Electronic Journal of Mathematical Analysis and Applications,
Vol. 2(2) July 2014, pp. 189-198.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

OSTROWSKI TYPE INEQUALITIES FOR HARMONICALLY QUASI-CONVEX FUNCTIONS

İMDAT İŞCAN, SELIM NUMAN

Abstract. In this paper, we give some new Ostrowski type inequalities for the class of functions whose derivatives in absolutely value at certain powers are harmonically quasi-convex functions.

1. Introduction

Let $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in I° (the interior of I) and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, for all $x \in [a, b]$, then the following inequality holds

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{M}{b-a} \left[\frac{(x-a)^2 + (b-x)^2}{2} \right] \tag{1}$$

for all $x \in [a, b]$. This inequality is known in the literature as the Ostrowski inequality (see [10]), which gives an upper bound for the approximation of the integral average $\frac{1}{b-a} \int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(\alpha x + (1-\alpha)y) \leq \sup \{f(x), f(y)\},$$

for any $x, y \in [a, b]$ and $\alpha \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [4]).

For some results which generalize, improve and extend the inequalities(1) concerning quasi-convex functions we refer the reader to see [1, 2, 4, 5, 6, 7, 11] and plenty of references therein.

In [8], the author gave harmonically convex and established Hermite-Hadamard's inequality for harmonically convex functions as follows:

2000 *Mathematics Subject Classification.* 26A51, 26D15, 26D10.
Key words and phrases. Harmonically quasi-convex function, Ostrowski type inequality, Hermite-Hadamard's inequality, hypergeometric function.
Submitted Oct, 19, 2013.

Definition 1. *Let $I\subseteq\mathbb{R}\backslash\{0\}$ be a real interval. A function $f:I\rightarrow\mathbb{R}$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq tf(y)+(1-t)f(x) \tag{2}$$

for all $x,y\in I$ and $t\in[0,1]$. If the inequality in (2) is reversed, then f is said to be harmonically concave.

Theorem 1. *Let $f:I\subseteq\mathbb{R}\backslash\{0\}\rightarrow\mathbb{R}$ be a harmonically convex function and $a,b\in I$ with $a<b$. If $f\in L[a,b]$ then the following inequalities hold*

$$f\left(\frac{2ab}{a+b}\right)\leq\frac{ab}{b-a}\int\limits_a^b\frac{f(x)}{x^2}dx\leq\frac{f(a)+f(b)}{2}. \tag{3}$$

The above inequalities are sharp.

In [11], Zhang et al. defined the harmonically quasi-convex function and supplied several properties of this kind of functions.

Definition 2. *A function $f:I\subseteq(0,\infty)\rightarrow[0,\infty)$ is said to be harmonically convex, if*

$$f\left(\frac{xy}{tx+(1-t)y}\right)\leq\sup\{f(x),f(y)\}$$

for all $x,y\in I$ and $t\in[0,1]$.

We would like to point out that any harmonically convex function on $I\subseteq(0,\infty)$ is a harmonically quasi-convex function, but not conversely. For example, the function

$$f(x)=\begin{cases}1, & x\in(0,1];\\ (x-2)^2, & x\in[1,4].\end{cases}$$

is harmonically quasi-convex on $(0,4]$, but it is not harmonically convex on $(0,4]$.

In [11], by using the following lemma, Zhang et al. obtained some new Hermite-Hadamard type inequalities for harmonically quasi-convex functions.

Lemma 1. *Let $f:I\subseteq(0,\infty)\rightarrow\mathbb{R}$ be a differentiable function on I° and $a,b\in I$ with $a<b$. If $f'\in L[a,b]$ then*

$$\frac{bf(b)-af(a)}{b-a}-\frac{1}{b-a}\int\limits_a^b f(x)dx=\int\limits_0^1\frac{a^2b^2}{(tb+(1-t)a)^3}f'\left(\frac{ab}{a+t\,(b-a)}\right)dt.$$

In order to prove our main results we need the following lemma:

Lemma 2. *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a differentiable function on I° and $a, b \in I$ with $a < b$. If $f' \in L[a,b]$ then*

$$\begin{aligned} & f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \\ = \; & \frac{ab}{b-a}\left\{(x-a)^2\int_0^1 \frac{t}{(ta+(1-t)x)^2} f'\left(\frac{ax}{ta+(1-t)x}\right)dt \right. \\ & \left. -(b-x)^2\int_0^1 \frac{t}{(tb+(1-t)x)^2} f'\left(\frac{bx}{tb+(1-t)x}\right)dt\right\}. \end{aligned}$$

A simple proof of equality can be given by performing integration by parts in the integrals of the right side and changing the variable (see [9]).

In this paper, by using Lemma 2, we obtained some new Ostrowski type inequalities for harmonically quasi-convex functions.

