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CLASSIFICATION OF 1-WEIERSTRASS POINTS ON KURIBAYASHI QUARTICS, I (WITH TWO PARAMETERS)

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Abstract. In this paper, we classify the 1-Weierstrass points of the Kuribayashi quartic curves with two parameters a and b defined by the equation

$$C_{a,b} : X^4 + Y^4 + Z^4 + aX^2Y^2 + b(X^2 + Y^2)Z^2 = 0,$$

such that $(a^2 - 4)(b^2 - 4)(a^2 - 1)(b^2 - a - 2) \neq 0$. Furthermore, the geometry of these points is investigated.

1. Introduction

The Weierstrass points of an algebraic curve, which can be defined over some extension of the base field, form a distinguished set of points of the curve having the property of being geometrically intrinsic. Curves of genus 0 or 1 have no Weierstrass points, and for hyperelliptic curves, the Weierstrass points can easily be characterized as the ramification points under the hyperelliptic involution, which generate the 2-torsion subgroup of the Jacobian. For non-hyperelliptic curves of genus 3, which admit a plane quartic model, there is an extensive literature on the topic. For instance, Weierstrass points and automorphism groups of Riemann surfaces of genus 3 are studied in [4, 16, 17]. In 1983, Vermeulen [23] constructed a stratification of the moduli space M_3 of curves of genus 3 in terms of the number of hyperflexes and their geometric configuration. Similar results were obtained independently around the same time by Lugert [19]. Subsequent work on curves with large number of hyperflexes has been done by Martine Girard and his collaborators. In [20], they described the group generated by the Weierstrass points in the Jacobian of the curve

$$X^4 + Y^4 + Z^4 + 3\big(X^2Y^2 + X^2Z^2 + Y^2Z^2\big) = 0.$$

This curve is the only curve of genus 3, apart from the fourth Fermat curve, possessing exactly twelve Weierstrass points. In [21], they determined the group generated by Weierstrass points for all smooth quartics with eight hyperflexes or more. Also,

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they got bounds on both the rank and the torsion part of this group for a generic quartic having a fixed number of hyperflexes in the moduli space M_3 of curves of genus 3.

Let C_a be the smooth plane quartic curves defined by the equation

$$C_a : X^4 + Y^4 + Z^4 + a\big(X^2Y^2 + X^2Z^2 + Y^2Z^2\big) = 0, \qquad a \neq -1, \pm 2.$$

These types of quartic curves are called *Kuribayashi quartics with one parameter.* Weierstrass points and automorphism groups of C_a are studied in [18]. Kuribayashi and his students used *the Wronskian method* to classify the number of the 1-Weierstrass points of C_a. Alwaleed [1, 2] used the *Wronskian method* together with the S_4 action on C_a to classify the number and investigate the geometry of the 2-Weierstrass points of this family.

Let $C_{a,b}$ be the smooth plane quartic curves defined by the equation

$$C_{a,b} : X^4 + Y^4 + Z^4 + aX^2Y^2 + b\big(X^2 + Y^2\big)Z^2 = 0,$$

where a and b are two parameters such that $(a^2-4)(b^2-4)(a^2-1)(b^2-a-2) \neq 0$. We call these quartic curves *Kuribayashi quartics with two parameter.* The Kuribayashi pencil is spanned by quartic curves with maximal number of hyperflexes. Hayakawa [15] investigated the conditions under which the number of the Weierstrass points of $C_{a,b}$ is 12 or 16.

In this paper, we use finite group actions on the Riemann surfaces $C_{a,b}$ to investigate the number of the 1-Weierstrass points of this family. Furthermore, we study the geometry of these points.

The present paper is organized in the following manner. In section 2, we present some preliminaries concerning the basic concepts that will be used throughout the work [13, 22]. In section 3, we establish our main results *(Theorems 11, 18)* that concern with the classification of the number of the 1-Weierstrass points of the quartic curves $C_{a,b}$ together with their geometry. In section 4, we illustrate, through examples, the cases mentioned in the main results. Finally, we conclude the paper with some remarks, comments and related problems.

2. Preliminaries

2.1. **q-Weierstrass points.** Let C be a smooth projective plane curve of genus $g := \frac{(d-1)(d-2)}{2} \geq 2$ and let D be a divisor on C with $dim|D| = r \geq 0$. We denote by $L(D)$ the $\mathbb{C}$-vector space of meromorphic functions f such that $div(f) + D \geq 0$ and by $l(D)$ the dimension of $L(D)$ over $\mathbb{C}$. Then, the notion of D-Weierstrass points [22] can be defined in the following way:

Definition 1. *Let $p \in C$. If n is a positive integer such that*

$$l\big(D - (n-1).p\big) > l\big(D - n.p\big),$$

we call the integer n a D-gap number at p.

