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# DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATIONS OF CERTAIN MEROMORPHIC FUNCTIONS ASSOCIATED WITH A FAMILIY OF MULTIPLIER TRANSFORMATIONS

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Abstract. Differential subordinations and superordinations results are obtained for certain meromorphic functions in the punctured unit disk which are associated with certain multiplier transformation and These results are obtained by investigating appropriate classes of admissible functions. Sandwich-type results are also obtained.

## 1. Introduction

Let $H(\mathbb{U})$ denotes the class of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, n]$ denotes the subclass of the functions $f \in H(\mathbb{U})$ of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots \quad (a \in \mathbb{C}),$$

with $H[1, 1] \equiv H$. If $f(z), F(z) \in H(\mathbb{U})$, we say that $f\ (z)$ is subordinate to $F(z)$, or $F(z)$ is superordinate to $f(z)$, if there exists a Schwarz function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = F(w(z))$. In such a case we write $f \prec F$ or $f(z) \prec F(z)$ $(z \in \mathbb{U})$. If $F(z)$ is univalent in $\mathbb{U}$, then the following equivalence relationship holds true

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

bf Definition 1. [[16], Definition 2.2a, p. 27] Denote by $Q$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$, where

$$E(q) = \{\varsigma \in \partial\mathbb{U} : \lim_{z \to \varsigma} q(z) = \infty\},$$

and are such that $q'(\varsigma) \neq 0$ for $\varsigma \in \partial\mathbb{U} \backslash E(q)$. Further let the subclass of $Q$ for which $q(0) = a$ be denoted by $Q(a)$,and $Q(1) = Q_1$.
Let $\sum$ denote the class of functions of the form:

$$f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k, \tag{1}$$

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which are analytic in the punctured disc $\mathbb{U}^* = \{z : z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$, with a simple pole at the origin.
Let $f, g \in \sum$, where $f$ is given by 1 and $g$ is given by

$$g(z) = \frac{1}{z} + \sum_{k=0}^{\infty} b_k z^k.$$

Then the Hadamard product (or convolution) $f * g$ of the functions $f$ and $g$ is defined by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=0}^{\infty} a_k b_k \, z^k := (g * f)(z).$$

In recent years, several families of integral operators and differential operators were introduced using Hadamard product (or convolution). For example, we choose to mention the Ruscheweyh derivative [20], the Carlson-Shaffer operator [7], the Dziok-Srivastava operator [10], the Noor integral operator [19] and so on (see[[11], [12], [8], [18]]).
For complex parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ $(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = 0, -1, -2, ...;$ $j = 1, ..., s)$, let

$$\phi(\alpha_1, \alpha_2, ..., \alpha_q, \beta_1, \beta_2, ..., \beta_s; z) := \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\alpha_1)_{k+1}(\alpha_2)_{k+1}...(\alpha_q)_{k+1}}{(\beta_1)_{k+1}(\beta_2)_{k+1}...(\beta s)_{k+1}} \frac{a_k z^k}{(k+1)!}$$

$(q \leq s+1; q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathbb{U})$,
where $(x)_k$ is the pochhammer symbol defined by

$$(x)_k = \begin{cases} 1 & if \;\; k = 0 \\ x(x+1)(x+2)...(x+k-1) & if \;\; k \in \mathbb{N}_0 = \{1, 2, ...\}. \end{cases}$$

Motivated by the work of Cho and Noor [9], Selvaraj and Karthikeyan [21] introduced and investigated the following family of integral operators
$I_\mu^P(\alpha_1, \alpha_2, ..., \alpha_q, \beta_1, \beta_2, ..., \beta_s) : \sum \to \sum$ as follows:
Let

$$F_{\mu,p}(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{k+\mu+1}{\mu}\right)^p a_k z^k,$$

$p \in \mathbb{N}_0$, $\mu \neq 0$ and let $F_{\mu,p}^{-1}(z)$ be defined such that

$$F_{\mu,p}(z) * F_{\mu,p}^{-1}(z) = \phi(\alpha_1, \alpha_2, ..., \alpha_q, \beta_1, \beta_2, ..., \beta_s; z).$$

Then

$$I_\mu^P \;(\alpha_1, \alpha_2, ..., \alpha_q, \beta_1, \beta_2, ..., \beta_s) f = F_{\mu,p}^{-1}(z) * f(z) \tag{2}$$

