CLASSIFICATION OF 1-WEIERSTRASS POINTS ON KURIBAYASHI QUARTICS, I (WITH TWO PARAMETERS)

M. A. SALEEM AND E. E. BADR

ABSTRACT. In this paper, we classify the 1-Weierstrass points of the Kuribayashi quartic curves with two parameters $a$ and $b$ defined by the equation

$$C_{a,b} : X^4 + Y^4 + Z^4 + aX^2Y^2 + b(X^2 + Y^2)Z^2 = 0,$$

such that $(a^2 - 4)(b^2 - 4)(a^2 - 1)(b^2 - a - 2) \neq 0$. Furthermore, the geometry of these points is investigated.

1. Introduction

The Weierstrass points of an algebraic curve, which can be defined over some extension of the base field, form a distinguished set of points of the curve having the property of being geometrically intrinsic. Curves of genus 0 or 1 have no Weierstrass points, and for hyperelliptic curves, the Weierstrass points can easily be characterized as the ramification points under the hyperelliptic involution, which generate the 2-torsion subgroup of the Jacobian. For non-hyperelliptic curves of genus 3, which admit a plane quartic model, there is an extensive literature on the topic. For instance, Weierstrass points and automorphism groups of Riemann surfaces of genus 3 are studied in [1, 10, 17]. In 1983, Vermeulen [23] constructed a stratification of the moduli space $M_3$ of curves of genus 3 in terms of the number of hyperflexes and their geometric configuration. Similar results were obtained independently around the same time by Lugert [19]. Subsequent work on curves with large number of hyperflexes has been done by Martine Girard and his collaborators. In [20], they described the group generated by the Weierstrass points in the Jacobian of the curve

$$X^4 + Y^4 + Z^4 + 3(X^2Y^2 + X^2Z^2 + Y^2Z^2) = 0.$$

This curve is the only curve of genus 3, apart from the fourth Fermat curve, possessing exactly twelve Weierstrass points. In [21], they determined the group generated by Weierstrass points for all smooth quartics with eight hyperflexes or more. Also,

1991 Mathematics Subject Classification. Primary 14H55, 14R20; Secondary 14H37, 14H45.

Key words and phrases. Kuribayashi quartics, 1-Weierstrass points, Flexes, Riemann surfaces, Automorphisms group.

they got bounds on both the rank and the torsion part of this group for a generic quartic having a fixed number of hyperflexes in the moduli space $M_3$ of curves of genus 3.

Let $C_a$ be the smooth plane quartic curves defined by the equation 

$$C_a : X^4 + Y^4 + Z^4 + a(X^2Y^2 + X^2Z^2 + Y^2Z^2) = 0, \quad a \neq -1, \pm 2.$$ 

These types of quartic curves are called Kuribayashi quartics with one parameter. Weierstrass points and automorphism groups of $C_a$ are studied in [18]. Kuribayashi and his students used the Wronskian method to classify the number of the 1-Weierstrass points of $C_a$. Alwaleed [1, 2] used the Wronskian method together with the $S_4$ action on $C_a$ to classify the number and investigate the geometry of the 2-Weierstrass points of this family.

Let $C_{a,b}$ be the smooth plane quartic curves defined by the equation 

$$C_{a,b} : X^4 + Y^4 + Z^4 + aX^2Y^2 + b(X^2 + Y^2)Z^2 = 0,$$

where $a$ and $b$ are two parameters such that $(a^2-4)(b^2-4)(a^2-1)(b^2-a-2) \neq 0$. We call these quartic curves Kuribayashi quartics with two parameter. The Kuribayashi pencil is spanned by quartic curves with maximal number of hyperflexes. Hayakawa [15] investigated the conditions under which the number of the Weierstrass points of $C_{a,b}$ is 12 or 16.

In this paper, we use finite group actions on the Riemann surfaces $C_{a,b}$ to investigate the number of the 1-Weierstrass points of this family. Furthermore, we study the geometry of these points.

The present paper is organized in the following manner. In section 2, we present some preliminaries concerning the basic concepts that will be used throughout the work [13, 22]. In section 3, we establish our main results (Theorems 11, 18) that concern with the classification of the number of the 1-Weierstrass points of the quartic curves $C_{a,b}$ together with their geometry. In section 4, we illustrate, through examples, the cases mentioned in the main results. Finally, we conclude the paper with some remarks, comments and related problems.

2. Preliminaries

2.1. q-Weierstrass points. Let $C$ be a smooth projective plane curve of genus $g := \frac{(d-1)(d-2)}{2} \geq 2$ and let $D$ be a divisor on $C$ with $\text{dim}(D) = r \geq 0$. We denote by $L(D)$ the $\mathbb{C}$-vector space of meromorphic functions $f$ such that $\text{div}(f) + D \geq 0$ and by $l(D)$ the dimension of $L(D)$ over $\mathbb{C}$. Then, the notion of D-Weierstrass points [22] can be defined in the following way:

**Definition 1.** Let $p \in C$. If $n$ is a positive integer such that 

$$l(D - (n-1)p) > l(D - np),$$

we call the integer $n$ a $D$-gap number at $p$.

**Lemma 1.** [22] Let $p \in C$, then there are exactly $r+1$ $D$-gap numbers $\{n_1, n_2, ..., n_{r+1}\}$ such that $n_1 < n_2 < ... < n_{r+1}$. The sequence $\{n_1, n_2, ..., n_{r+1}\}$ is called the $D$-gap sequence at $p$. 
Definition 2. The integer \( \omega_D(p) := \sum_{i=1}^{r+1} (n_i - 1) \) is called \( D \)-weight at \( p \). If \( \omega_D(p) > 0 \), we call the point \( p \) a \( D \)-Weierstrass point on \( C \). In particular, for the canonical divisor \( K \), the \( qK \)-Weierstrass points \( (q \geq 1) \) are called \( q \)-Weierstrass points and the \( qK \)-weight is called \( q \)-weight, denoted by \( \omega(q)(p) \).