Lemma 1. [22] *Let $p \in C$, then there are exactly $r+1$ D-gap numbers $\{n_1, n_2, ..., n_{r+1}\}$ such that $n_1 < n_2 < ... < n_{r+1}$. The sequence $\{n_1, n_2, ..., n_{r+1}\}$ is called the D-gap sequence at p.*

Definition 2. *The integer* $\omega_D(p) := \sum_{i=1}^{r+1}(n_i - 1)$ *is called D-weight at p. If* $\omega_D(p) > 0$*, we call the point p a D-Weierstrass point on C. In particular, for the canonical divisor K, the qK-Weierstrass points* ($q \geq 1$) *are called q-Weierstrass points and the qK-weight is called q-weight, denoted by* $\omega^{(q)}(p)$*.*

Definition 3. [2] A point p on a smooth plane curve C is said to be a flex point if the tangent line L_p meets C at p with contact order $I_p(C, L_p)$ at least three. We say that p is i-flex, if $I_p(C, L_p) - 2 = i$. The positive integer i is called the flex order of p.

Lemma 2. [13] Let $C : F(X, Y, Z) = 0$ be a smooth projective plane curve. A point p on C is a flex point if, and only if, $H_F(p) = 0$, where H_F is the Hessian curve of C defined by

$$H_F := det \begin{pmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{pmatrix}.$$

Lemma 3. [23] Let C be a smooth projective plane quartic curve, the 1-Weierstrass points on C are nothing but flexes and divided into two types: ordinary flex and hyperflex points. Moreover, we have

$\omega^{(1)}(p)$	1-gap Sequence	Geometry
1	$\{1, 2, 4\}$	ordinary flex
2	$\{1, 2, 5\}$	hyperflex

Lemma 4. [12, 22] Let C be a smooth projective plane curve of genus g. The number of q-Weierstrass points $N^{(q)}(C)$, counted with their q-weights, is given by

$$N^{(q)}(C) = \begin{cases} g(g^2 - 1), & if\, q = 1 \\ \\ (2q - 1)^2(g - 1)^2 g, & if\, q \geq 2. \end{cases}$$

In particular, for smooth projective plane quartics (i.e. $g = 3$), the number of 1-Weierstrass points is 24 counted with their weights.

Let $W^{(q)}(C)$ be the set of q-Weierstrass points on C and $G^{(q)}(p)$ the q-gap sequence at the point $p \in C$.

Lemma 5. [1] Let τ be an automorphism on C, then we have

$$\tau\big(W^{(q)}(C)\big) = W^{(q)}(C) \quad and \quad G^{(q)}(\tau(p)) = G^{(q)}(p).$$

2.2. **Group Action on Riemann Surfaces** [12]**.**

Definition 4. *An action of a finite group G on a Riemann surface C is a map*

$$\cdot : G \times C \longrightarrow C : (g, p) \longmapsto g \cdot p$$

such that: $(gh) \cdot p = g \cdot (h \cdot p)$ *and* $e \cdot p = p$*, for all* $g, h \in G$ *and* $p \in C$*. Here e denotes the identity element of G.*

Definition 5. *The orbit of a point* $p \in C$ *is the set* $Orb_G(p) := \{g \cdot p : g \in G\}$*.*

Definition 6. *The stabilizer of a point* $p \in C$ *is the subgroup*

$$G_p := \{g \in G : g \cdot p = p\}.$$

It is often called the isotropy subgroup of p.

Remark 1. It is well known that G_p is cyclic and points in the same orbit have conjugate stabilizers; Indeed, $G_{g\cdot p} = gG_pg^{-1}$. Moreover $|Orb_G(p)|\,|G_p| = |G|$.

Notation. [1] The set of points $p \in C$ such that $|G_p| > 1$ is denoted by $X(C)$. Also

$$X_i(C) := \{p \in C : |G_p| = i\}.$$

3. Main results

In what follows, we investigate the number of the 1-Weierstrass points of *Kuribayashi quartic curves with two parameter* a and b defined by the equation

$$C_{a,b} : X^4 + Y^4 + Z^4 + aX^2Y^2 + b\big(X^2 + Y^2\big)Z^2,$$

such that $(a^2 - 4)(b^2 - 4)(a^2 - 1)(b^2 - a - 2) \neq 0$. Moreover, the geometry of these points is studied. We have two cases, either $b = 0$ or $b \neq 0$. In the following we treat each of these cases.

3.1. **Case $b = 0$.** In this case, we consider the one parameter family of smooth plane quartics $C_{a,0}$ defined by

$$C_{a,0} : \quad X^4 + Y^4 + Z^4 + aX^2Y^2, \qquad (a^2 - 1)(a^2 - 4) \neq 0.$$

A group action of order 16 on $C_{a,0}$ can be defined as follows. Let G be the projective transformation group generated by the three elements σ, τ, ρ of orders $2, 4, 4$ respectively, where

$$\sigma := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau := \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It has been shown by Francesc [4] that, $G \cong C_4 \circledcirc (C_2 \times C_2)$, where C_m denotes the cyclic group of order m and $C_4 \circledcirc (C_2 \times C_2)$ denotes the group $(16, 13)$ in GAP library of small groups which is a group in $Ext^1\,(C_2 \times C_2, C_4)$. Now, computing the fixed points of the automorphisms of G on $C_{a,0}$ and their corresponding orbits gives rise to the following result.