2. Main Results

Theorem 2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically quasi- convex on $[a,b]$ for $q \geq 1$, then for all $x \in [a,b]$, we have*

$$\left|f(x) - \frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \tag{4}$$

$$\begin{aligned} \leq \; & \frac{ab}{b-a}\left\{(x-a)^2\left(C_1(a,x,q,q)\sup\left\{\left|f'(x)\right|^q, \left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right. \\ & \left. +(b-x)^2\left(C_2(b,x,q,q)\sup\left\{\left|f'(x)\right|^q, \left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}, \end{aligned}$$

where

$$C_1(a,x,\vartheta,\rho) = \frac{\beta(\rho+1,1)}{x^{2\vartheta}}.{}_2F_1\left(2\vartheta, \rho+1; \rho+2; 1-\frac{a}{x}\right),$$

$$C_2(b,x,\vartheta,\rho) = \frac{\beta(1,\rho+1)}{b^{2\vartheta}}.{}_2F_1\left(2\vartheta, 1; \rho+2; 1-\frac{x}{b}\right),$$

β is Euler Beta function defined by

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x, y > 0,$$

and ${}_2F_1$ is hypergeometric function defined by

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, \; c > b > 0, \; |z| < 1 \textit{ (see [3])}.$$

Proof. From Lemma 2, Power mean inequality and the harmonically quasi-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left| f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \right|$$

$$\leq \frac{ab}{b-a}\left\{ (x-a)^2\int_0^1 \frac{t}{(ta+(1-t)x)^2}\left| f'\left(\frac{ax}{ta+(1-t)x}\right)\right| dt + (b-x)^2\int_0^1 \frac{t}{(tb+(1-t)x)^2}\left| f'\left(\frac{bx}{tb+(1-t)x}\right)\right| dt \right\}$$

$$\leq \frac{ab\,(x-a)^2}{b-a}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}}\sup\left\{ |f'(x)|^q, |f'(a)|^q \right\} dt\right)^{\frac{1}{q}} \tag{5}$$

$$+\frac{ab\,(b-x)^2}{b-a}\left(\int_0^1 1dt\right)^{1-\frac{1}{q}} \times \left(\int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}}\sup\left\{ |f'(x)|^q, |f'(b)|^q \right\} dt\right)^{\frac{1}{q}},$$

where an easy calculation gives

$$\int_0^1 \frac{t^q}{(ta+(1-t)x)^{2q}}dt = \frac{\beta(q+1,1)}{x^{2q}}.{}_2F_1\left(2q,q+1;q+2;1-\frac{a}{x}\right), \tag{6}$$

$$\int_0^1 \frac{t^q}{(tb+(1-t)x)^{2q}}dt = \frac{\beta(1,q+1)}{b^{2q}}.{}_2F_1\left(2q,1;q+2;1-\frac{x}{b}\right). \tag{7}$$

Hence, If we use (6) and (7) in (5), we obtain the desired result. This completes the proof. □

Theorem 3. *Let $f: I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a<b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically quasi- convex on $[a,b]$ for $q \geq 1$, then for all $x \in [a,b]$, we have*

$$\left| f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du \right| \tag{8}$$

$$\leq \frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{(x-a)^2\left(C_1(a,x,q,1)\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^2\left(C_2(b,x,q,1)\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}$$

where C_1 and C_2 are defined as in Theorem 2.

Proof. From Lemma 2, Power mean inequality and the harmonically s-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right|$$
$$\leq \frac{ab\,(x-a)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{t}{(ta+(1-t)x)^{2q}}\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}dt\right)^{\frac{1}{q}}$$
$$+\frac{ab\,(b-x)^2}{b-a}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{t}{(tb+(1-t)x)^{2q}}\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}}$$
$$\leq \frac{ab}{b-a}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left\{(x-a)^2\left(C_1(a,x,q,1)\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+(b-x)^2\left(C_2(b,x,q,1)\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}$$

This completes the proof. □

Theorem 4. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, and $f'\in L[a,b]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q\geq 1$, then for all $x\in[a,b]$, we have*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right| \tag{9}$$
$$\leq \frac{ab}{b-a}\left\{C_3(a,x)\,(x-a)^2\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{1/q}\right.$$
$$\left.+C_3(b,x)\,(b-x)^2\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{1/q}\right\}$$

where

$$C_3(\theta,x)=\frac{1}{x-\theta}\left\{\frac{1}{\theta}-\frac{\ln x-\ln\theta}{x-\theta}\right\}.$$

