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# THE $k$-ANALOGUE OF SOME INEQUALITIES FOR THE GAMMA FUNCTION

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ABSTRACT. In this paper, we present and prove the $k$-analogue of the Inequalities obtained by A. Sh. Shabani [3] and N. V. Vinh, N. P. N. Ngoc [4]. We also present some new results involving the $k$-analogue of the digamma function.

## 1. INTRODUCTION

We begin by recalling some definitions related to the Gamma function.

The classical Euler's Gamma function, $\Gamma(t)$ is defined as

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\,dx, \qquad t > 0. \tag{1}$$

The digamma function, $\psi(t)$ also known as the logarithmic derivative of the Gamma function is defined as

$$\psi(t) = \frac{d}{dt}\ln(\Gamma(t)) = \frac{\Gamma'(t)}{\Gamma(t)}, \qquad t > 0. \tag{2}$$

The $k$-analogue of the Gamma Function $\Gamma_k(t)$ is defined as

$$\Gamma_k(t) = \int_0^\infty e^{-\frac{x^k}{k}} x^{t-1}\,dx, \quad k > 0, \quad t > 0. \tag{3}$$

For several properties and other representation of $\Gamma_k(t)$, see [1].

Similarly, the $k$-analogue of $\psi(t)$ is defined as follows. (See [2])

$$\psi_k(t) = \frac{d}{dt}\ln(\Gamma_k(t)) = \frac{\Gamma_k'(t)}{\Gamma_k(t)}, \quad k > 0, \quad t > 0. \tag{4}$$

and

$$\lim_{k\to 1}\Gamma_k(t) = \Gamma(t), \qquad \lim_{k\to 1}\psi_k(t) = \psi(t) \tag{5}$$

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In an effort to generalize some ealier results, A. S. Shabani [3] established the following.

$$\frac{\Gamma(a+b)^r}{\Gamma(\alpha+\beta)^q} \le \frac{\Gamma(a+bt)^r}{\Gamma(\alpha+\beta t)^q} \le \frac{\Gamma(a)^r}{\Gamma(\alpha)^q}, \qquad t \in [0,1] \tag{6}$$

where $a, b, r, \alpha, \beta, q$ are positive real numbers such that $a + bt > 0$, $\alpha + \beta t > 0$, $a + bt \le \alpha + \beta t$, $0 < br \le \beta q$ and $\psi(a + bt) > 0$ or $\psi(\alpha + \beta t) > 0$.

Also, by using the Dirichlet's integral, N. V. Vinh and N. P. N. Ngoc [4] proved the following results.

$$\frac{\prod_{i=1}^n \Gamma(1+a_i)}{\Gamma(b+\sum_{i=1}^n a_i)} \le \frac{\prod_{i=1}^n \Gamma(1+a_i t)}{\Gamma(b+\sum_{i=1}^n a_i t)} \le \frac{1}{\Gamma(b)} \tag{7}$$

where $t \in [0, 1]$, $b \ge 1$, $a_i > 0$, $n \in \aleph$.

Our aim in this paper is to establish and prove the $k$-analogues of inequalities (6) and (7) presented in [3] and [4] respectively. Further, we present some new results involving the $k$-digamma function.

## 2. Preliminaries

Here, we give some Lemmas that will be used to aid the proofs of our main results.

**Lemma 2.1.** *The function $\psi_k(t)$ as defined by inequality (4) has the following series representation.*

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \left( \frac{1}{nk} - \frac{1}{t+nk} \right) \tag{8}$$

*where $\gamma$ is the Euler-Mascheroni's constant.*

*Proof.* In [1] and [2], we have the following representation of $\Gamma_k(t)$

$$\frac{1}{\Gamma_k(t)} = tk^{-\frac{t}{k}} e^{\frac{t}{k}\gamma} \prod_{n=1}^{\infty} \left[ \left(1 + \frac{t}{nk}\right) e^{-\frac{t}{nk}} \right] \tag{9}$$

Taking the logarithmic derivative of (9) gives

$$-\ln \Gamma_k(t) = \ln t - \frac{t}{k}\ln k + \frac{t}{k}\gamma + \sum_{n=1}^{\infty} \left[ \ln\left(1 + \frac{t}{nk}\right) - \frac{t}{nk} \right]$$

$$-\frac{d}{dt}\ln(\Gamma_k(t)) = \frac{1}{t} - \frac{\ln k}{k} + \frac{\gamma}{k} + \sum_{n=1}^{\infty} \left( \frac{1}{t+nk} - \frac{1}{nk} \right)$$