Lemma 6. *For the quartics $C_{a,0}$, we have:*

$$X(C_{a,0}) = X_2(C_{a,0}) \cup X_4(C_{a,0}).$$

Moreover,

$$X(C_{a,0}) = Orb_G[\beta : 1 : 0] \cup Orb_G[\alpha : \alpha : 1] \cup Orb_G[0 : \delta : 1] \cup Orb_G[i\lambda : \lambda : 1],$$

where β is a root of the equation $x^4 + ax^2 + 1 = 0$, α is a root of the equation $(a+2)x^4 + 1 = 0$, δ is a root of the equation $x^4 + 1 = 0$, and λ is a root of the equation $(a+2)x^4 + 1 = 0$.

Proof. Since non-trivial elements of G have orders 2 or 4, then we have

$$X(C_{a,0}) = X_2(C_{a,0}) \cup X_4(C_{a,0}).$$

Since points in the same orbit have conjugate stabilizers, it suffices to compute the fixed points of the automorphisms φ_1, φ_2, φ_3, φ_5, φ_7, φ_{10}, φ_{12} and φ_{14} on the quartics $C_{a,0}$. Indeed, we have

$$Fix\big(\varphi_1, C_{a,0}\big) = \{[0 : \delta_i : 1] : i = 1, 2, 3, 4\},$$

where δ_i $(i = 1, 2, 3, 4)$ are the roots of the equation $\delta^4 + 1 = 0$. In other words,

$$\delta_1 = \frac{1}{\sqrt{2}}(1+i), \quad \delta_2 = \frac{-1}{\sqrt{2}}(1+i), \quad \delta_3 = \frac{1}{\sqrt{2}}(1-i), \quad \delta_4 = \frac{-1}{\sqrt{2}}(1-i).$$

The four roots are of the form δ, $-\delta$, $-i\delta$, $i\delta$ which gives rise to 1_8 orbit, namely,

$$Orb_G[0:\delta:1] = \{[0:\pm\delta:1], [\pm\delta:0:1], [0:\pm i\delta:1], [\pm i\delta:0:1]\}.$$

On the other hand,

$$Fix\big(\varphi_3, C_{a,0}\big) = \{[\beta_i:1:0] : i = 1, 2, 3, 4\} = Fix\big(\varphi_7, C_{a,0}\big),$$

where β_i $(i = 1, 2, 3, 4)$ are the roots of the equation $\beta^4 + a\beta^2 + 1 = 0$. The four roots are of the form

$$\beta, -\beta, \frac{1}{\beta}, \frac{-1}{\beta},$$

such that $\beta \neq 0, \pm 1, \pm i$, which in turns gives just 1_4 orbit, namely,

$$Orb_G[\beta:1:0] = \{[\beta:1:0], [1:\beta:0], [-\beta:1:0], [1:-\beta:0]\}.$$

Moreover, φ_2, φ_5 and φ_{12} have no fixed points on $C_{a,0}$, since

$$\begin{aligned} Fix\big(\varphi_2, \mathbb{P}^2(\mathbb{C})\big) &= \{[1:0:0], [0:1:0], [0:0:1]\}, \\ Fix\big(\varphi_5, \mathbb{P}^2(\mathbb{C})\big) &= \{[\pm i:1:0], [0:0:1]\}, \\ Fix\big(\varphi_{12}, \mathbb{P}^2(\mathbb{C})\big) &= \{[\pm 1:1:0], [0:0:1]\}. \end{aligned}$$

Concerning φ_{14}, all the points $[i\lambda : \lambda : 1]$ with $\lambda \in \mathbb{C}$ are in $Fix\big(\varphi_{14}, \mathbb{P}^2(\mathbb{C})\big)$, in particular always we obtain 4 fixed points on $C_{a,0}$: for satisfying $C_{a,0}$ we obtain $(a+2)\lambda^4 + 1 = 0$. On the other hand,

$$Fix\big(\varphi_{10}, C_{a,0}\big) = \{[\alpha_i : \alpha_i : 1] : i = 1, 2, 3, 4\},$$

where α_i $(i = 1, 2, 3, 4)$ are the four roots of the equation $(a+2)\alpha^4 + 1 = 0$. The four roots are of the form $\alpha, -\alpha, i\alpha, -i\alpha$, such that $\alpha \neq 0$. Hence, we have only 1_8 orbit, namely,

$$Orb_G[\alpha:\alpha:1] = \{[\pm\alpha:\pm\alpha:1], [\pm i\alpha:\pm i\alpha:1]\}.$$

Consequently,

$$X(C_{a,0}) = Orb_G[\beta:1:0] \cup Orb_G[\alpha:\alpha:1] \cup Orb_G[0:\delta:1] \cup Orb_G[i\lambda:\lambda:1],$$

□

Proposition 7. *For the quartics $C_{a,0}$, we have*

$$X(C_{a,0}) \cap W_1(C_{a,0}) = Orb_G[\beta:1:0] \cup \begin{cases} Orb_G[0:\delta:1], & if\ a = 0 \\ Orb_G[\alpha:\alpha:1], & if\ a = 6 \\ Orb_G[i\lambda:\lambda:1], & if\ a = -6 \\ \phi, & otherwise \end{cases}$$