From 2 it can be easily seen that

$$I_\mu^P \;(\alpha_1, \alpha_2, ..., \alpha_q, \beta_1, \beta_2, ..., \beta_s) f = \frac{1}{z} + \sum_{k=0}^{\infty} \left(\frac{\mu}{k+\mu+1}\right)^p \frac{(\alpha_1)_{k+1}(\alpha_2)_{k+1}...(\alpha_q)_{k+1}}{(\beta_1)_{k+1}(\beta_2)_{k+1}...(\beta s)_{k+1}} \frac{a_k z^k}{(k+1)!}. \tag{3}$$

For convenience, we shall henceforth denote

$$I_\mu^P(\alpha_1, \alpha_2, ..., \alpha_q, \beta_1, \beta_2, ..., \beta_s) f = I_\mu^P(\alpha_1, \beta_1) f. \tag{4}$$

For the choice of the parameters $p=0,\ q=2,\ s=1$, the operator $I_{\mu}^{P}(\alpha_1,\beta_1)f$ is reduced to an operator introduced by N. E. Cho and K. I. Noor in [9] and when $p=0,\ q=2,\ s=1,\ \alpha_1=\lambda,\ \alpha_2=1,\ \beta_1=(n+1)$, the operator $I_{\mu}^{P}(\alpha_1,\beta_1)f$ is reduced to an operator recently introduced by S.-M. Yuan et. al. in [22].
It can be easily verified from the above definition of the operator $I_{\mu}^{P}(\alpha_{1,}\beta_1)f$ that

$$z(I_{\mu}^{P+1}(\alpha_1,\beta_1)f(z))'=\mu I_{\mu}^{P}(\alpha_1,\beta_1)f(z)-(\mu+1)I_{\mu}^{P+1}(\alpha_1,\beta_1)f(z). \tag{5}$$

**Definition 2.** [[16], Definition 2.3a, p. 27] Let $\Omega$ be a set in $\mathbb{C}$, $q\in Q$ and $n$ be a positive integer. The class of admissible functions $\Psi_n[\Omega,q]$ consists of these functions $\psi:\mathbb{C}^3\times\mathbb{U}\to\mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z)\notin\Omega$ whenever $r=q(\varsigma),\ s=k\varsigma q'(\varsigma)$, and

$$\Re\left\{\frac{t}{s}+1\right\}\geq k\Re\left\{1+\frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right\},$$

where $z\in\mathbb{U},\ \varsigma\in\partial\mathbb{U}\backslash E(q)$ and $k\geq n$. We write $\Psi_1[\Omega,q]$ as $\Psi[\Omega,q]$.
**Definition 3.** [[17], Definition 3, p. 817] Let $\Omega$ be a set in $\mathbb{C}$, $q\in H[a,n]$ with $q'(z)\neq 0$. The class of admissible functions $\Psi_n'[\Omega,q]$ consists of these functions $\psi:\mathbb{C}^3\times\overline{\mathbb{U}}\to\mathbb{C}$ that satisfy the admissibility condition $\psi(r,s,t;z)\in\Omega$ whenever $r=q(\varsigma),\ s=\varsigma q'(\varsigma)/m$, and

$$\Re\left\{\frac{t}{s}+1\right\}\leq \frac{1}{m}\Re\left\{1+\frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right\},$$

where $z\in\mathbb{U},\ \varsigma\in\partial\mathbb{U}$ and $m\geq n\geq 1$. We write $\Psi_1'[\Omega,q]$ as $\Psi'[\Omega,q]$.
For the above two classes of admissible functions, Miller and Mocanu proved the following lemmas.
**Lemma 1.** [[16], Theorem 2.3b, p. 28] Let $\psi\in\Psi_n[\Omega,q]$ with $q(0)=a$. If the analytic function

$$g(z)=a+a_nz^n+a_{n+1}z^{n+1}+\cdots$$

satisfies

$$\psi(g(z),zg'(z),z^2g''(z);z)\in\Omega,$$

then $g(z)\prec q(z)$.
**Lemma 2.** [[17], Theorem 1, p. 818] Let $\psi\in\Psi_n'[\Omega,q]$ with $q(0)=a$. If $g(z)\in Q(a)$ and