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{t}{nk(nk+t)}.$$

**Lemma 2.2.** *Let $s > 0$, $t > 0$ with $s \le t$, then*

$$\psi_k(s) \le \psi_k(t). \tag{10}$$

*Proof.* From (8), we have the following.

$$\begin{aligned}\psi_k(s)-\psi_k(t) &= \frac{1}{t}-\frac{1}{s}+\sum_{n=1}^{\infty}\left(\frac{1}{nk}-\frac{1}{s+nk}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{nk}-\frac{1}{t+nk}\right)\\ &= \frac{s-t}{st}+\sum_{n=1}^{\infty}\left(\frac{1}{t+nk}-\frac{1}{s+nk}\right)\\ &= \frac{s-t}{st}+\sum_{n=1}^{\infty}\frac{(s-t)}{(s+nk)(t+nk)}\leq 0\end{aligned}$$

Hence the proof.

By differentiating (8), we have the following representation.

$$\psi_k'(t)=\sum_{n=0}^{\infty}\frac{1}{(nk+t)^2},\quad k>0,\quad t>0. \tag{11}$$

**Lemma 2.3.** *Let $s>0$, $t>0$ with $s\leq t$, then*

$$\psi_k'(s)\geq\psi_k'(t). \tag{12}$$

*Proof.* From (11) we have,

$$\begin{aligned}\psi_k'(s)-\psi_k'(t) &= \sum_{n=0}^{\infty}\frac{1}{(nk+s)^2}-\sum_{n=0}^{\infty}\frac{1}{(nk+t)^2}\\ &= \sum_{n=0}^{\infty}\left[\frac{1}{(nk+s)^2}-\frac{1}{(nk+t)^2}\right]\\ &= \sum_{n=0}^{\infty}\frac{2nk(t-s)+(t^2-s^2)}{(nk+s)^2(nk+t)^2}\geq 0.\end{aligned}$$

ending the proof.

**Lemma 2.4.** *Let $a,b,\alpha,\beta$ be real numbers such that $a+bt>0$, $\alpha+\beta t>0$. Then $a+bt\leq\alpha+\beta t$ implies $\psi_k(a+bt)\leq\psi_k(\alpha+\beta t)$.*

*Proof.* A direct consequence of Lemma 2.2.

## 3. Results and Discussion

Now we state and prove the results of the paper. We begin with a Lemma.

**Lemma 3.1.** *Let $a,b,\alpha,\beta,r,q$, be real numbers such that $a+bt>0$, $\alpha+\beta t>0$, $a+bt\leq\alpha+\beta t$ and $q\beta\geq rb$. If $\psi_k(a+bt)>0$ or $\psi_k(\alpha+\beta t)>0$, then*

$$rb\psi_k(a+bt)-q\beta\psi_k(\alpha+\beta t)\leq 0.$$

*Proof.* Let $\psi_k(a+bt)>0$, $\psi_k(\alpha+\beta t)>0$. Multiplying both sides of $q\beta\geq rb$ by $\psi_k(\alpha+\beta t)$ yields

$$q\beta\psi_k(\alpha+\beta t)\geq rb\psi_k(\alpha+\beta t)\geq rb\psi_k(a+bt)\quad\text{(By Lemma 2.4)}.$$

Thus

$$rb\psi_k(a+bt)-q\beta\psi_k(\alpha+\beta t)\leq 0.$$

**Lemma 3.2.** *Let $t \in [0, \infty)$, $a_i > 0$, $b \geq 1$, $n \in \aleph$ then,*
$1 + a_i t \leq \beta + \sum_{i=1}^{n} a_i t$ *implies* $\psi_k(1 + a_i t) \leq \psi_k(\beta + \sum_{i=1}^{n} a_i t)$.

*Proof.* A direct consequence of Lemma 2.2.

**Theorem 3.3.** *Define a function $\Omega$ by*

$$\Omega(t) = \frac{\Gamma_k(a+bt)^r}{\Gamma_k(\alpha+\beta t)^q}, \quad t \in [0, \infty) \tag{13}$$

*where $a, b, r, \alpha, \beta, q$ are positive real numbers such that $a + bt > 0$, $\alpha + \beta t > 0$, $a + bt \leq \alpha + \beta t$, $0 < br \leq \beta q$ and $\psi_k(a + bt) > 0$ or $\psi_k(\alpha + \beta t) > 0$ then $\Omega$ is decreasing and for every $t \in [0, 1]$, the following inequalities hold.*

$$\frac{\Gamma_k(a+b)^r}{\Gamma_k(\alpha+\beta)^q} \leq \frac{\Gamma_k(a+bt)^r}{\Gamma_k(\alpha+\beta t)^q} \leq \frac{\Gamma_k(a)^r}{\Gamma_k(\alpha)^q}. \tag{14}$$