Definition 3. [12] A point \( p \) on a smooth plane curve \( C \) is said to be a flex point if the tangent line \( L_p \) meets \( C \) at \( p \) with contact order \( I_p(C, L_p) \) at least three. We say that \( p \) is \( i \)-flex, if \( I_p(C, L_p) - 2 = i \). The positive integer \( i \) is called the flex order of \( p \).

Lemma 2. [13] Let \( C : F(X,Y,Z) = 0 \) be a smooth projective plane curve. A point \( p \) on \( C \) is a flex point if, and only if, \( H_F(p) = 0 \), where \( H_F \) is the Hessian curve of \( C \) defined by

\[
H_F := \det \begin{pmatrix} F_{XX} & F_{XY} & F_{XZ} \\ F_{YX} & F_{YY} & F_{YZ} \\ F_{ZX} & F_{ZY} & F_{ZZ} \end{pmatrix}.
\]

Lemma 3. [23] Let \( C \) be a smooth projective plane quartic curve, the 1-Weierstrass points on \( C \) are nothing but flexes and divided into two types: ordinary flex and hyperflex points. Moreover, we have

<table>
<thead>
<tr>
<th>( \omega^{(1)}(p) )</th>
<th>1-gap Sequence</th>
<th>Geometry</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 [1, 2, 4]</td>
<td>ordinary flex</td>
<td></td>
</tr>
<tr>
<td>2 [1, 2, 5]</td>
<td>hyperflex</td>
<td></td>
</tr>
</tbody>
</table>

Lemma 4. [12] [22] Let \( C \) be a smooth projective plane curve of genus \( g \). The number of \( q \)-Weierstrass points \( N^{(q)}(C) \), counted with their \( q \)-weights, is given by

\[
N^{(q)}(C) = \begin{cases} g(g^2 - 1), & \text{if } q = 1 \\ (2q - 1)^2(g - 1)^2g, & \text{if } q \geq 2. \end{cases}
\]

In particular, for smooth projective plane quartics (i.e. \( g = 3 \)), the number of 1-Weierstrass points is 24 counted with their weights.

Let \( W^{(q)}(C) \) be the set of \( q \)-Weierstrass points on \( C \) and \( G^{(q)}(p) \) the \( q \)-gap sequence at the point \( p \in C \).

Lemma 5. [1] Let \( \tau \) be an automorphism on \( C \), then we have

\[
\tau(W^{(q)}(C)) = W^{(q)}(C) \quad \text{and} \quad G^{(q)}(\tau(p)) = G^{(q)}(p).
\]

2.2. Group Action on Riemann Surfaces [12].

Definition 4. An action of a finite group \( G \) on a Riemann surface \( C \) is a map

\[
\cdot : G \times C \longrightarrow C : (g, p) \mapsto g \cdot p
\]

such that: \( (gh) \cdot p = g \cdot (h \cdot p) \) and \( e \cdot p = p \), for all \( g, h \in G \) and \( p \in C \). Here \( e \) denotes the identity element of \( G \).

Definition 5. The orbit of a point \( p \in C \) is the set \( \text{Orb}_G(p) := \{ g \cdot p : g \in G \} \).

Definition 6. The stabilizer of a point \( p \in C \) is the subgroup

\[
G_p := \{ g \in G : g \cdot p = p \}.
\]

It is often called the isotropy subgroup of \( p \).
Remark 1. It is well known that $G_p$ is cyclic and points in the same orbit have conjugate stabilizers; Indeed, $G_{g,p} = gG_pg^{-1}$. Moreover $|Orb_G(p)| |G_p| = |G|$.

Notation. [1] The set of points $p \in C$ such that $|G_p| > 1$ is denoted by $X(C)$. Also $X_i(C) := \{p \in C : |G_p| = i\}$.

3. Main results

In what follows, we investigate the number of the $1$-Weierstrass points of Kuribayashi quartic curves with two parameter points is studied. We have two cases, either $b = 0$ or $b \neq 0$. In the following we treat each of these cases.

3.1. Case $b = 0$. In this case, we consider the one parameter family of smooth plane quartics $C_{a,0}$ defined by

$$C_{a,0} : X^4 + Y^4 + Z^4 + aX^2Y^2 + b(X^2 + Y^2)Z^2,$$

such that $(a^2 - 4)(b^2 - 4)(a^2 - 1)(b^2 - a - 2) \neq 0$. Moreover, the geometry of these points is studied. We have two cases, either $b = 0$ or $b \neq 0$. In the following we treat each of these cases.

A group action of order 16 on $C_{a,0}$ can be defined as follows. Let $G$ be the projective transformation group generated by the three elements $\sigma$, $\tau$, $\rho$ of orders 2, 4, 4 respectively, where

$$\sigma := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau := \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It has been shown by Francesc [4] that, $G \cong C_4 \oplus (C_2 \times C_2)$, where $C_m$ denotes the cyclic group of order $m$ and $C_4 \oplus (C_2 \times C_2)$ denotes the group $(16, 13)$ in GAP library of small groups which is a group in $Ext^1(C_2 \times C_2, C_4)$. Now, computing the fixed points of the automorphisms of $G$ on $C_{a,0}$ and their corresponding orbits gives rise to the following result.