$$\psi(g(z),zg'(z),z^2g''(z);z),$$

is univalent in $\mathbb{U}$ then

$$\Omega\subset\left\{\psi(g(z),zg'(z),z^2g''(z);z):z\in\mathbb{U}\right\},$$

implies $q(z)\prec g(z)$.
In the present investigation, the differential subordination result of Miller and Mocanu [[16], Theorem 2.3b, p. 28] is extended for functions associated with the multiplier transformation $I_{\mu}^{P}(\alpha_{1,}\beta_1)$, and we obtain certain other related results. A similar problem for analytic functions was studied by Aghalary et al. [1], Ali et al. [3], Aouf [4], Aouf et al. [5], Aouf and Seoudy [6], and Kim and Srivastava [14]. Also Ali et al. [2], Liu and Owa [15] and Kamali [13] investigated a subordination problem for meromorphic functions. Additionally, the corresponding superordination problem is investigated, and several differential sandwich-type results are obtained.

## 2. Subordination Results Involving The Operator $I_\mu^p$

**Definition 4.** Let $\Omega$ be a set in $\mathbb{C}$ and $q(z) \in Q_1 \cap H$. The class of admissible functions $\Phi_H[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition $\varphi(u, \upsilon, w; z) \notin \Omega$ whenever

$$u = q(\varsigma),\ \upsilon = \frac{\mu q(\varsigma) + k\varsigma q'(\varsigma)}{\mu},$$

$$\Re\left\{\frac{\mu(w-u)}{\upsilon - u} - 2\mu\right\} \geq k\Re\left\{1 + \frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right\},$$

where $z \in \mathbb{U},\ \varsigma \in \partial\mathbb{U}\backslash E(q)$ and $k \geq 1$.

**Theorem 1**. Let $\varphi \in \Phi_H[\Omega, q]$. If $f \in \sum$ and

$$\varphi\left\{\left(zI_\mu^{p+2}(\alpha_1, \beta_1)f(z), zI_\mu^{p+1}(\alpha_1, \beta_1)f(z), zI_\mu^{p}(\alpha_1, \beta_1)f(z); z\right) : z \in \mathbb{U}\right\} \subset \Omega, \quad (6)$$

then

$$zI_\mu^{p+2}(\alpha_1, \beta_1)f(z) \prec q(z).$$

**Proof.** Define the analytic function $g(z)$ in $\mathbb{U}$ by

$$g(z) := zI_\mu^{p+2}(\alpha_1, \beta_1)f(z). \quad (7)$$

In view of the relation 5, it follows from 7 that

$$zI_\mu^{p+1}(\alpha_1, \beta_1)f(z) = \frac{zg'(z) + \mu g(z)}{\mu}. \quad (8)$$

Further computations show that

$$zI_\mu^{p}(\alpha_1, \beta_1)f(z) = \frac{z^2 g''(z) + (2\mu+1)zg'(z) + \mu^2 g(z)}{\mu^2}. \quad (9)$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r,s,t) = r,\ \upsilon(r,s,t) = \frac{\mu r + s}{\mu},\ w(r,s,t) = \frac{\mu^2 r + (2\mu+1)s + t}{\mu^2}. \quad (10)$$

Let

$$\psi(r,s,t;z) := \varphi(u, \upsilon, w; z) = \varphi\left(r, \frac{\mu r + s}{\mu}, \frac{\mu^2 r + (2\mu+1)s + t}{\mu^2}; z\right). \quad (11)$$

The proof will make use of Lemma 1. Using equations 7, 8 and 9, it follows from 11 that

$$\begin{aligned} &\psi(g(z), zg'(z), z^2 g''(z); z) \\ =\ &\varphi\left(zI_\mu^{p+2}(\alpha_1, \beta_1)f(z), zI_\mu^{p+1}(\alpha_1, \beta_1)f(z), zI_\mu^{p}(\alpha_1, \beta_1)f(z); z\right). \end{aligned} \quad (12)$$

Hence 6 becomes

$$\psi(g(z), zg'(z), z^2 g''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\varphi \in \Phi_H[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 2. Note that

$$\frac{t}{s} + 1 = \left[\frac{\mu(w-u)}{\upsilon - u} - 2\mu\right],$$

and hence $\psi \in \Psi[\Omega, q]$. By Lemma 1,

$$g(z) \prec q(z) \text{ or } zI_\mu^{p+2}f(z) \prec q(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto $\Omega$. In this case the class $\Phi_H[h(\mathbb{U}), q]$ is written as $\Phi_H[h, q]$. The following result is an immediate consequence of Theorem 1.