*Proof.* Let $u(t) = \ln \Omega(t)$ for every $t \in [0, \infty)$. Then,

$$\begin{aligned} u(t) &= \ln \frac{\Gamma_k(a+bt)^r}{\Gamma_k(\alpha+\beta t)^q} \\ &= r \ln \Gamma_k(a+bt) - q \ln \Gamma_k(\alpha+\beta t) \end{aligned}$$

Then,

$$\begin{aligned} u'(t) &= br \frac{\Gamma'_k(a+bt)}{\Gamma_k(a+bt)} - \beta q \frac{\Gamma'_k(\alpha+\beta t)}{\Gamma_k(\alpha+\beta t)} \\ &= br\psi_k(a+bt) - \beta q \psi_k(\alpha+\beta t) \leq 0. \quad \text{(by Lemma 3.1).} \end{aligned}$$

That implies $u$ is decreasing on $t \in [0, \infty)$. Hence, $\Omega$ is decreasing for every $t \in [0, \infty)$. Then for every $t \in [0, 1]$ we have,

$$\Omega(1) \leq \Omega(t) \leq \Omega(0) \qquad \text{yielding,}$$

$$\frac{\Gamma_k(a+b)^r}{\Gamma_k(\alpha+\beta)^q} \leq \frac{\Gamma_k(a+bt)^r}{\Gamma_k(\alpha+\beta t)^q} \leq \frac{\Gamma_k(a)^r}{\Gamma_k(\alpha)^q}.$$

**Corollary 3.4.** *If $t \in (1, \infty)$, then the following inequality holds.*

$$\frac{\Gamma_k(a+bt)^r}{\Gamma_k(\alpha+\beta t)^q} \leq \frac{\Gamma_k(a+b)^r}{\Gamma_k(\alpha+\beta)^q}.$$

*Proof.* If $t \in (1, \infty)$, then we have $\Omega(t) \leq \Omega(1)$ yielding the result.

**Theorem 3.5.** *Define a function $\Phi$ by*

$$\Phi(t) = \frac{\prod_{i=1}^{n} \Gamma_k(1 + a_i t)}{\Gamma_k(b + \sum_{i=1}^{n} a_i t)}, \quad t \in [0, \infty) \tag{15}$$

*where $b \geq 1$, $a_i > 0$, $n \in \aleph$. Then $\Phi$ is decreasing and for every $t \in [0, 1]$, the following inequalities hold.*

$$\frac{\prod_{i=1}^{n}\Gamma_k(1+a_i)}{\Gamma_k(b+\sum_{i=1}^{n}a_i)} \leq \frac{\prod_{i=1}^{n}\Gamma_k(1+a_it)}{\Gamma_k(b+\sum_{i=1}^{n}a_it)} \leq \frac{1}{\Gamma_k(b)}. \tag{16}$$

*Proof.* Let $v(t) = \ln\Phi(t)$ for every $t \in [0,\infty)$. Then,

$$\begin{aligned} v(t) &= \ln\frac{\prod_{i=1}^{n}\Gamma_k(1+a_it)}{\Gamma_k(b+\sum_{i=1}^{n}a_it)} \\ &= \ln\prod_{i=1}^{n}\Gamma_k(1+a_it) - \ln\Gamma_k(b+\sum_{i=1}^{n}a_it) \end{aligned}$$

Then,

$$\begin{aligned} v'(t) &= \sum_{i=1}^{n}\left(a_i\frac{\Gamma_k'(1+a_it)}{\Gamma_k(1+a_it)}\right) - \left(\sum_{i=1}^{n}a_i\right)\frac{\Gamma_k'(b+\sum_{i=1}^{n}a_it)}{\Gamma_k(b+\sum_{i=1}^{n}a_it)} \\ &= \sum_{i=1}^{n}\left(a_i\psi_k(1+a_it)\right) - \left(\sum_{i=1}^{n}a_i\right)\psi_k(b+\sum_{i=1}^{n}a_it) \\ &= \sum_{i=1}^{n}a_i\left[\psi_k(1+a_it) - \psi_k(b+\sum_{i=1}^{n}a_it)\right] \leq 0. \quad \text{(by Lemma 3.2)}. \end{aligned}$$