Lemma 6. For the quartics $C_{a,0}$, we have:

$$X(C_{a,0}) = X_2(C_{a,0}) \cup X_4(C_{a,0}).$$

Moreover,

$$X(C_{a,0}) = Orb_G[\beta : 1 : 0] \cup Orb_G[\alpha : 1 : 0] \cup Orb_G[\delta : 1] \cup Orb_G[i\lambda : 1],$$

where $\beta$ is a root of the equation $x^4 + ax^2 + 1 = 0$, $\alpha$ is a root of the equation $(a + 2)x^4 + 1 = 0$, $\delta$ is a root of the equation $x^4 + 1 = 0$, and $\lambda$ is a root of the equation $(a + 2)x^4 + 1 = 0$.

Proof. Since non-trivial elements of $G$ have orders 2 or 4, then we have

$$X(C_{a,0}) = X_2(C_{a,0}) \cup X_4(C_{a,0}).$$

Since points in the same orbit have conjugate stabilizers, it suffices to compute the fixed points of the automorphisms $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_5$, $\varphi_7$, $\varphi_{10}$, $\varphi_{12}$ and $\varphi_{14}$ on the quartics $C_{a,0}$. Indeed, we have

$$Fix(\varphi_1, C_{a,0}) = \{[0 : \delta_i : 1] : i = 1, 2, 3, 4\},$$

where $[0 : \delta_i : 1]$ are the fixed points of the automorphisms $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_5$, $\varphi_7$, $\varphi_{10}$, $\varphi_{12}$ and $\varphi_{14}$ on the quartics $C_{a,0}$. In this case, we consider the one parameter family of smooth plane quartics $C_{a,0}$ defined by
where \( \delta_i (i = 1, 2, 3, 4) \) are the roots of the equation \( \delta^4 + 1 = 0 \). In other words,

\[
\begin{align*}
\delta_1 &= \frac{1}{\sqrt{2}}(1 + i), \\
\delta_2 &= -\frac{1}{\sqrt{2}}(1 + i), \\
\delta_3 &= \frac{1}{\sqrt{2}}(1 - i), \\
\delta_4 &= -\frac{1}{\sqrt{2}}(1 - i).
\end{align*}
\]

The four roots are of the form \( \delta, -\delta, -i\delta, i\delta \) which gives rise to \( 1_8 \) orbit, namely,

\[
Orb_G[0 : \delta : 1] = \{ [0 : \pm\delta : 1], [\pm\delta : 0 : 1], [0 : \pm i\delta : 1], [\pm i\delta : 0 : 1] \}.
\]

On the other hand,

\[
Fix(\varphi_3, C_{a,0}) = \{ [\beta_i : 1 : 0] : i = 1, 2, 3, 4 \} = Fix(\varphi_7, C_{a,0}),
\]

where \( \beta_i (i = 1, 2, 3, 4) \) are the roots of the equation \( \beta^4 + a\beta^2 + 1 = 0 \). The four roots are of the form

\[
\beta, -\beta, \frac{1}{\beta}, -\frac{1}{\beta},
\]

such that \( \beta \neq 0, \pm 1, \pm i \), which in turns gives just \( 1_4 \) orbit, namely,

\[
Orb_G[\beta : 1 : 0] = \{ [\beta : 1 : 0], [1 : \beta : 0], [-\beta : 1 : 0], [1 : -\beta : 0] \}.
\]

Moreover, \( \varphi_2, \varphi_5 \) and \( \varphi_{12} \) have no fixed points on \( C_{a,0} \), since

\[
\begin{align*}
Fix(\varphi_2, \mathbb{P}^2(\mathbb{C})) &= \{ [1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1] \}, \\
Fix(\varphi_5, \mathbb{P}^2(\mathbb{C})) &= \{ [\pm i : 1 : 0], [0 : 0 : 1] \}, \\
Fix(\varphi_{12}, \mathbb{P}^2(\mathbb{C})) &= \{ [\pm 1 : 1 : 0], [0 : 0 : 1] \}.
\end{align*}
\]

Concerning \( \varphi_{14} \), all the points \( [i\lambda : \lambda : 1] \) with \( \lambda \in \mathbb{C} \) are in \( Fix(\varphi_{14}, \mathbb{P}^2(\mathbb{C})) \), in particular always we obtain 4 fixed points on \( C_{a,0} \): for satisfying \( C_{a,0} \) we obtain \( (a + 2)\lambda^4 + 1 = 0 \). On the other hand,

\[
Fix(\varphi_{10}, C_{a,0}) = \{ [\alpha_i : \alpha_i : 1] : i = 1, 2, 3, 4 \},
\]

where \( \alpha_i (i = 1, 2, 3, 4) \) are the four roots of the equation \( (a + 2)\alpha^4 + 1 = 0 \). The four roots are of the form \( \alpha, -\alpha, i\alpha, -i\alpha \), such that \( \alpha \neq 0 \). Hence, we have only \( 1_8 \) orbit, namely,

\[
Orb_G[\alpha : \alpha : 1] = \{ [\pm\alpha : \pm\alpha : 1], [\pm i\alpha : \pm i\alpha : 1] \}.
\]

Consequently,

\[
X(C_{a,0}) = Orb_G[\beta : 1 : 0] \cup Orb_G[\alpha : \alpha : 1] \cup Orb_G[0 : \delta : 1] \cup Orb_G[i\lambda : \lambda : 1],
\]

\[
\square
\]