Proof. The Hessian curve H_F of $C_{a,0}$ is given by

$$H_F(x,y,z) = -144\left(\left(-12+a^2\right)x^2y^2 - 2a\left(x^4+y^4\right)\right)z^2.$$

Hence, $H_F(\beta,1,0)=0$, that is

$$Orb_G[\beta:1:0] \subset W_1(C_{a,0}).$$

Also, the resultant of $F(x,y,1)$ and $H_F(x,y,1)$ with respect to y is given by

$$Res\left(H_F(x,y,1),F(x,y,1);y\right) = constant\left(h(x)\right)^2,$$

where

$$h(x) := 4a^2 + \left(144-48a^2+3a^4\right)x^4 + \left(144-72a^2+9a^4\right)x^8.$$

Hence

$$Orb_G[0:\delta:1] \subset W_1(C_{a,0}) \text{ if and only if } a=0.$$

On the other hand,

$$H_F[\alpha:\alpha:1] = -144(a-6)(a+2)\alpha^4.$$

Consequently,

$$Orb_G[\alpha:\alpha:1] \subset W_1(C_{a,0}) \text{ if and only if } a=6.$$

Moreover,

$$H_F[i\lambda:\lambda:1] = 144(a+6)(a-2)\lambda^4.$$

Therefore,

$$Orb_G[i\lambda:\lambda:1] \subset W_1(C_{a,0}) \text{ if and only if } a=-6.$$

This completes the proof. ◻

Lemma 8. *If* $[\gamma:\zeta:1] \notin X(C_{a,0})$, *then*

$$Orb_G[\gamma:\zeta:1] = \{[\pm\gamma:\pm\zeta:1],[\pm i\gamma:\pm i\zeta:1],[\pm\zeta:\pm\gamma:1],[\pm i\zeta:\pm i\gamma:1]\}$$

and

$$|Orb_G[\gamma:\zeta:1]| = 16.$$

Proof. Since the elements of G have orders $1,2$ or 4, then stabilizers subgroups have orders $16,8$ or 4. Hence, if $[\gamma:\zeta:1] \notin X(C_{a,0})$, then $|G_{[\gamma:\zeta:1]}| \neq 8,4$. ◻

Lemma 9. *The intersection* $X(C_{a,0}) \cap W_1(C_{a,0})$ *consists of hyperflex points.*

Proof. It suffices to prove the result for the point $[\beta:1:0]$. Indeed, the tangent line to $C_{a,0}$ at $[\beta:1:0]$ is $L_{[\beta:1:0]}: x-\beta=0$. The resultant of $L_{[\beta:1:0]}$ and $C_{a,0}$ with respect to x is $Res\big(C_{a,0},L_{[\beta:1:0]};x\big)=z^4$. Thus, $L_{[\beta:1:0]}$ meets $C_{a,0}$ at $[\beta:1:0]$ with intersection multiplicity 4. Hence, $Orb_G[\beta:1:0]$ consists of hyperflex points. ◻ Similarly, for $Orb_G[0:\delta:1]$, $Orb_G[\alpha:\alpha:1]$ and $Orb_G[i\lambda:\lambda:1]$.

Lemma 10. *If* $a \neq 0,\pm 6$, *then there exists an ordinary flex point* $[\gamma:\zeta:1] \notin X(C_{a,0})$ *such that*

$$W_1(C_{a,0}) = Orb_G[\beta:1:0] \cup Orb_G[\gamma:\zeta:1].$$

Proof. It follows, by Proposition 7, that if $a \neq 0,\pm6$, then

$$X(C_{a,0}) \cap W_1(C_{a,0}) = Orb_G[\beta:1:0].$$

Moreover, by Lemma 9, $Orb_G[\beta:1:0]$ consists of hyperflex points. Finally, it follows, by Lemma 8, and the fact that the number of the 1-Weierstrass points is 24 counted with their weights, that there exists an ordinary flex point $[\gamma:\zeta:1] \notin X(C_{a,0})$ such that $W_1(C_{a,0}) = Orb_G[\beta:1:0] \cup Orb_G[\gamma:\zeta:1]$. ◻

Notation. Let O_r be the orbits classification, where O denotes the number of orbits and r the number of points in these orbits. For example 2_4 means: two orbits each of 4 points. Now, our main result for the case $b=0$ is the following.

Theorem 11. *For the quartics $C_{a,0}$, the number of the 1-Weierstrass points of the quartics $C_{a,0}$ together with their geometry are given by the following table. where*

TABLE 1. Number and Orbit Classification on C_0

	Ordinary flexes	Hyperflexes
$a=0,\pm 6$	**0**	**12** 1_4 1_8
Otherwise	**16** 1_{16}	**4** 1_4

the boldfaced numbers denote the number of points.