**Theorem 2**. Let $\varphi \in \Phi_H[h, q]$ with $q(0) = 1$. If $f \in \sum$ and

$$\varphi\left(zI_{\mu}^{p+2}(\alpha_1, \beta_1)f(z), zI_{\mu}^{p+1}(\alpha_1, \beta_1)f(z), zI_{\mu}^{p}(\alpha_1, \beta_1)f(z); z\right) \prec h(z) \quad (z \in \mathbb{U}), \tag{13}$$

then

$$zI_{\mu}^{p+2}(\alpha_{1,}\beta_1)f(z) \prec q(z).$$

Our next result is an extension of Theorem 1 to the case where the behavior of $q(z)$ on $\partial\mathbb{U}$ is not known.

**Corollary 1.** Let $\Omega \subset \mathbb{C}$ and let $q(z)$ be univalent in $\mathbb{U}$, $q(0) = 1$. Let $\varphi \in \Phi_H[\Omega, q_\rho]$ for some $\rho \in (0, 1)$ where $q_\rho(z) = q(\rho z)$. If $f \in \sum$ and

$$\varphi\left(zI_{\mu}^{p+2}(\alpha_1, \beta_1)f(z), zI_{\mu}^{p+1}(\alpha_1, \beta_1)f(z), zI_{\mu}^{p}(\alpha_1, \beta_1)f(z); z\right) \in \Omega \quad (z \in \mathbb{U}),$$

then

$$zI_{\mu}^{p+2}(\alpha_{1,}\beta_1)f(z) \prec q(z).$$

**Proof.** Theorem 1 yields $zI_{\mu}^{p+2}(\alpha_{1,}\beta_1)f(z) \prec q_\rho(z)$. The result is now deduced from $q_\rho(z) \prec q(z)$.

In the particular case $q(z) = 1 + Mz,\ M > 0$, and in view of Definition 3, the class of admissible functions $\Phi_H[\Omega, q]$ denoted by $\Phi_H[\Omega, M]$ can be expressed in the following form:

**Definition 5.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\varphi\left(1 + Me^{i\theta}, 1 + \frac{(\mu + k)}{\mu}Me^{i\theta}, 1 + \frac{[\mu^2 + (2\mu + 1)k)]Me^{i\theta} + L}{\mu^2}; z\right) \notin \Omega$$

whenever $z \in \mathbb{U},\ \theta \in R,\ \Re(Le^{-i\theta}) \geq k(k-1)M$ and $k \geq 1$.

**Corollary 2**. Let $\varphi \in \Phi_{H,1}[\Omega, M]$. If $f \in \sum$ and

$$\varphi\left(zI_{\mu}^{p+2}(\alpha_1, \beta_1)f(z), zI_{\mu}^{p+1}(\alpha_1, \beta_1)f(z), zI_{\mu}^{p}(\alpha_1, \beta_1)f(z); z\right) \in \Omega \quad (z \in \mathbb{U}), \tag{14}$$

then

$$\left|zI_{\mu}^{p+2}(\alpha_{1,}\beta_1)f(z) - 1\right| < M.$$

**Definition 6.** Let $\Omega$ be a set in $\mathbb{C}$ and $q(z) \in Q_1 \cap H$. The class of admissible functions $\Phi_{H,1}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition $\varphi(u, \upsilon, w; z) \notin \Omega$ whenever

$$u = q(\varsigma),\ \upsilon = \frac{1}{\mu}\left(\mu q(\varsigma) + \frac{k\varsigma q'(\varsigma)}{q(\varsigma)}\right) \quad (q(\varsigma) \neq 0),$$

$$\Re\left\{\frac{\mu\upsilon\,(w - u)}{\upsilon - u} + \mu(\upsilon - 2u)\right\} \geq k\Re\left\{1 + \frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right\},$$

where $z \in \mathbb{U},\ \varsigma \in \partial\mathbb{U}\backslash E(q)$ and $k \geq 1$.