**Proposition 7.** For the quartics \( C_{a,0} \), we have

\[
X(C_{a,0}) \cap W_1(C_{a,0}) = Orb_G[\beta : 1 : 0] \cup
\begin{cases}
Orb_G[0 : \delta : 1], & \text{if } a = 0 \\
Orb_G[\alpha : \alpha : 1], & \text{if } a = 6 \\
Orb_G[i\lambda : \lambda : 1], & \text{if } a = -6 \\
\phi, & \text{otherwise}
\end{cases}
\]
Proof. The Hessian curve $H_F$ of $C_{a,0}$ is given by

$$H_F(x, y, z) = -144 \left( -12 + a^2 \right) x^2 y^2 - 2a \left( x^4 + y^4 \right) z^2.$$ 

Hence, $H_F(\beta, 1, 0) = 0$, that is

$$Orb_G[\beta : 1 : 0] \subset W_1(C_{a,0}).$$

Also, the resultant of $F(x, y, 1)$ and $H_F(x, y, 1)$ with respect to $y$ is given by

$$Res(H_F(x, y, 1), F(x, y, 1); y) = constant \left( h(x) \right)^2,$$

where

$$h(x) := 4a^2 + \left( 144 - 48a^2 + 3a^4 \right) x^4 + \left( 144 - 72a^2 + 9a^4 \right) x^8.$$ 

Hence

$$Orb_G[0 : \delta : 1] \subset W_1(C_{a,0}) \text{ if and only if } a = 0.$$ 

On the other hand,

$$H_F[\alpha : \alpha : 1] = -144(a - 6)(a + 2)\alpha^4.$$ 

Consequently,

$$Orb_G[\alpha : \alpha : 1] \subset W_1(C_{a,0}) \text{ if and only if } a = 6.$$ 

Moreover,

$$H_F[i\lambda : \lambda : 1] = 144(a + 6)(a - 2)\lambda^4.$$ 

Therefore,

$$Orb_G[i\lambda : \lambda : 1] \subset W_1(C_{a,0}) \text{ if and only if } a = -6.$$ 

This completes the proof. \hfill \square

**Lemma 8.** If $[\gamma : \zeta : 1] \notin X(C_{a,0})$, then

$$Orb_G[\gamma : \zeta : 1] = \{ [\pm\gamma : \pm\zeta : 1], [\pm i\gamma : \pm i\zeta : 1], [\pm \gamma : \pm \zeta : 1], [\pm i\gamma : \pm i\gamma : 1] \}$$

and

$$|Orb_G[\gamma : \zeta : 1]| = 16.$$ 

**Proof.** Since the elements of $G$ have orders 1, 2 or 4, then stabilizers subgroups have orders 16, 8 or 4. Hence, if $[\gamma : \zeta : 1] \notin X(C_{a,0})$, then $|G_{[\gamma : \zeta : 1]}| \neq 8, 4$. \hfill \square

**Lemma 9.** The intersection $X(C_{a,0}) \cap W_1(C_{a,0})$ consists of hyperflex points.

**Proof.** It suffices to prove the result for the point $[\beta : 1 : 0]$. Indeed, the tangent line to $C_{a,0}$ at $[\beta : 1 : 0]$ is $L_{[\beta : 1 : 0]} : x - \beta = 0$. The resultant of $L_{[\beta : 1 : 0]}$ and $C_{a,0}$ with respect to $x$ is $Res(C_{a,0}, L_{[\beta : 1 : 0]}; x) = z^4$. Thus, $L_{[\beta : 1 : 0]}$ meets $C_{a,0}$ at $[\beta : 1 : 0]$ with intersection multiplicity 4. Hence, $Orb_G[\beta : 1 : 0]$ consists of hyperflex points. Similarly, for $Orb_G[0 : \delta : 1]$, $Orb_G[i\alpha : \alpha : 1]$ and $Orb_G[i\lambda : \lambda : 1]$.

**Lemma 10.** If $a \neq 0, \pm 6$, then there exists an ordinary flex point $[\gamma : \zeta : 1] \notin X(C_{a,0})$ such that

$$W_1(C_{a,0}) = Orb_G[\beta : 1 : 0] \cup Orb_G[\gamma : \zeta : 1].$$

**Proof.** It follows, by Proposition [2] that if $a \neq 0, \pm 6$, then

$$X(C_{a,0}) \cap W_1(C_{a,0}) = Orb_G[\beta : 1 : 0].$$

Moreover, by Lemma [3] $Orb_G[\beta : 1 : 0]$ consists of hyperflex points. Finally, it follows, by Lemma [3] and the fact that the number of the 1-Weierstrass points is 24 counted with their weights, that there exists an ordinary flex point $[\gamma : \zeta : 1] \notin X(C_{a,0})$ such that $W_1(C_{a,0}) = Orb_G[\beta : 1 : 0] \cup Orb_G[\gamma : \zeta : 1]$. \hfill \square
Notation. Let $O_r$ be the orbits classification, where $O$ denotes the number of orbits and $r$ the number of points in these orbits. For example $2_4$ means: two orbits each of 4 points. Now, our main result for the case $b = 0$ is the following.

