], Magesh and Yamini [15], and others.

In our investigation of the estimates for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the above-defined general bi-univalent function class $\mathcal{M}_{\Sigma}^{\beta}(\delta,\mu,\lambda,h)$, which indeed provides a bridge between the classes of bi-λ-convex functions in $\mathbb{U}$ and bi-μ-starlike functions in $\mathbb{U}$, we shall need each of the following lemmas.

Lemma 1. (see [20]). *If $p \in \mathcal{P}$, then $|p_j| \leqq 2$ for each $j \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{U}$, for which*

$$\Re\{p(z)\} > 0, \qquad (z \in \mathbb{U}),$$

where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}),$$

$\mathbb{N}$ being the set of positive integers.

Lemma 2. (see [21] and [28]). *Let the function $\varphi(z)$ given by*

$$\varphi(z) = \sum_{n=1}^{\infty} B_n z^n \qquad (z \in \mathbb{U}) \tag{11}$$

be convex in $\mathbb{U}$. *Suppose also that the function* $h(z)$ *given by*

$$h(z) = \sum_{n=1}^{\infty} h_n z^n \qquad (z \in \mathbb{U})$$

is holomorphic in $\mathbb{U}$. *If*

$$h(z) \prec \varphi(z) \qquad (z \in \mathbb{U}),$$

then

$$|h_n| \leqq B_1 \qquad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}). \tag{12}$$

2. A Set of Main Results

In this section, we find the estimates for the coefficients $|a_2|$ and $|a_3|$ for functions in the general bi-univalent function class $\mathcal{M}_{\Sigma}^{\beta}(\delta, \mu, \lambda, h)$.

Theorem 1. *Let the function* $f(z)$ *given by* (1) *be in the class* $\mathcal{M}_{\Sigma}^{\beta}(\delta, \mu, \lambda, h)$. *Also let*

$$-\frac{\pi}{2} < \beta < \frac{\pi}{2}, \quad 0 \leqq \lambda < 1, \quad 0 \leqq \mu \leqq 1 \qquad \text{and} \qquad 1 \leqq \delta \leqq 2.$$

Then

$$|a_2| \leqq \sqrt{\frac{2B_1 \cos\beta}{(1-\lambda)\delta(1+\delta) + \lambda(\mu^2 - 3\mu + 4)}} \tag{13}$$

and

$$|a_3| \leqq \frac{B_1^2 \cos^2\beta}{[(1-\lambda)\delta + \lambda(2-\mu)]^2} + \frac{B_1 \cos\beta}{2(1-\lambda)\delta + 2\lambda(3-2\mu)}, \tag{14}$$

where the coefficient B_1 *is given as in* (3).

Proof. It follows from (6) and (7) that

$$e^{i\beta}\left((1-\lambda)\left[\frac{zf'(z)}{f(z)}\right]^{\delta} + \lambda\left[\frac{zf'(z)}{f(z)}\right]^{\mu}\left[1 + \frac{zf''(z)}{f'(z)}\right]^{1-\mu}\right) = p(z)\cos\beta + i\sin\beta \tag{15}$$

and

$$e^{i\beta}\left((1-\lambda)\left[\frac{wg'(w)}{g(w)}\right]^{\delta} + \lambda\left[\frac{wg'(w)}{g(w)}\right]^{\mu}\left[1 + \frac{wg''(w)}{g'(w)}\right]^{1-\mu}\right) = p(w)\cos\beta + i\sin\beta, \tag{16}$$

where

$$p(z) \prec h(z) \quad (z \in \mathbb{U}) \qquad \text{and} \qquad q(w) \prec h(w) \quad (w \in \mathbb{U})$$

have the following forms:

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathbb{U}) \tag{17}$$

and

$$q(z) = 1 + q_1 w + q_2 w^2 + \cdots \qquad (w \in \mathbb{U}). \tag{18}$$

Equating the coefficients in (15) and (16), we get

$$e^{i\beta}[(1-\lambda)\delta + \lambda(2-\mu)]a_2 = p_1 \cos\beta, \tag{19}$$

$$e^{i\beta}\left[2[(1-\lambda)\delta + \lambda(3-2\mu)]a_3 - [(1-\lambda)\delta(3-\delta) - \lambda(\mu^2 + 5\mu - 8)]\frac{a_2^2}{2}\right] = p_2\cos\beta, \tag{20}$$

$$-e^{i\beta}[(1-\lambda)\delta+\lambda(2-\mu)]a_2=q_1\cos\beta \tag{21}$$

and

$$e^{i\beta}\left[[(1-\lambda)\delta(5+\delta)+\lambda(\mu^2-11\mu+16)]\frac{a_2^2}{2}-2[(1-\lambda)\delta+\lambda(3-2\mu)]a_3\right]=q_2\cos\beta. \tag{22}$$