**Theorem 3**. Let $\varphi \in \Phi_{H,1}[\Omega, q]$. If $f \in \sum$ and

$$\left\{\varphi\left(\frac{I_{\mu}^{p+2}(\alpha_1, \beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1, \beta_1)f(z)}, \frac{I_{\mu}^{p+1}(\alpha_1, \beta_1)f(z)}{I_{\mu}^{p+2}(\alpha_1, \beta_1)f(z)}, \frac{I_{\mu}^{p}(\alpha_1, \beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1, \beta_1)f(z)}; z\right) : z \in \mathbb{U}\right\} \subset \Omega, \tag{15}$$

then

$$\frac{I_\mu^{p+2}(\alpha_1,\beta_1)f(z)}{I_\mu^{p+3}(\alpha_1,\beta_1)f(z)} \prec q(z).$$

**Proof.** Define the analytic function $g(z)$ in $\mathbb{U}$ by

$$g(z) := \frac{I_\mu^{p+2}(\alpha_1,\beta_1)f(z)}{I_\mu^{p+3}(\alpha_1,\beta_1)f(z)}. \tag{16}$$

Then

$$\frac{zg^{'}(z)}{g(z)} = \frac{z\left(I_\mu^{p+2}(\alpha_1,\beta_1)f(z)\right)'}{I_\mu^{p+2}(\alpha_1,\beta_1)f(z)} - \frac{z\left(I_\mu^{p+3}(\alpha_1,\beta_1)f(z)\right)'}{I_\mu^{p+3}(\alpha_1,\beta_1)f(z)}. \tag{17}$$

In view of the relation 5, it follows from 17 that

$$\frac{I_\mu^{p+1}(\alpha_1,\beta_1)f(z)}{I_\mu^{p+2}(\alpha_1,\beta_1)f(z)} = g(z) + \frac{1}{\mu}\left[\frac{zg^{'}(z)}{g(z)}\right]. \tag{18}$$

Differentiating logarithmically 18, further computations show that

$$\frac{I_\mu^{p}(\alpha_1,\beta_1)f(z)}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z)} = g(z) + \frac{1}{\mu}\left[\frac{zg'(z)}{g(z)} + \frac{\mu zg'(z) + z^2\frac{g''(z)}{g(z)} + \frac{zg'(z)}{g(z)} - \left(\frac{zg'(z)}{g(z)}\right)^2}{\mu g(z) + \frac{zg'(z)}{g(z)}}\right]. \tag{19}$$

Define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u(r,s,t) = r,\ v(r,s,t) = r + \frac{1}{\mu}\left(\frac{s}{r}\right),\ w(r,s,t) = r + \frac{1}{\mu}\left[\frac{s}{r} + \frac{\mu s + \frac{t}{r} + \frac{s}{r} - \left(\frac{s}{r}\right)^2}{\mu r + \frac{s}{r}}\right]. \tag{20}$$

Let

$$\begin{aligned}\psi(r,s,t;z) \quad &: = \varphi(u,v,w;z)\\ &= \varphi\left(r, \frac{1}{\mu}\left[\mu r + \frac{s}{r}\right], \frac{1}{\mu}\left[\mu r + \frac{s}{r} + \frac{\mu s + \frac{t}{r} + \frac{s}{r} - \left(\frac{s}{r}\right)^2}{\mu r + \frac{s}{r}}\right]; z\right).\end{aligned} \tag{21}$$

The proof will make use of Lemma 1. Using equations 16, 18 and 19, it follows from 21 that

$$\begin{aligned}&\psi(g(z), zg'(z), z^2g''(z); z)\\ = \quad &\varphi\left(\frac{I_\mu^{p+2}(\alpha_1,\beta_1)f(z)}{I_\mu^{p+3}(\alpha_1,\beta_1)f(z)}, \frac{I_\mu^{p+1}(\alpha_1,\beta_1)f(z)}{I_\mu^{p+2}(\alpha_1,\beta_1)f(z)}, \frac{I_\mu^{p}(\alpha_1,\beta_1)f(z)}{I_\mu^{p+1}(\alpha_1,\beta_1)f(z)}; z\right).\end{aligned} \tag{22}$$