**Theorem 11.** For the quartics $C_{a,0}$, the number of the 1-Weierstrass points of the quartics $C_{a,0}$ together with their geometry are given by the following table. where

**Table 1. Number and Orbit Classification on $C_0$**

<table>
<thead>
<tr>
<th></th>
<th>Ordinary flexes</th>
<th>Hyperflexes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 0, \pm 6$</td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1_4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1_8</td>
</tr>
<tr>
<td>Otherwise</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>1_16</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1_4</td>
<td></td>
</tr>
</tbody>
</table>

*the boldfaced numbers denote the number of points.*

Proof. Recall that the number of the 1-Weierstrass points is 24 counted with their weights. Now, if $a = 0$ or $a = \pm 6$, then, by Proposition 7 and Lemma 9 the intersection $X(C_{a,0}) \cap W_1(C_{a,0})$ consists of 1_4 and 1_8 of hyperflex points. On the other hand, if $a \neq 0$ and $a \neq \pm 6$, then, again by Proposition 7 and Lemma 9 $X(C_{a,0}) \cap W_1(C_{a,0})$ consists of 1_4 of hyperflex points and it follows, by Lemma 10, that there is 1_16 of ordinary flex points.

3.2. Case $b \neq 0$. A group action of order 8 on $C_{a,b}$ can be defined as follows. Let $G_1 \cong D_4$ be the projective transformation group generated by the two elements $\sigma_1$ and $\tau_1$ of orders 2 and 4 respectively, where

$$\sigma_1 := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tau_1 := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, computing the fixed points of the automorphisms of $G_1$ on $C_{a,b}$ and their corresponding orbits gives rise to the following result.

**Lemma 12.** For the quartics $C_{a,b}$ such that $b \neq 0$, we have:

$$X(C_{a,b}) = Orb_{G_1}[\alpha_1 : \alpha_1 : 1] \cup Orb_{G_1}[\alpha_3 : \alpha_3 : 1] \cup Orb_{G_1}[\beta : 0 : 1] \cup Orb_{G_1}[\frac{1}{\beta} : 0 : 1] \cup Orb_{G_1}[\delta : 1 : 0],$$

where $\alpha_1$ and $\alpha_3$ are two distinct roots of the equation $(a + 2)x^4 + 2bx^2 + 1 = 0$ such that $\alpha_1 \neq \pm \alpha_3$, $\beta$ is a root of the equation $x^4 + bx^2 + 1 = 0$ and $\delta$ is a root of the equation $x^4 + ax^2 + 1 = 0$.

**Remark 2.** Each of the above orbits satisfies $|Orb_{G_1} P| = 4$. This means that

$$Orb_{G_1}[\alpha_{1,3} : \alpha_{1,3} : 1] = \{[\pm \alpha_{1,3} : \pm \alpha_{1,3} : 1] \},$$

$$Orb_{G_1}[\beta : 0 : 1] = \{[\pm \beta : 0 : 1], [0 : \pm \beta : 1] \},$$

$$Orb_{G_1}[\frac{1}{\beta} : 0 : 1] = \{[\pm \frac{1}{\beta} : 0 : 1], [0 : \pm \frac{1}{\beta} : 1] \}.$$
\[ \text{Orb}_{G_1} [\delta : 1 : 0] = \{[\pm \delta : 1 : 0], [1 : \pm \delta : 0]\}. \]

Moreover, if \([\zeta : \varepsilon : 1] \notin X(C_{a,b}), \) then \(|\text{Orb}_{G_1}[\zeta : \varepsilon : 1]| = 8.\]

Now, the intersections \(X(C_{a,b}) \cap W_1(C_{a,b})\) are given by the following three lemmas.

**Lemma 13.** Let

\[ A := \text{Orb}_{G_1}[\beta : 0 : 1] \cup \text{Orb}_{G_1}[\frac{1}{\beta} : 0 : 1], \]

then for the quartics \(C_{a,b}\) such that \(b \neq 0,\) we have:

\[ A \cap W_1(C_{a,b}) \neq \phi \quad \text{if and only if} \quad a^2 + b^2 - ab^2 = 0. \]

Moreover,

\[ A \cap W_1(C_{a,b}) = \begin{cases} \text{Orb}_{G_1}[\beta : 0 : 1], & \text{if } a = \frac{1}{2}(b^2 - b\sqrt{b^2 - 4}) \\ \text{Orb}_{G_1}[\frac{1}{\beta} : 0 : 1], & \text{if } a = \frac{1}{2}(b^2 + b\sqrt{b^2 - 4}) \\ \phi, & \text{otherwise} \end{cases} \]

where \(\beta = \frac{\sqrt{2}}{-b + \sqrt{b^2 - 4}}.\)

**Proof.** The Hessian curve \(H_F(x, 0, 1)\) is given by

\[ H_F(x, 0, 1) = (2b + 12x^2)(2b + 2ax^2)(12 + 2bx^2) + 4bx(-8b^2x - 8abx^3) \]

and so the resultant of \(H_F(x, 0, 1)\) and \(F(x, 0, 1)\) with respect to \(x\) is

\[ \text{Res}(H_F(x, 0, 1), F(x, 0, 1); x) = \text{const.} \cdot \left(g(a, b)\right)^2, \]

where

\[ g(a, b) = \left(b^2 - 4\right)^2(a^2 + b^2 - ab^2). \]

From the last equation, we have

\[ A \cap W_1(C_{a,b}) \neq \phi \quad \text{if and only if} \quad a^2 + b^2 - ab^2 = 0. \]

Moreover, substituting \(a = \frac{1}{2}(b^2 + b\sqrt{b^2 - 4})\) into \(H_F(x, 0, 1) = 0 = F(x, 0, 1)\) yields the system

\[ x^4 + x^2 + 1 = 0, \]

\[ \left(2 + bx^2 + \sqrt{-4 + b^2x^2}\right)(-12x^2 + b^2x^2 - 2b(1 + x^4)) = 0, \]

which has the two solutions \(\pm \frac{\sqrt{2}}{-b - \sqrt{b^2 - 4}}.\)

Also, substituting \(a = \frac{1}{2}(b^2 - b\sqrt{b^2 - 4})\) into \(H_F(x, 0, 1) = 0 = F(x, 0, 1)\) yields the system

\[ x^4 + x^2 + 1 = 0, \]

\[ \left(2 + bx^2 - \sqrt{-4 + b^2x^2}\right)(-12x^2 + b^2x^2 - 2b(1 + x^4)) = 0, \]

which again has the two roots \(\pm \frac{\sqrt{2}}{-b + \sqrt{b^2 - 4}}.\) \(\square\)
**Lemma 14.** For the quartics $C_{a,b}$, such that $b \neq 0$, the points of $\text{Orb}_{G_i}[\delta : 1 : 0]$ are not 1-Weierstrass.