Proof. Recall that the number of the 1-Weierstrass points is 24 counted with their weights. Now, if $a=0$ or $a=\pm 6$, then, by Proposition 7 and Lemma 9, the intersection $X(C_{a,0})\cap W_1(C_{a,0})$ consists of 1_4 and 1_8 of hyperflex points. On the other hand, if $a\neq 0$ and $a\neq\pm 6$, then, again by Proposition 7 and Lemma 9, $X(C_{a,0})\cap W_1(C_{a,0})$ consists of 1_4 of hyperflex points and it follows, by Lemma 10, that there is 1_{16} of ordinary flex points. □

3.2. **Case $b\neq 0$.** A group action of order 8 on $C_{a,b}$ can be defined as follows. Let $G_1\cong D_4$ be the projective transformation group generated by the two elements σ_1 and τ_1 of orders 2 and 4 respectively, where

$$\sigma_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_1 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, computing the fixed points of the automorphisms of G_1 on $C_{a,b}$ and their corresponding orbits gives rise to the following result.

Lemma 12. *For the quartics $C_{a,b}$ such that $b\neq 0$, we have:*

$$\begin{aligned} X(C_{a,b}) &= Orb_{G_1}[\alpha_1:\alpha_1:1]\cup Orb_{G_1}[\alpha_3:\alpha_3:1]\cup Orb_{G_1}[\beta:0:1]\cup \\ &\quad \cup Orb_{G_1}[\frac{1}{\beta}:0:1]\cup Orb_{G_1}[\delta:1:0], \end{aligned}$$

where α_1 and α_3 are two distinct roots of the equation $(a+2)x^4+2bx^2+1=0$ such that $\alpha_1\neq\pm\alpha_3$, β is a root of the equation $x^4+bx^2+1=0$ and δ is a root of the equation $x^4+ax^2+1=0$.

Remark 2. Each of the above orbits satisfies $|Orb_{G_1}P|=4$. This means that

$$Orb_{G_1}[\alpha_{1,3}:\alpha_{1,3}:1]=\{[\pm\alpha_{1,3}:\pm\alpha_{1,3}:1]\},$$
$$Orb_{G_1}[\beta:0:1]=\{[\pm\beta:0:1],[0:\pm\beta:1]\},$$
$$Orb_{G_1}[\frac{1}{\beta}:0:1]=\{[\pm\frac{1}{\beta}:0:1],[0:\pm\frac{1}{\beta}:1]\},$$

$$Orb_{G_1}[\delta:1:0]=\{[\pm\delta:1:0],[1:\pm\delta:0]\}.$$

Moreover, if $[\zeta:\varepsilon:1]\notin X(C_{a,b})$, then $|Orb_{G_1}[\zeta:\varepsilon:1]|=8$.

Now, the intersections $X(C_{a,b})\cap W_1(C_{a,b})$ are given by the following three lemmas.

Lemma 13. *Let*

$$A:=Orb_{G_1}[\beta:0:1]\cup Orb_{G_1}[\frac{1}{\beta}:0:1],$$

then for the quartics $C_{a,b}$ such that $b\neq 0$, we have:

$$A\cap W_1(C_{a,b})\neq\phi \quad \text{if and only if} \quad a^2+b^2-ab^2=0.$$

Moreover,

$$A\cap W_1(C_{a,b})=\begin{cases} Orb_{G_1}[\beta:0:1], & \text{if } a=\frac{1}{2}(b^2-b\sqrt{b^2-4})\\ Orb_{G_1}[\frac{1}{\beta}:0:1], & \text{if } a=\frac{1}{2}(b^2+b\sqrt{b^2-4})\\ \phi, & \text{otherwise}\end{cases}$$

where $\beta=\dfrac{\sqrt{2}}{\sqrt{-b+\sqrt{b^2-4}}}$.

Proof. The Hessian curve $H_F(x,0,1)$ is given by

$$H_F(x,0,1)=\left(2b+12x^2\right)\left(2b+2ax^2\right)\left(12+2bx^2\right)+4bx\left(-8b^2x-8abx^3\right)$$

and so the resultant of $H_F(x,0,1)$ and $F(x,0,1)$ with respect to x is

$$Res\left(H_F(x,0,1),F(x,0,1);x\right)=const.\left(g(a,b)\right)^2,$$

where

$$g(a,b)=\left(b^2-4\right)^2\left(a^2+b^2-ab^2\right).$$

From the last equation, we have

$$A\cap W_1(C_{a,b})\neq\phi \quad \text{if and only if} \quad a^2+b^2-ab^2=0.$$

Moreover, substituting $a=\frac{1}{2}(b^2+b\sqrt{b^2-4})$ into $H_F(x,0,1)=0=F(x,0,1)$ yields the system

$$x^4+x^2+1=0,$$
$$\left(2+bx^2+\sqrt{-4+b^2}x^2\right)\left(-12x^2+b^2x^2-2b(1+x^4)\right)=0,$$

which has the two solutions $\pm\dfrac{\sqrt{2}}{\sqrt{-b-\sqrt{b^2-4}}}$.