Hence 15 implies

$$\psi(g(z), zg'(z), z^2g''(z); z) \in \Omega.$$

The proof is completed if it can be shown that the admissibility condition for $\varphi \in \Phi_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 2. For this purpose, note that

$$\frac{s}{r} = \mu\left(v - u\right),\ \frac{t}{r} = \mu^2 v(w-v) - (\mu v - \mu u)(\mu v - 2\mu u) - (\mu v - \mu u),$$

and thus

$$\frac{t}{s} + 1 = \left[\frac{\mu v\left(w-u\right)}{v-u} + \mu(v-2u)\right].$$

Hence $\psi \in \Psi[\Omega, q]$. By Lemma 1, $g(z) \prec q(z)$ or

$$\frac{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+3}(\alpha_1, \beta_1) f(z)} \prec q(z). \quad (z \in \mathbb{U})$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain with $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto $\Omega$. In this case the class $\Phi_{H,1}[h(\mathbb{U}), q]$ is written as $\Phi_{H,2}[h, q]$. The following result is an immediate consequence of Theorem 3.

**Theorem 4**. Let $\varphi \in \Phi_{H,1}[h, q]$. If $f \in \sum$ and

$$\varphi\left(\frac{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+3}(\alpha_1, \beta_1) f(z)}, \frac{I_\mu^{p+1}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}, \frac{I_\mu^{p}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+1}(\alpha_1, \beta_1) f(z)}; z\right) \prec h(z) \quad (z \in \mathbb{U}), \tag{23}$$

then

$$\frac{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+3}(\alpha_1, \beta_1) f(z)} \prec q(z).$$

In the particular case $q(z) = 1 + Mz,\ M > 0$, the class of admissible functions $\Phi_{H,1}[\Omega, q]$ denoted by $\Phi_{H,1}[\Omega, M]$.

**Definition 7.** Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{H,1}[\Omega, M]$ consists of those functions $\varphi : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$ that satisfy the admissibility condition

$$\varphi\Bigg(1 + Me^{i\theta}, 1 + \frac{[\mu(1 + Me^{i\theta}) + k]}{\mu(1 + Me^{i\theta})} Me^{i\theta}, 1 + \frac{[\mu(1 + Me^{i\theta}) + k]}{\mu(1 + Me^{i\theta})} Me^{i\theta}$$

$$+ \frac{(M + e^{-i\theta})\left[(\mu + 1)kM + \mu k M^2 e^{i\theta} + Le^{-i\theta}\right] - k^2 M^2}{\mu(M + e^{-i\theta})\left[\mu e^{-i\theta} + (2\mu + k) M + \mu M^2 e^{i\theta}\right]}; z\Bigg) \notin \Omega$$

whenever $z \in \mathbb{U},\ \theta \in R,\ \Re(Le^{-i\theta}) \geq k(k-1)M$ for all real $\theta$ and $k \geq 1$.

**Corollary 3**. Let $\varphi \in \Phi_{H,1}[\Omega, M]$. If $f \in \sum$ and

$$\varphi\left(\frac{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+3}(\alpha_1, \beta_1) f(z)}, \frac{I_\mu^{p+1}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}, \frac{I_\mu^{p}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+1}(\alpha_1, \beta_1) f(z)}; z\right) \in \Omega \quad (z \in \mathbb{U}),$$

then

$$\left|\frac{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+3}(\alpha_1, \beta_1) f(z)} - 1\right| < M.$$

In the special case $\Omega = q(\mathbb{U}) = \{w : |w - 1| < M\}$, the class $\Phi_{H,1}[\Omega, M]$ is denoted by $\Phi_{H,1}[M]$, and Corollary 3 takes the following form:

**Corollary 4**. Let $\varphi \in \Phi_{H,1}[M]$. If $f \in \sum$ and

$$\left|\varphi\left(\frac{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+3}(\alpha_1, \beta_1) f(z)}, \frac{I_\mu^{p+1}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}, \frac{I_\mu^{p}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+1}(\alpha_1, \beta_1) f(z)}; z\right) - 1\right| < M \quad (z \in \mathbb{U}),$$

then

$$\left|\frac{I_\mu^{p+2}(\alpha_1, \beta_1) f(z)}{I_\mu^{p+3}(\alpha_1, \beta_1) f(z)} - 1\right| < M.$$

## 3. Superordination Results Involving the operator$I_\mu^p$

The dual problem of differential subordination, that is, differential superordination of the operator $I_\mu^p$ is investigated in this section. For this purpose the class of admissible functions is given in the following definition.