Proof. From Lemma 2, Power mean inequality and the harmonically quasi-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \tag{10}$$

$$\leq \frac{ab(x-a)^2}{b-a} \left(\int_0^1 \frac{t}{(ta+(1-t)x)^2} dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \frac{t}{(ta+(1-t)x)^2} \sup\left\{ |f'(x)|^q, |f'(a)|^q \right\} dt \right)^{\frac{1}{q}}$$

$$+ \frac{ab(b-x)^2}{b-a} \left(\int_0^1 \frac{t}{(tb+(1-t)x)^2} dt \right)^{1-\frac{1}{q}}$$

$$\times \left(\int_0^1 \frac{t}{(tb+(1-t)x)^2} \sup\left\{ |f'(x)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}}.$$

It is easily checked that

$$\int_0^1 \frac{t}{(ta+(1-t)x)^2} dt = \frac{1}{x-a} \left\{ \frac{1}{a} - \frac{\ln x - \ln a}{x-a} \right\}, \tag{11}$$

$$\int_0^1 \frac{t}{(tb+(1-t)x)^2} dt = \frac{1}{b-x} \left\{ \frac{\ln b - \ln x}{b-x} - \frac{1}{b} \right\}, \tag{12}$$

Hence, If we use (11) and (12) in (10), we obtain the desired result. This completes the proof. □

For $q \geq 1$, we can give the following result:

Corollary 1. *Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L[a, b]$. If $|f'|^q$ is harmonically quasi-convex on $[a, b]$ for $q \geq 1$. If $|f'(x)| \leq M$, $x \in [a, b]$ then*

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{abM}{b-a} \min\{I_1, I_2, I_3\}$$

where

$$I_1 = (x-a)^2 C_1^{1/q}(a, x, s, q, q) + (b-x)^2 C_2^{1/q}(b, x, s, q, q)$$

$$I_2 = \left(\frac{1}{2}\right)^{1-\frac{1}{q}} \left\{ (x-a)^2 C_1^{1/q}(a, x, q, 1) + (b-x)^2 C_2^{1/q}(b, x, q, 1) \right\}$$

$$I_3 = C_3(a, x)(x-a)^2 + C_3(b, x)(b-x)^2$$

Theorem 5. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab}{b-a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \tag{13}$$

$$\times \left\{ (x-a)^2 \left(C_1(a,x,q,0) \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} + (b-x)^2 \left(C_2(b,x,q,0) \sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \right\}.$$

where C_1 and C_2 are defined as in Theorem 2.

Proof. From Lemma 2, Hölder's inequality and the harmonically convexity of $|f'|^q$ on $[a,b]$,we have

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right| \leq \frac{ab\,(x-a)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \frac{1}{(ta+(1-t)x)^{2q}} \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} dt \right)^{\frac{1}{q}}$$

$$+ \frac{ab\,(b-x)^2}{b-a} \left(\int_0^1 t^p dt \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^1 \frac{1}{(tb+(1-t)x)^{2q}} \sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} dt \right)^{\frac{1}{q}}$$

$$\leq \frac{ab}{b-a} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left\{ (x-a)^2 \left(C_1(a,x,q,0) \sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} + (b-x)^2 \left(C_2(b,x,q,0) \sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \right\}.$$

This completes the proof. □

Theorem 6. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left| f(x) - \frac{ab}{b-a} \int_a^b \frac{f(u)}{u^2} du \right|$$

$$\leq \frac{ab}{b-a} \left\{ (C_1(a,x,p,p))^{\frac{1}{p}} (x-a)^2 \left(\sup \left\{ |f'(x)|^q, |f'(a)|^q \right\} \right)^{\frac{1}{q}} + (C_2(b,x,p,p))^{\frac{1}{p}} (b-x)^2 \left(\sup \left\{ |f'(x)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \right\}.$$

where C_1 and C_2 are defined as in Theorem 2.

Proof. From Lemma 2, Hölder's inequality and the harmonically quasi-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \leq \frac{ab\,(x-a)^2}{b-a}\left(\int_0^1 \frac{t^p}{(ta+(1-t)x)^{2p}}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}dt\right)^{\frac{1}{q}}$$

$$+\frac{ab\,(b-x)^2}{b-a}\left(\int_0^1 \frac{t^p}{(tb+(1-t)x)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1 \sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab}{b-a}\Big\{\left(C_1(a,x,p,p)\right)^{\frac{1}{p}}(x-a)^2\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+\left(C_2(b,x,p,p)\right)^{\frac{1}{p}}(b-x)^2\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\Big\}.$$

This completes the proof. □

Theorem 7. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$, and $f' \in L[a,b]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q > 1$, $\frac{1}{p}+\frac{1}{q}=1$, then*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \leq \frac{ab}{b-a}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}$$

$$\times\Big\{C_1^{1/p}(a,x,p,0)\,(x-a)^2\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}\right)^{\frac{1}{q}}$$
$$+C_2^{1/p}(b,x,p,0)\,(b-x)^2\left(\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}\right)^{\frac{1}{q}}\Big\}.$$

where C_1 and C_2 are defined as in Theorem 2.