Also, substituting $a=\frac{1}{2}(b^2-b\sqrt{b^2-4})$ into $H_F(x,0,1)=0=F(x,0,1)$ yields the system

$$x^4+x^2+1=0,$$
$$\left(2+bx^2-\sqrt{-4+b^2}x^2\right)\left(-12x^2+b^2x^2-2b(1+x^4)\right)=0,$$

which again has the two roots $\pm\dfrac{\sqrt{2}}{\sqrt{-b+\sqrt{b^2-4}}}$. □

Lemma 14. *For the quartics $C_{a,b}$, such that $b \neq 0$, the points of $Orb_{G_1}[\delta : 1 : 0]$ are not 1-Weierstrass.*

Proof. The Hessian H_F at the point $[x : 1 : 0]$ is given by

$$H_F(x, 1, 0) = 2b(1 + x^2)\left(-16a^2x^2 + \left(2a + 12x^2\right)\left(12 + 2ax^2\right)\right).$$

So, the resultant with $F(x, 1, 0)$ is given by

$$Res\left(H_F(x, 1, 0), F(x, 1, 0), x\right) = constant.\left(g(a, b)\right)^2,$$

where

$$g(a, b) = (a - 2)^3(a + 2)^2b^2 \neq 0.$$

We have done. □

Lemma 15. *Let*

$$B := Orb_{G_1}[\alpha_1 : \alpha_1 : 1] \cup Orb_{G_1}[\alpha_3 : \alpha_3 : 1],$$

then for the quartics $C_{a,b}$ such that $b \neq 0$, we have:

$$B \cap W_1(C_{a,b}) \neq \phi \quad \text{if and only if} \quad 36 - 12a + a^2 - 10b^2 + 3ab^2 = 0.$$

Moreover,

$$B \cap W_1(C_{a,b}) = \begin{cases} Orb_{G_1}[\alpha_1 : \alpha_1 : 1], & \textit{if } a = \frac{1}{2}(12 - 3b^2 - \sqrt{9b^2 - 32}) \\ Orb_{G_1}[\alpha_3 : \alpha_3 : 1], & \textit{if } a = \frac{1}{2}(12 - 3b^2 + \sqrt{9b^2 - 32}) \\ \phi, & \textit{otherwise} \end{cases}$$

where

$$\alpha_1 = \frac{1}{4}\sqrt{\sqrt{9b^2 - 32} - 3b} \ \textit{ and } \ \alpha_3 = \frac{i}{4}\sqrt{\sqrt{9b^2 - 32} + 3b}.$$

Proof. The resultant of $H_F(x, x, 1)$ and $F(x, x, 1)$ with respect to x is given by

$$Res\left(H_F(x, x, 1), F(x, x, 1), x\right) = constant.\left(g(a, b)\right)^2,$$

where

$$g(a, b) = (a + 2)\left(2 + a - b^2\right)^2\left(36 - 12a + a^2 - 10b^2 + 3a^2\right).$$

Thus,

$$B \cap W_1(C_{a,b}) \neq \phi \text{ if and only if } 36 - 12a + a^2 - 10b^2 + 3ab^2 = 0.$$

Moreover, we have

$$F(x, x, 1) = (2 + a)x^4 + 2bx^2 + 1,$$

$$H_F(x, x, 1) = -64b^2x^2\left(b - (-6 + a)x^2\right) + 16(3 + bx^2)\left(-4a^2x^4 + \left(b + (6 + a)x^2\right)^2\right).$$

Hence, substituting $a = \frac{1}{2}(12 - 3b^2 \pm \sqrt{9b^2 - 32})$ in the above system and solving, the required result is obtained. □

Notation. We denote by Γ the set of common zeros of the equations

$$a^2 + b^2 - ab^2 = 0 \quad \textit{and} \quad 36 - 12a + a^2 - 10b^2 + 3ab^2 = 0.$$

Now, the next result follows from of the previous three lemmas.

Proposition 16. *If $b \neq 0$, we have the following two cases for $C_{a,b}$:*
(i) If $(a,b) \in \Gamma$, *then*

$$X(C_{a,b}) \cap W_1(C_{a,b}) = Orb_{G_1}[\alpha : \alpha : 1] \cup Orb_{G_1}[\beta_0 : 0 : 1],$$

where $\alpha \in \{\alpha_1, \alpha_3\}$, $\beta_0 \in \{\beta, \dfrac{1}{\beta}\}$.

(ii) If $(a,b) \notin \Gamma$, *then*

$$X(C_{a,b}) \cap W_1(C_{a,b}) = \begin{cases} Orb_{G_1}[\beta : 0 : 1], & \text{if } a = \frac{1}{2}(b^2 - b\sqrt{b^2-4}) \\ Orb_{G_1}[\frac{1}{\beta} : 0 : 1], & \text{if } a = \frac{1}{2}(b^2 + b\sqrt{b^2-4}) \\ Orb_{G_1}[\alpha_1 : \alpha_1 : 1], & \text{if } a = \frac{1}{2}(12 - 3b^2 - \sqrt{9b^2-32}) \\ Orb_{G_1}[\alpha_3 : \alpha_3 : 1], & \text{if } a = \frac{1}{2}(12 - 3b^2 + \sqrt{9b^2-32}) \\ \phi, & \text{otherwise} \end{cases}$$

Lemma 17. *If $X(C_{a,b}) \cap W_1(C_{a,b}) \neq \phi$, then the intersection points are necessarily hyperflex points.*

Now, our main result for the case $b \neq 0$ is the following.