**Definition 8.** Let $\Omega$ be a set in $\mathbb{C}$ and $q(z) \in H$ with $\varsigma q'(\varsigma) \neq 0$. The class of admissible functions $\Phi'_H[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\varphi(u, \upsilon, w; z) \in \Omega$ whenever

$$u = q(\varsigma),\ \ \upsilon = q(\varsigma) + \frac{\varsigma q'(\varsigma)}{\mu m},$$

$$\Re\left\{\frac{\mu(w-u)}{\upsilon - u} - 2\mu\right\} \leq \frac{1}{m}\Re\left\{1 + \frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right\},$$

where $z \in \mathbb{U},\ \varsigma \in \partial\mathbb{U}$ and $m \geq 1$.

**Theorem 5**. Let $\varphi \in \Phi'_H[\Omega, q]$. If $f \in \sum,\ zI_\mu^{p+2}(\alpha_1, \beta_1)f(z) \in Q_1$ and

$$\varphi\left(zI_\mu^{p+2}(\alpha_1, \beta_1)f(z), zI_\mu^{p+1}(\alpha_1, \beta_1)f(z), zI_\mu^{p}(\alpha_1, \beta_1)f(z); z\right)$$

is univalent in , then

$$\Omega \subset \left\{\varphi\left(zI_\mu^{p+2}(\alpha_1, \beta_1)f(z), zI_\mu^{p+1}(\alpha_1, \beta_1)f(z), zI_\mu^{p}(\alpha_1, \beta_1)f(z); z\right) : z \in \mathbb{U}\right\} \quad (24)$$

implies

$$q(z) \prec zI_\mu^{p+2}(\alpha_1, \beta_1)f(z).$$

**Proof.** Let $g$ be defined by 7 and $\psi$ by 12. Since $\varphi \in \Phi'_H[\Omega, q]$, 12 and 24 yield

$$\Omega \subset \left\{\psi(g(z), zg'(z), z^2 g''(z); z) : z \in \mathbb{U}\right\}.$$

From 11, the admissibility condition for $\varphi \in \Phi'_H[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 3. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 2, $q(z) \prec g(z)$ or

$$q(z) \prec zI_\mu^{p+2}(\alpha_1, \beta_1)f(z).$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto $\Omega$. Then the class $\Phi'_H[h(\mathbb{U}), q]$ is written as $\Phi'_H[h, q]$. Proceeding similarly as in the previous section 2, the following results are an immediate consequence of Theorem 5.

**Theorem 6**. Let $q(z) \in H,\ h(z)$ be analytic in $\mathbb{U}$ and $\varphi \in \Phi'_H[h, q]$. If $f \in \sum,\ zI_\mu^{p+2}(\alpha_1, \beta_1)f(z) \in Q_1$
and

$$\varphi\left(zI_\mu^{p+2}(\alpha_1, \beta_1)f(z), zI_\mu^{p+1}(\alpha_1, \beta_1)f(z), zI_\mu^{p}(\alpha_1, \beta_1)f(z); z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \varphi\left(zI_\mu^{p+2}(\alpha_1, \beta_1)f(z), zI_\mu^{p+1}(\alpha_1, \beta_1)f(z), zI_\mu^{p}(\alpha_1, \beta_1)f(z); z\right) \quad (z \in \mathbb{U}), \quad (25)$$

implies

$$q(z) \prec zI_\mu^{p+2}(\alpha_1, \beta_1)f(z).$$

Combining Theorems 2 and 6, we obtain the following sandwich-type theorem.

**Corollary 5.** Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, $h_2(z)$ be univalent in $\mathbb{U}$, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$, and $\varphi \in \Phi_H[h_2, q_2] \cap \Phi'_H[h_1, q_1]$. If $f \in \sum,\ zI_\mu^{p+2}(\alpha_1, \beta_1)f(z) \in H \cap Q_1$

and $\varphi\left(zI_{\mu}^{p+2}(\alpha_1,\beta_1)f(z), zI_{\mu}^{p+1}(\alpha_1,\beta_1)f(z), zI_{\mu}^{p}(\alpha_1,\beta_1)f(z);z\right)$ is univalent in $\mathbb{U}$, then