**Proof.** The Hessian $H_F$ at the point $[x : 1 : 0]$ is given by

$$H_F(x, 1, 0) = 2b(1 + x^2) \left(-16a^2x^2 + (2a + 12x^2)(12 + 2ax^2)\right).$$

So, the resultant with $F(x, 1, 0)$ is given by

$$\text{Res}(H_F(x, 1, 0), F(x, 1, 0), x) = \text{constant}. (g(a, b))^2,$$

where

$$g(a, b) = (a - 2)^3(a + 2)2b^2 \neq 0.$$ 

We have done. 

**Lemma 15.** Let $B := \text{Orb}_{G_i}[\alpha_1 : \alpha_1 : 1] \cup \text{Orb}_{G_i}[\alpha_3 : \alpha_3 : 1]$, then for the quartics $C_{a,b}$ such that $b \neq 0$, we have:

$B \cap W_1(C_{a,b}) \neq \emptyset$ if and only if

$$36 - 12a + a^2 - 10b^2 + 3ab^2 = 0.$$ 

Moreover,

$$B \cap W_1(C_{a,b}) = \begin{cases} \text{Orb}_{G_i}[\alpha_1 : \alpha_1 : 1], & \text{if } a = \frac{1}{2}(12 - 3b^2 - \sqrt{9b^2 - 32}) \\ \text{Orb}_{G_i}[\alpha_3 : \alpha_3 : 1], & \text{if } a = \frac{1}{2}(12 - 3b^2 + \sqrt{9b^2 - 32}) \\ \emptyset, & \text{otherwise} \end{cases}$$

where

$$\alpha_1 = \frac{1}{4}\sqrt{9b^2 - 32 - 3b} \text{ and } \alpha_3 = \frac{i}{4}\sqrt{9b^2 - 32 + 3b}.$$ 

**Proof.** The resultant of $H_F(x, x, 1)$ and $F(x, x, 1)$ with respect to $x$ is given by

$$\text{Res}(H_F(x, x, 1), F(x, x, 1), x) = \text{constant}. (g(a, b))^2,$$

where

$$g(a, b) = (a + 2)(2 + a - b^2)^2(36 - 12a + a^2 - 10b^2 + 3a^2).$$

Thus,

$$B \cap W_1(C_{a,b}) \neq \emptyset \text{ if and only if } 36 - 12a + a^2 - 10b^2 + 3ab^2 = 0.$$ 

Moreover, we have

$$F(x, x, 1) = (2 + a)x^4 + 2bx^2 + 1,$$

$$H_F(x, x, 1) = -64b^2x^2(b - (-6 + a)x^2) + 16(3 + bx^2)(-4a^2x^4 + (b + (6 + a)x^2)^2).$$

Hence, substituting $a = \frac{1}{2}(12 - 3b^2 \pm \sqrt{9b^2 - 32})$ in the above system and solving, the required result is obtained.

**Notation.** We denote by $\Gamma$ the set of common zeros of the equations

$$a^2 + b^2 - ab^2 = 0 \text{ and } 36 - 12a + a^2 - 10b^2 + 3ab^2 = 0.$$ 

Now, the next result follows from of the previous three lemmas.
Proposition 16. If \( b \neq 0 \), we have the following two cases for \( C_{a,b} \):

(i) If \( (a,b) \in \Gamma \), then \( X(C_{a,b}) \cap W_1(C_{a,b}) = \text{Orb}_{G_1}[\alpha : \alpha : 1] \cup \text{Orb}_{G_1}[\beta_0 : 0 : 1] \),

where \( \alpha \in \{\alpha_1, \alpha_3\} \), \( \beta_0 \in \{\beta, \frac{1}{\beta}\} \).

(ii) If \( (a,b) \notin \Gamma \), then

\[
X(C_{a,b}) \cap W_1(C_{a,b}) = \begin{cases} 
\text{Orb}_{G_1}[\beta : 0 : 1], & \text{if } a = \frac{1}{2}(b^2 - b\sqrt{b^2 - 4}) \\
\text{Orb}_{G_1}[\frac{1}{\beta} : 0 : 1], & \text{if } a = \frac{1}{2}(b^2 + b\sqrt{b^2 - 4}) \\
\text{Orb}_{G_1}[\alpha_1 : \alpha_1 : 1], & \text{if } a = \frac{1}{2}(12 - 3b^2 - \sqrt{9b^2 - 32}) \\
\text{Orb}_{G_1}[\alpha_3 : \alpha_3 : 1], & \text{if } a = \frac{1}{2}(12 - 3b^2 + \sqrt{9b^2 - 32}) \\
\phi, & \text{otherwise}
\end{cases}
\]

Lemma 17. If \( X(C_{a,b}) \cap W_1(C_{a,b}) \neq \phi \), then the intersection points are necessarily hyperflex points.

Now, our main result for the case \( b \neq 0 \) is the following.