Theorem 18. *For the quartics $C_{a,b}$ such that $b \neq 0$, the number of the 1-Weierstrass points together with their geometry are given in the following tables.*

Number Classification on $C_{a,b}$

	Ordinary flexes	Hyperflexes
$(a,b) \in \Gamma$	**8**	**8**
$P(a,b) = 0,\ Q(a,b) \neq 0$	**0** (Resp. **16**)	**12** (Resp. **4**)
$P(a,b) \neq 0,\ Q(a,b) = 0$	**0** (Resp. **16**)	**12** (Resp. **4**)
$P(a,b) \neq 0,\ Q(a,b) \neq 0$	**24** (Resp. **8**)	**0** (Resp. **8**)

Orbit Classification on $C_{a,b}$

	Ordinary flexes	*Hyperflexes*
$(a,b) \in \Gamma$	1_8	2_4
$P(a,b) = 0,\ Q(a,b) \neq 0$	None (Resp. 2_8)	1_4, 1_8 (Resp. 1_4)
$P(a,b) \neq 0,\ Q(a,b) = 0$	None (Resp. 2_8)	1_4, 1_8 (Resp. 1_4)
$P(a,b)Q(a,b) \neq 0$	3_8 (Resp. 1_8)	None (Resp. 1_8)

where

$$P(a,b) := a^2 + b^2 - ab^2, \qquad Q(a,b) := 36 - 12a + a^2 - 10b^2 + 3ab^2.$$

Proof. Recall that the number of the 1-Weierstrass points is 24 counted with their weights. Now, if $(a,b) \in \Gamma$, then by Proposition 16 (i) and Lemma 17, the intersections $X(C_{a,b}) \cap W_1(C_{a,b})$ consists of 2_4 of hyperflex points. Consequently, by Remark 2, there exist 1_8 of ordinary flexes.

If $P(a,b)=0,\ \ Q(a,b)\neq 0$ or $P(a,b)\neq 0,\ \ Q(a,b)=0$, then by Proposition 16 (ii) and Lemma 17, the intersections $X(C_{a,b})\cap W_1(C_{a,b})$ consists of 1_4 of hyperflex points. Consequently, by Remark 2, there exist either 1_8 of hyperflex points or 2_8 of ordinary flexes.

If $P(a,b)Q(a,b)\neq 0$, then by Proposition 16 (ii), $X(C_{a,b})\cap W_1(C_{a,b})=\phi$. Hence, $W_1\big(C_{a,b}\big)$ consists of orbits each of 8 points. Consequently, we have two cases: either 3_8 of ordinary flexes or 1_8 of ordinary flexes together with 1_8 of hyperflex points. This completes the proof. □

4. Examples

This section is devoted to illustrate through examples the cases mentioned in Theorem 18. It should be noted that, under a given condition, more than one case arise. So, it is convenient to investigate whether these cases can occur.

- For the case $P(a,b)=0$ and $Q(a,b)\neq 0$, we give the following examples:

  (1): If $a=\dfrac{6}{5},\ b=\dfrac{6}{\sqrt{5}}$, then we have

  $$W_1(C_{a,b})=Orb_{G_1}[1.495...i:0:1]\cup Orb_{G_1}[\zeta_5:\epsilon_5:1],$$

  where

  $$\zeta_5:=0.748...(1+i),\ \ \epsilon_5:=0.748...(1-i).$$

  That is, $C_{a,b}$ has 12 hyperflex points classified into 1_4 and 1_8.

  (2): If $a=3,\ b=\dfrac{3}{\sqrt{2}}$, then we have

  $$W_1(C_{a,b})=Orb_{G_1}[0.841...i:0:1]\cup Orb_{G_1}[\zeta_3:\epsilon_3:1]\cup Orb_{G_1}[\zeta_4:\epsilon_4:1],$$

  where

  $$\begin{aligned}\zeta_3 &:= 0.089...-0.454...i,\ \ \epsilon_3:=0.188...+0.947...i,\\ \zeta_4 &:= 0.089...+0.454...i,\ \ \epsilon_4:=0.188...-0.947...i.\end{aligned}$$

  Hence, $C_{a,b}$ has 20 flex points classified into 1_4 of hyperflex points and 2_8 of ordinary flexes.

- For the case $P(a,b)\neq 0$ and $Q(a,b)=0$, we get the following two examples:

  (3): If $a=3,\ b=3$, then $C_{a,b}$ has 12 hyperflex points [1, 18].