$$h_1(z) \prec \varphi\left(zI_{\mu}^{p+2}(\alpha_1,\beta_1)f(z), zI_{\mu}^{p+1}(\alpha_1,\beta_1)f(z), zI_{\mu}^{p}(\alpha_1,\beta_1)f(z);z\right) \prec h_2(z)\,(z\in\mathbb{U}),$$

implies

$$q_1(z) \prec zI_{\mu}^{p+2}(\alpha_1,\beta_1)f(z) \prec q_2(z).$$

**Definition 9.** Let $\Omega$ be a set in $\mathbb{C}$ and $q(z) \in H$ with $\varsigma q'(\varsigma) \neq 0$. The class of admissible functions $\Phi'_{H,1}[\Omega, q]$ consists of those functions $\varphi : \mathbb{C}^3 \times \overline{\mathbb{U}} \to \mathbb{C}$ that satisfy the admissibility condition $\varphi(u, v, w; \varsigma) \in \Omega$, whenever

$$u = q(\varsigma),\ \ \upsilon = q(\varsigma) + \frac{1}{\mu}\left(\frac{\varsigma q'(\varsigma)}{mq(\varsigma)}\right),$$

$$\Re\left\{\frac{\mu\upsilon\,(w-u)}{\upsilon-u} + \mu(\upsilon-2u)\right\} \leq \frac{1}{m}\Re\left\{1+\frac{\varsigma q''(\varsigma)}{q'(\varsigma)}\right\}.$$

$z \in \mathbb{U},\ \varsigma \in \partial\mathbb{U}$ and $m \geq 1$.

Now we will give the dual result of Theorem 3 for differential superordination.

**Theorem 7**. Let $\varphi \in \Phi'_{H,1}[\Omega, q]$. If $f \in \sum,\ \frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)} \in Q_1$ and

$$\varphi\left(\frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)};z\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\varphi\left(\frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}, \frac{\mu I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}, \frac{\mu I_{\mu}^{p}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)};z\right) : z \in \mathbb{U}\right\} \tag{26}$$

implies

$$q(z) \prec \frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}.$$

**Proof.** Let $g$ be defined by 16 and $\psi$ by 21. Since $\varphi \in \Phi'_{H,1}[\Omega, q]$, it follows from 22 and 26 that

$$\Omega \subset \left\{\psi(g(z), zg'(z), z^2g''(z); z) : z \in \mathbb{U}\right\}.$$

From 21, the admissibility condition for $\varphi \in \Phi'_{H,1}[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 3. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 2, $q(z) \prec g(z)$ or

$$q(z) \prec \frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}.$$

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping $h(z)$ of $\mathbb{U}$ onto $\Omega$. In this case the class $\Phi'_{H,1}[h(\mathbb{U}), q]$ is written as $\Phi'_{H,1}[h, q]$. Proceeding similarly, the following results are an immediate consequence of Theorem 7.

**Theorem 8**. Let $q(z) \in H,\ h(z)$ be analytic in $\mathbb{U}$ and $\varphi \in \Phi'_{H,1}[\Omega, q]$. If $f \in \sum,\ \frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)} \in Q_1$and

$$\varphi\left(\frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)};z\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec \varphi\left(\frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}; z\right) \quad (z \in \mathbb{U}),$$

implies

$$q(z) \prec \frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}.$$

Combining Theorems 4 and 8, gives the following sandwich-type theorem.

**Corollary 6.** Let $h_1(z)$ and $q_1(z)$ be analytic functions in $\mathbb{U}$, $h_2(z)$ be univalent in $\mathbb{U}$, $q_1 \in Q_1$ with $q_1(0) = q_2(0) = 1$, and $\varphi \in \Phi_{H,1}[h_2, q_2] \cap \Phi'_{H,1}[h_1, q_1]$. If $f \in \sum$, $\frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)} \in H \cap Q_1$ and

$$\varphi\left(\frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}; z\right)$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec \varphi\left(\frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}, \frac{I_{\mu}^{p}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+1}(\alpha_1,\beta_1)f(z)}; z\right) \prec h_2(z) \quad (z \in \mathbb{U}),$$

implies

$$q_1(z) \prec \frac{I_{\mu}^{p+2}(\alpha_1,\beta_1)f(z)}{I_{\mu}^{p+3}(\alpha_1,\beta_1)f(z)} \prec q_2(z).$$

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