That implies $v$ is decreasing on $t \in [0,\infty)$. Hence, $\Phi$ is decreasing for every $t \in [0,\infty)$. Then for every $t \in [0,1]$ we have,

$$\Phi(1) \leq \Phi(t) \leq \Phi(0) \qquad \text{yielding,}$$

$$\frac{\prod_{i=1}^{n}\Gamma_k(1+a_i)}{\Gamma_k(b+\sum_{i=1}^{n}a_i)} \leq \frac{\prod_{i=1}^{n}\Gamma_k(1+a_it)}{\Gamma_k(b+\sum_{i=1}^{n}a_it)} \leq \frac{1}{\Gamma_k(b)}.$$

**Corollary 3.6.** *If $t \in (1,\infty)$, then the following inequality holds.*

$$\frac{\prod_{i=1}^{n}\Gamma_k(1+a_it)}{\Gamma_k(b+\sum_{i=1}^{n}a_it)} \leq \frac{\prod_{i=1}^{n}\Gamma_k(1+a_i)}{\Gamma_k(b+\sum_{i=1}^{n}a_i)}$$

*Proof.* If $t \in (1,\infty)$, then we have $\Phi(t) \leq \Phi(1)$ giving the result.

**Theorem 3.7.** *Define a function $H(t)$ by*

$$H(t) = \frac{[\psi_k(a+bt)]^\alpha}{[\psi_k(c+dt)]^\beta}, \quad t \in [0,\infty),\ k > 0 \tag{17}$$

*where $a$, $b$, $c$, $d$, $\alpha$, $\beta$ are positive real numbers such that $a \leq c$, $b \leq d$, $\beta d \leq \alpha b$, $0 < a+bt \leq c+dt$, $\psi_k(a+bt) > 0$ and $\psi_k(c+dt) > 0$. Then $H(t)$ is increasing on $t \in [0,\infty)$ and the inequalities*

$$\frac{[\psi_k(a)]^\alpha}{[\psi_k(c)]^\beta} \leq \frac{[\psi_k(a+bt)]^\alpha}{[\psi_k(c+dt)]^\beta} \leq \frac{[\psi_k(a+b)]^\alpha}{[\psi_k(c+d)]^\beta} \tag{18}$$

*holds for every $t \in [0,1]$.*

*Proof.* Let $w(t) = \ln H(t)$ for every $t \in [0, \infty)$. Then,

$$w(t) = \ln \frac{[\psi_k(a+bt)]^\alpha}{[\psi_k(c+dt)]^\beta} = \alpha \ln \psi_k(a+bt) - \beta \ln \psi_k(c+dt)$$

and

$$\begin{aligned} w'(t) &= \alpha b \frac{\psi_k'(a+bt)}{\psi_k(a+bt)} - \beta d \frac{\psi_k'(c+dt)}{\psi_k(c+dt)} \\ &= \frac{\alpha b \psi_k'(a+bt)\psi_k(c+dt) - \beta d \psi_k'(c+dt)\psi_k(a+bt)}{\psi_k(a+bt)\psi_k(c+dt)}. \end{aligned}$$

Since $0 < a + bt \le c + dt$, then by Lemmas (2.2) and (2.3) we have,
$\psi_k(a+bt) \le \psi_k(c+dt)$ and $\psi_k'(a+bt) \ge \psi_k'(c+dt)$. Then that implies;
$\psi_k(c+dt)\psi_k'(a+bt) \ge \psi_k(c+dt)\psi_k'(c+dt) \ge \psi_k(a+bt)\psi_k'(c+dt)$.
Further, $\alpha b \ge \beta d$ implies;
$\alpha b \psi_k(c+dt)\psi_k'(a+bt) \ge \alpha b \psi_k(a+bt)\psi_k'(c+dt) \ge \beta d \psi_k(a+bt)\psi_k'(c+dt)$. Hence,
$\alpha b \psi_k(c+dt)\psi_k'(a+bt) - \beta d \psi_k(a+bt)\psi_k'(c+dt) \ge 0$. Therefore $w'(t) \ge 0$.
That implies $w(t)$ and $H(t)$ are increasing on $t \in [0, \infty)$. Thus, for every $t \in [0, 1]$ we have,

$$H(0) \le H(t) \le H(1)$$

yielding the result.

*Remark* 3.8. If we let $a \ge c$, $b \ge d$, $\beta d \ge \alpha b$ and $a + bt \ge c + dt > 0$ in Theorem 3.7, then the function $H(t)$ is decreasing and the inequality (18) is reversed.

## 4. Conclusion

We have proved that inequalities (6) and (7) also hold for the k-analogue of the gamma function as shown by inequalities (14) and (16). In addition, results involving the $k$-analogue of the digamma function, $\psi_k$ are also proved and thus shown by inequalities (18).

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