  (4): If $a=0,\ b=3\sqrt{\dfrac{2}{5}}$, then we have

  $$W_1(C_{a,b})=Orb_{G_1}[0.562...i:0.562...i:1]\cup Orb_{G_1}[\zeta_1:\epsilon_1:1]\cup Orb_{G_1}[\zeta_2:\epsilon_2:1],$$

  where

  $$\begin{aligned}\zeta_1 &:= 0.204...-1.151...i,\ \ \epsilon_1=0.302...-0.269...i,\\ \zeta_2 &:= 0.204...+1.151...i,\ \ \epsilon_2=0.302...+0.269...i.\end{aligned}$$

  Consequently, $C_{a,b}$ has 20 flex points classified into 1_4 of hyperflex points and 2_8 of ordinary flexes.

- For the case $P(a,b)Q(a,b)\neq 0$, we have the following examples:

  (5): If $a=4,\ b=4$, then $C_{a,b}$ has 24 ordinary flex points [1, 18].

(6): If $a = -5,\ b = 1$, then we have

$$W_1(C_{a,b}) = Orb_{G_1}\left[0.44...(1+i) : 0.44...(i-1) : 1\right] \cup Orb_{G_1}\left[0.93...i : 2.27...i : 1\right]$$

Consequently, $C_{a,b}$ has 16 flex points classified into 2_8, one consists of hyperflex points and the other consists of ordinary flexes.

Concluding remarks.

We conclude the paper by the following remarks and comments.

- The computations included in the present paper have been performed by the use of MATHEMATICA program. The source code files are available.
- The classification of the 1-Weierstrass points of *Kuribayashi quartics with one parameter,* treated in [1, 18] is a particular case of our main theorems. In fact, letting $a = b$, one gets the following table.

  Number classification of 1-Weierstrass points of $C_{a,a}$

	Ordinary flex	Hyperflex
$a = 0, 3$	0	12
Otherwise	24	0

- The geometry of the 1-Weierstrass points of the one parameter quartic family defined by the equation

$$X^4 + Y^4 + Z^4 + b\big(X^2 + Y^2\big)Z^2 = 0, \qquad (b^2 - 4)(b^2 - 2) \neq 0,$$

  is a particular case by substituting $a = 0$ in *Theorem 2.6* and *Theorem 2.14* to get the following table.

  Number and Orbit Classification on $C_{a,0}$

	Ordinary flexes	*Hyperflexes*
$b = 0$	**0**	**12** 1_4 1_8
$b^2 = \frac{18}{5}$	**16** 2_8	**4** 1_4
Otherwise	**24** (Resp. **8**) 3_8 (Resp. 1_8)	**0** (Resp. **8**) None (Resp. 1_8)

The following examples illustrates that under the condition $b^2 \neq 0,\ \frac{18}{5}$, both cases mentioned above can occur. For instance, let $b = \sqrt{6}$, then $C_{0,b}$ has 16 flex points or, equivalently,

$$W_1\big(C_{0,b}\big) = Orb_{G_1}\left[-0.6...(1+i) : 0.6...(1-i) : 1\right] \cup Orb_{G_1}\left[0.2...i : 0.2...i : 1\right].$$

On the other hand, let $b = \sqrt{5}$, then $C_{0,b}$ has 24 flex points or, equivalently,

$$W_1\big(C_{0,b}\big) = Orb_{G_1}[\nu_1 : \lambda_1 : 1] \ \cup\ Orb_{G_1}[\nu_2 : \lambda_2 : 1] \ \cup\ Orb_{G_1}[\nu_3 : \lambda_3 : 1],$$

where

$$\begin{aligned}
\nu_1 : &= 0.524132... + 0.316563...i, \ \ \lambda_1 := -0.417502... + 0.812472...i,\\
\nu_2 : &= 0.524132... - 0.316563...i, \ \ \lambda_2 := 0.417502... + 0.812472...i,\\
\nu_3 : &= -0.410813...i, \ \ \lambda_3 := -1.37547...i.
\end{aligned}$$

- The main theorems constitute a motivation to solve more general problems. One of these problems is the investigation of the geometry of the 1-Weierstrass points of *Kuribayashi quartics with three parameters family* defined by the equation

$$C_{a,b,c} : X^4 + Y^4 + Z^4 + aX^2Y^2 + bX^2Z^2 + cY^2Z^2 = 0.$$

  Hayakawa [14] studied the conditions under which the number of the Weierstrass points of this family is exactly 12 or < 24. However, this problem will be the object of a forthcoming work.
  Another problem is the investigation of the geometry of higher order and multiple Weierstrass points of $C_{a,b}$ to generalize the classification of the 2-Weierstrass points of *Kuribayashi quartics with one parameter family* [1, 2].
- The technique used in this paper is completely different from that followed by Hayakawa [15]. Our technique consists of dividing the quartics by group actions into finite orbits and investigate the geometry of these orbits. The results obtained are more informative than those obtained by Hayakawa.
- There are extra geometrical structures that can be used to make the classification more transparent. Examples, include the Eisenbud-Harris theory on monodromy and limits of Weierstrass points [9, 10, 11], associated del Pezzo surfaces of degree two (arising as double covers of the plane branched along the quartic) , and $K3$ surfaces (arising as cyclic degree-four covers) [5, 6]. However, this extension will be the object of a forthcoming work.

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