Theorem 18. For the quartics \( C_{a,b} \) such that \( b \neq 0 \), the number of the 1-Weierstrass points together with their geometry are given in the following tables.

### Number Classification on \( C_{a,b} \)

<table>
<thead>
<tr>
<th>((a,b) \in \Gamma)</th>
<th>Ordinary Flexes</th>
<th>Hyperflexes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(a,b) = 0, \ Q(a,b) \neq 0)</td>
<td>8 (Resp. 16)</td>
<td>12 (Resp. 4)</td>
</tr>
<tr>
<td>(P(a,b) \neq 0, \ Q(a,b) = 0)</td>
<td>0 (Resp. 16)</td>
<td>12 (Resp. 4)</td>
</tr>
<tr>
<td>(P(a,b) \neq 0, \ Q(a,b) \neq 0)</td>
<td>24 (Resp. 8)</td>
<td>0 (Resp. 8)</td>
</tr>
</tbody>
</table>

### Orbit Classification on \( C_{a,b} \)

<table>
<thead>
<tr>
<th>((a,b) \in \Gamma)</th>
<th>Ordinary Flexes</th>
<th>Hyperflexes</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P(a,b) = 0, \ Q(a,b) \neq 0)</td>
<td>(1_s)</td>
<td>(1_4, 1_8) (Resp. 14)</td>
</tr>
<tr>
<td>(P(a,b) \neq 0, \ Q(a,b) = 0)</td>
<td>None (Resp. 2_8)</td>
<td>(1_4, 1_8) (Resp. 14)</td>
</tr>
<tr>
<td>(P(a,b)Q(a,b) = 0)</td>
<td>(3_8) (Resp. 1_8)</td>
<td>None (Resp. 1_8)</td>
</tr>
</tbody>
</table>

where \( P(a,b) := a^2 + b^2 - ab^2 \), \( Q(a,b) := 36 - 12a + a^2 - 10b^2 + 3ab^2 \).

**Proof.** Recall that the number of the 1-Weierstrass points is 24 counted with their weights. Now, if \( (a,b) \in \Gamma \), then by Proposition 16 (i) and Lemma 17, the intersections \( X(C_{a,b}) \cap W_1(C_{a,b}) \) consists of \( 1_4 \) of hyperflex points. Consequently, by Remark 2 there exist \( 1_8 \) of ordinary flexes.
If $P(a, b) = 0$, $Q(a, b) \neq 0$ or $P(a, b) \neq 0$, $Q(a, b) = 0$, then by Proposition \textbf{16} (ii) and Lemma \textbf{17}, the intersections $X(C_{a,b}) \cap W_1(C_{a,b})$ consists of $1_4$ of hyperflex points. Consequently, by Remark \textbf{2}, there exist either $1_8$ of hyperflex points or $2_8$ of ordinary flexes.

If $P(a, b)Q(a, b) \neq 0$, then by Proposition \textbf{16} (ii), $X(C_{a,b}) \cap W_1(C_{a,b}) = \phi$. Hence, $W_1(C_{a,b})$ consists of orbits each of $8$ points. Consequently, we have two cases: either $3_8$ of ordinary flexes or $1_8$ of ordinary flexes together with $1_8$ of hyperflex points. This completes the proof.

\section*{4. Examples}

This section is devoted to illustrate through examples the cases mentioned in Theorem \textbf{18}. It should be noted that, under a given condition, more than one case arise. So, it is convenient to investigate whether these cases can occur.

- For the case $P(a, b) = 0$ and $Q(a, b) \neq 0$, we give the following examples:
  \begin{enumerate}
  \item[(1):] If $a = \frac{6}{5}$, $b = \frac{6}{\sqrt{5}}$, then we have
  $$W_1(C_{a,b}) = Orb_{G_1}[1.495...i : 0 : 1] \cup Orb_{G_1}[\zeta_5 : \epsilon_5 : 1],$$
  where
  $$\zeta_5 := 0.748...(1 + i), \quad \epsilon_5 := 0.748...(1 - i).$$
  That is, $C_{a,b}$ has $12$ hyperflex points classified into $1_4$ and $1_8$.
  \item[(2):] If $a = 3$, $b = \frac{3}{\sqrt{2}}$, then we have
  $$W_1(C_{a,b}) = Orb_{G_1}[0.841...i : 0 : 1] \cup Orb_{G_1}[\zeta_3 : \epsilon_3 : 1] \cup Orb_{G_1}[\zeta_4 : \epsilon_4 : 1],$$
  where
  $$\zeta_3 := 0.089... - 0.454...i, \quad \epsilon_3 := 0.188... + 0.947...i,$$
  $$\zeta_4 := 0.089... + 0.454...i, \quad \epsilon_4 := 0.188... - 0.947...i.$$
  Hence, $C_{a,b}$ has $20$ flex points classified into $1_4$ of hyperflex points and $2_8$ of ordinary flexes.
  \item For the case $P(a, b) \neq 0$ and $Q(a, b) = 0$, we get the following two examples:
    \begin{enumerate}
    \item[(3):] If $a = 3$, $b = 3$, then $C_{a,b}$ has $12$ hyperflex points \textbf{[18]}.\textbf{[18]}
    \item[(4):] If $a = 0$, $b = 3\sqrt{\frac{2}{5}}$, then we have
    $$W_1(C_{a,b}) = Orb_{G_1}[0.562...i : 0.562...i : 1] \cup Orb_{G_1}[\zeta_1 : \epsilon_1 : 1] \cup Orb_{G_1}[\zeta_2 : \epsilon_2 : 1],$$
    where
    $$\zeta_1 := 0.204... - 1.151...i, \quad \epsilon_1 = 0.302... - 0.269...i,$$
    $$\zeta_2 := 0.204... + 1.151...i, \quad \epsilon_2 = 0.302... + 0.269...i.$$
    Consequently, $C_{a,b}$ has $20$ flex points classified into $1_4$ of hyperflex points and $2_8$ of ordinary flexes.
  \item For the case $P(a, b)Q(a, b) \neq 0$, we have the following examples:
    \begin{enumerate}
    \item[(5):] If $a = 4$, $b = 4$, then $C_{a,b}$ has $24$ ordinary flex points \textbf{[18]}.\textbf{[18]}.
(6): If $a = -5$, $b = 1$, then we have