Proof. From Lemma 2, Hölder's inequality and the harmonically quasi-convexity of $|f'|^q$ on $[a,b]$,we have

$$\left|f(x)-\frac{ab}{b-a}\int_a^b \frac{f(u)}{u^2}du\right| \leq \frac{ab\,(x-a)^2}{b-a}\left(\int_0^1 \frac{1}{(ta+(1-t)x)^{2p}}dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 t^q\sup\left\{\left|f'(x)\right|^q,\left|f'(a)\right|^q\right\}dt\right)^{\frac{1}{q}}$$

$$+\frac{ab\,(b-x)^2}{b-a}\left(\int_0^1 \frac{1}{(tb+(1-t)x)^{2p}}dt\right)^{\frac{1}{p}}\left(\int_0^1 t^q\sup\left\{\left|f'(x)\right|^q,\left|f'(b)\right|^q\right\}dt\right)^{\frac{1}{q}}$$

$$\leq \frac{ab}{b-a}\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left\{C_1^{1/p}(a,x,p,0)\,(x-a)^2\left(\sup\left\{|f'(x)|^q,|f'(a)|^q\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.+\,C_2^{1/p}(b,x,p,0)\,(b-x)^2\left(\sup\left\{|f'(x)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\right\}.$$

This completes the proof. □

For $q>1$, we can give the following result:

Corollary 2. *Let $f: I\subseteq(0,\infty)\to\mathbb{R}$ be a differentiable function on I°, $a,b\in I^\circ$ with $a<b$, and $f'\in L[a,b]$. If $|f'|^q$ is harmonically quasi-convex on $[a,b]$ for $q>1$, $\frac{1}{p}+\frac{1}{q}=1$, if $|f'(x)|\leq M$, $x\in[a,b]$ then*

$$\left|f(x)-\frac{ab}{b-a}\int_a^b\frac{f(u)}{u^2}du\right|\leq\frac{abM}{b-a}\min\left\{J_1,J_2,J_3\right\} \tag{14}$$

where

$$J_1=\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{(x-a)^2\,C_1^{1/q}(a,x,q,0)+(b-x)^2\,C_2^{1/q}(b,x,q,0)\right\}$$

$$J_2=\left(C_1(a,x,p,p)\right)^{\frac{1}{p}}(x-a)^2+\left(C_2(b,x,p,p)\right)^{\frac{1}{p}}(b-x)^2$$

$$J_3=\left(\frac{1}{q+1}\right)^{\frac{1}{q}}\left\{C_1^{1/p}(a,x,p,0)\,(x-a)^2+C_2^{1/p}(b,x,p,0)\,(b-x)^2\right\}.$$

References

[1] Alomari, M. W., Darus, M., Kirmaci, U. S, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Computers and Mathematics with Applications, 59 (2010), 225-232.

[2] Alomari, M., Hussain, S., Two inequalities of Simpson type for quasi-convex functions and applications, Applied Mathematics E-Notes, 11 (2011), 110-117.

[3] M. Abramowitz and I.A. Stegun (Eds.), Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover, New York, 1965.

[4] Ion, D.A., Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Comp. Sci. Ser. 34 (2007), 82-87.

[5] İşcan, İ., Generalization of different type integral inequalities via fractional integrals for functions whose second derivatives absolute values are quasi-convex, Konuralp journal of Mathematics, 1(2) (2013) 67–79.

[6] İşcan, İ., New general integral inequalities for quasi-geometrically convex functions via fractional integrals, Journal of Inequalities and Applications, 2013(491) (2013), 15 pages. doi:10.1186/1029-242X-2013-491.

[7] İşcan, İ., On generalization of some integral inequalities for quasi-convex functions and their applications, International Journal of Engineering and Applied sciences (EAAS), 3(1) (2013), 37-42.

[8] İşcan, İ., Hermite-Hadamard type inequalities for harmonically convex functions. Hacet. J. Math. Stat., (2013a). Accepted for publication.

[9] İşcan, İ., Ostrowski type inequalities for harmonically s-convex functions, arXiv preprint arXiv:1307.5201 (2013). Submitted.

[10] Ostrowski, A., Über die Absolutabweichung einer differentiebaren funktion von ihren integralmittelwert, Comment. Math. Helv., 10 (1938), 226–227.

[11] Zhang, T.-Y., Ji, A.-P. and Qi, F., Integral inequalities of Hermite-Hadamard type for harmonically quasi-convex functions. Proc. Jangjeon Math. Soc, 16(3) (2013), 399-407.

İmdat İşcan, Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun, Turkey

E-mail address: imdati@yahoo.com, imdat.iscan@giresun.edu.tr

Selim Numan, Department of Mathematics, Faculty of Arts and Sciences, Giresun University, Giresun, Turkey

E-mail address: selim.numan@giresun.edu.tr