From (19) and (21), we find that

$$p_1=-q_1 \tag{23}$$

and

$$2e^{i2\beta}[(1-\lambda)\delta+\lambda(2-\mu)]^2a_2^2=(p_1^2+q_1^2)\cos^2\beta. \tag{24}$$

Also, from (20) and (22), we obtain

$$a_2^2=\frac{e^{-i\beta}(p_2+q_2)\cos\beta}{(1-\lambda)\delta(1+\delta)+\lambda(\mu^2-3\mu+4)}. \tag{25}$$

Since $p,q\subset h(\mathbb{U})$, by applying Lemma 2, we immediately have

$$|p_m|=\left|\frac{p^{(m)}(0)}{m!}\right|\leqq B_1 \qquad (m\in\mathbb{N}) \tag{26}$$

and

$$|q_m|=\left|\frac{q^{(m)}(0)}{m!}\right|\leqq B_1 \qquad (m\in\mathbb{N}). \tag{27}$$

Now, if we apply (26), (27) and Lemma 2 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_2|\leqq\sqrt{\frac{2B_1\cos\beta}{(1-\lambda)\delta(1+\delta)+\lambda(\mu^2-3\mu+4)}},$$

which gives the bound on $|a_2|$ as asserted in (13).

Next, in order to find the bound on $|a_3|$, by subtracting (22) from (20), we get

$$4[(1-\lambda)\delta+\lambda(3-2\mu)]a_3-4[(1-\lambda)\delta+\lambda(3-2\mu)]a_2^2=e^{-i\beta}(p_2-q_2)\cos\beta. \tag{28}$$

It follows from (24) and (28) that

$$a_3=\frac{(p_1^2+q_1^2)e^{-i2\beta}\cos^2\beta}{2[(1-\lambda)\delta+\lambda(2-\mu)]^2}+\frac{e^{-i\beta}(p_2-q_2)\cos\beta}{4[(1-\lambda)\delta+\lambda(3-2\mu)]}. \tag{29}$$

Applying (26), (27) and Lemma 2 once again for the coefficients p_1, p_2, q_1 and q_2, we readily obtain

$$|a_3|\leqq\frac{B_1^2\cos^2\beta}{[(1-\lambda)\delta+\lambda(2-\mu)]^2}+\frac{B_1\cos\beta}{2(1-\lambda)\delta+2\lambda(3-2\mu)},$$

which evidently completes the proof of Theorem 1. □

In view of Remarks 1 and 4, if we replace $h(z)$ by $h_{A,B}(z)$ given by (9) and $h_\alpha(z)$ given by (10) in Theorem 1, we can easily deduce Theorems 2 and 3, respectively, which we choose to merely state here *without* proof.

Theorem 2. *Let the function $f(z)$ given by (1) be in the class $\mathcal{M}_\Sigma^\beta(\delta,\mu,\lambda,h_{A,B})$. Then*

$$|a_2|\leqq\sqrt{\frac{2(A-B)\cos\beta}{(1-\lambda)\delta(1+\delta)+\lambda(\mu^2-3\mu+4)}} \tag{30}$$

and

$$|a_3| \leqq \frac{(A-B)^2\cos^2\beta}{[(1-\lambda)\delta+\lambda(2-\mu)]^2} + \frac{(A-B)\cos\beta}{2(1-\lambda)\delta+2\lambda(3-2\mu)}. \tag{31}$$

Theorem 3. *Let the function $f(z)$ given by (1) be in the class $\mathcal{M}_{\Sigma}^{\beta}(\delta,\mu,\lambda,h_\alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{4(1-\alpha)\cos\beta}{(1-\lambda)\delta(1+\delta)+\lambda(\mu^2-3\mu+4)}} \tag{32}$$

and

$$|a_3| \leqq \frac{4(1-\alpha)^2\cos^2\beta}{[(1-\lambda)\delta+\lambda(2-\mu)]^2} + \frac{2(1-\alpha)\cos\beta}{2(1-\lambda)\delta+2\lambda(3-2\mu)}. \tag{33}$$

Remark 8. For $\delta = 1$, $\mu = 0$ and $\beta = 0$, the estimates for the coefficients $|a_2|$ and $|a_3|$ given by Theorem 3 are reduced at once to the estimates obtained earlier by Li and Wang [12, Theorem 3.2].