$$W_1(C_{a,b}) = Orb_{G_1} [0.44...(1 + i) : 0.44...(i - 1) : 1] \cup Orb_{G_1} [0.93...i : 2.27...i : 1]$$

Consequently, $C_{a,b}$ has 16 flex points classified into $2_8$, one consists of hyperflex points and the other consists of ordinary flexes.

**Concluding remarks.**

We conclude the paper by the following remarks and comments.

- The computations included in the present paper have been performed by the use of MATHEMATICA program. The source code files are available.
- The classification of the 1-Weierstrass points of Kuribayashi quartics with one parameter, treated in [1, 18] is a particular case of our main theorems.

In fact, letting $a = b$, one gets the following table.

<table>
<thead>
<tr>
<th>$a = 0, 3$</th>
<th>Ordinary flex</th>
<th>Hyperflex</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>Otherwise</td>
<td>24</td>
<td>0</td>
</tr>
</tbody>
</table>

- The geometry of the 1-Weierstrass points of the one parameter quartic family defined by the equation

$$X^4 + Y^4 + Z^4 + b(X^2 + Y^2)Z^2 = 0, \quad (b^2 - 4)(b^2 - 2) \neq 0,$$

is a particular case by substituting $a = 0$ in Theorem 2.6 and Theorem 2.14 to get the following table.

**Number and Orbit Classification on $C_{a,0}$**

<table>
<thead>
<tr>
<th>$b = 0$</th>
<th>Ordinary flexes</th>
<th>Hyperflexes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td></td>
<td>1_4</td>
<td>1_8</td>
</tr>
<tr>
<td>$b^2 = \frac{18}{5}$</td>
<td>16</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>2_8 (Resp. 8)</td>
<td>1_4</td>
</tr>
<tr>
<td>Otherwise</td>
<td>24 (Resp. 8)</td>
<td>0 (Resp. 8)</td>
</tr>
<tr>
<td></td>
<td>3_8 (Resp. 1_8)</td>
<td>None (Resp. 1_8)</td>
</tr>
</tbody>
</table>

The following examples illustrates that under the condition $b^2 \neq 0$, $\frac{18}{5}$, both cases mentioned above can occur. For instance, let $b = \sqrt{6}$, then $C_{0,b}$ has 16 flex points or, equivalently,

$$W_1(C_{0,b}) = Orb_{G_1} [-0.6...(1 + i) : 0.6...(1 - i) : 1] \cup Orb_{G_1} [0.2...i : 0.2...i : 1].$$

On the other hand, let $b = \sqrt{5}$, then $C_{0,b}$ has 24 flex points or, equivalently,

$$W_1(C_{0,b}) = Orb_{G_1} [\nu_1 : \lambda_1 : 1] \cup Orb_{G_1} [\nu_2 : \lambda_2 : 1] \cup Orb_{G_1} [\nu_3 : \lambda_3 : 1],$$

where

$$\nu_1 := 0.524132... + 0.316563...i, \quad \lambda_1 := -0.417502... + 0.812472...i,$$

$$\nu_2 := 0.524132... - 0.316563...i, \quad \lambda_2 := 0.417502... + 0.812472...i,$$

$$\nu_3 := -0.410813...i, \quad \lambda_3 := -1.37547...i.$$
The main theorems constitute a motivation to solve more general problems. One of these problems is the investigation of the geometry of the 1-Weierstrass points of Kuribayashi quartics with three parameters family defined by the equation
\[ C_{a,b,c} : X^4 + Y^4 + Z^4 + aX^2Y^2 + bX^2Z^2 + cY^2Z^2 = 0. \]

Hayakawa [14] studied the conditions under which the number of the Weierstrass points of this family is exactly 12 or < 24. However, this problem will be the object of a forthcoming work.

Another problem is the investigation of the geometry of higher order and multiple Weierstrass points of Kuribayashi quartics with one parameter family [1, 2].

The technique used in this paper is completely different from that followed by Hayakawa [15]. Our technique consists of dividing the quartics by group actions into finite orbits and investigate the geometry of these orbits. The results obtained are more informative than those obtained by Hayakawa.

There are extra geometrical structures that can be used to make the classification more transparent. Examples, include the Eisenbud-Harris theory on monodromy and limits of Weierstrass points [9, 10, 11], associated del Pezzo surfaces of degree two (arising as double covers of the plane branched along the quartic), and K3 surfaces (arising as cyclic degree-four covers) [5, 6]. However, this extension will be the object of a forthcoming work.

Acknowledgment The authors would like express their sincere gratitude to Prof. Nabil Youssef of Cairo university, Egypt for his guidance throughout the preparation of this work. We are thankful to Prof. Brendan Hassett of Rice university, Houston, USA and Dr. Fransese Bars of Universitat Auto' noma de Barcelona, Bellaterra, Spain for their comments and suggestions due to the paper is improved to a great extent.

References


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