3. Corollaries and Consequences

In view of Remark 2, we have the following corollaries.

Corollary 1. *Let the function $f(z)$ given by (1) be in the class $\mathcal{M}_{\Sigma}^{\beta}(\lambda,h)$. Then*

$$|a_2| \leqq \sqrt{\frac{B_1\cos\beta}{1+\lambda}} \tag{34}$$

and

$$|a_3| \leqq \frac{B_1^2\cos^2\beta}{(1+\lambda)^2} + \frac{B_1\cos\beta}{2+4\lambda}. \tag{35}$$

Corollary 2. *Let the function $f(z)$ given by (1) be in the class $\mathcal{M}_{\Sigma}(\lambda,h)$. Then*

$$|a_2| \leqq \sqrt{\frac{B_1}{1+\lambda}} \tag{36}$$

and

$$|a_3| \leqq \frac{B_1^2}{(1+\lambda)^2} + \frac{B_1}{2+4\lambda}. \tag{37}$$

Remark 9. The estimates in Corollary **??** provide improvement over the estimates obtained by Ali *et al.* [1, Theorem 2.3].

In light of Remarks 2 to 5, we have following corollaries.

Corollary 3. *Let the function $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{\beta}(\mu,h)$. Then*

$$|a_2| \leqq \sqrt{\frac{2B_1\cos\beta}{\mu^2-3\mu+4}} \tag{38}$$

and

$$|a_3| \leqq \frac{B_1^2\cos^2\beta}{(2-\mu)^2} + \frac{B_1\cos\beta}{2(3-2\mu)}. \tag{39}$$

Corollary 4. *Let the function $f(z)$ given by (1) be in the class $\mathcal{L}_{\Sigma}^{\beta}(\mu,h_\alpha)$. Then*

$$|a_2| \leqq \sqrt{\frac{4(1-\alpha)\cos\beta}{\mu^2-3\mu+4}} \tag{40}$$

and

$$|a_3| \leqq \frac{4(1-\alpha)^2\cos^2\beta}{(2-\mu)^2} + \frac{(1-\alpha)\cos\beta}{3-2\mu}. \tag{41}$$

Corollary 5. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{L}_\Sigma(\mu, h)$. Then*

$$|a_2| \leqq \sqrt{\frac{2B_1}{\mu^2 - 3\mu + 4}} \tag{42}$$

and

$$|a_3| \leqq \frac{B_1^2}{(2-\mu)^2} + \frac{B_1}{2(3-2\mu)}. \tag{43}$$

Remark 10. The estimates in Corollary 5 provide improvement over the estimates derived by Ali *et al.* [1, Theorem 2.4]

In view of Remarks 6 and 7, we have the following corollaries. **Corollary 6.** *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{S}^*_\Sigma(\beta, h)$. Then*

$$|a_2| \leqq \sqrt{B_1 \cos\beta} \tag{44}$$

and

$$|a_3| \leqq B_1^2 \cos^2\beta + \frac{B_1 \cos\beta}{2}. \tag{45}$$

Corollary 7. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{S}^*_\Sigma(h)$. Then*

$$|a_2| \leqq \sqrt{B_1} \tag{46}$$

and

$$|a_3| \leqq B_1^2 + \frac{B_1}{2}. \tag{47}$$

Remark 11. For the function $h(z)$ replaced by $h_\alpha(z)$ as given in (10), the estimates in Corollary 7 reduce to a result proven earlier by Li and Wang [12, Corollary 3.3].

Corollary 8. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{K}_\Sigma(\beta, h)$. Then*

$$|a_2| \leqq \sqrt{\frac{B_1 \cos\beta}{2}} \tag{48}$$

and

$$|a_3| \leqq \frac{B_1^2 \cos^2\beta}{4} + \frac{B_1 \cos\beta}{6}. \tag{49}$$

Corollary 9. *Let the function $f(z)$ given by* (1) *be in the class $\mathcal{K}_\Sigma(h)$. Then*

$$|a_2| \leqq \sqrt{\frac{B_1}{2}} \tag{50}$$

and

$$|a_3| \leqq \frac{B_1^2}{4} + \frac{B_1}{6}. \tag{51}$$

Remark 12. In the special case when we replace the function $h(z)$ by $h_\alpha(z)$ given by (10), the estimates in Corollary 9 would reduce to a known result in [3, Theorem 4.1].

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