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SOME SUBORDINATION AND SUPERORDINATION RESULTS WITH AN INTEGRAL OPERATOR

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Abstract. In this article, we obtain some subordination and superordination preserving properties of meromorphic multivalent functions in the punctured open unit disc associated with an integral operator. Sandwich-type result is also obtained.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disc

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $n \in \mathbb{N} = \{1, 2, ...\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H} : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + ...\}.$$

Let f and g be members of $\mathcal{H}$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a function w analytic in $\mathbb{U}$, with $w(0) = 0$ and $|w(z)| < 1 (z \in \mathbb{U})$, such that $f(z) = g(w(z)) (z \in \mathbb{U})$.

In such a case, we write

$$f \prec g \quad (z \in \mathbb{U}) \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

If the function g is univalent in $\mathbb{U}$, then we have (cf. [5]),

$$f \prec g \quad (z \in \mathbb{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Definition 1 [5]**.** Let $\phi : \mathbb{C}^2 \to \mathbb{C}$ and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the differential subordination:

$$\phi(p(z); zp'(z)) \prec h(z) \quad (z \in \mathbb{U}), \tag{1.1}$$

then $p(z)$ is called a solution of the differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant.

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Definition 2 [6]**.** Let $\varphi : \mathbb{C}^2 \to \mathbb{C}$ and let $h(z)$ be analytic in $\mathbb{U}$. If $p(z)$ and $\varphi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the differential superordination:

$$h(z) \prec \varphi(p(z), zp'(z)) \ (z \in \mathbb{U}), \tag{1.2}$$

then $p(z)$ is called a solution of the differential superordination. An analytic function $q(z)$ is called a subordinant of the solutions of the differential superordination, or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.2). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 3 [5]**.** Denote by $\mathcal{F}$ the set of all functions $q(z)$ that are analytic and injective on $\overline{\mathbb{U}} \backslash E(q)$, where

$$E(q) = \left\{ \zeta \in \partial\mathbb{U} : \lim_{z \to \zeta} q(z) = \infty \right\},$$

and are such that

$$q'(\zeta) \neq 0 \quad (\zeta \in \partial\mathbb{U} \backslash E(q)).$$

Further let the subclass of $\mathcal{F}$ for which $q(0) = a$ be denoted by $\mathcal{F}(a)$, $\mathcal{F}(0) \equiv \mathcal{F}_0$ and $\mathcal{F}(1) \equiv \mathcal{F}_1$.

Definition 4 [6]**.** A function $L(z,t)$ $(z \in \mathbb{U}, t \geq 0)$ is said to be a subordination chain if $L(.,t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$, $L(z,.)$ is continuously differentiable on $[0,+\infty)$ for all $z \in \mathbb{U}$ and $L(z,t_1) \prec L(z,t_2)$ for all $0 \leq t_1 \leq t_2$.

Let Σ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k \tag{1.3}$$

which are analytic in the punctured unit open unit disc $\mathbb{U}^*$. For functions $f \in \Sigma$ given by (1.3), and $g \in \Sigma$ given by

$$g(z) := \frac{1}{z} + \sum_{k=1}^{\infty} b_k z^k,$$

define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) := \frac{1}{z} + \sum_{k=1}^{\infty} a_k b_k z^k = (g * f)(z).$$

Analogous to the integral operator defined by Jung et al. [1], Lashin [2] introduced and investigated the following integral operator

$$Q_{\alpha,\beta} : \Sigma \to \Sigma \tag{1.4}$$

defined in terms of the familiar Gamma function by

$$Q_{\alpha,\beta} f(z) = \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)\Gamma(\alpha)} \frac{1}{z^{\beta+1}} \int_0^z t^{\beta}(1-\frac{t}{z})^{\alpha-1} f(t)dt$$

$$= \frac{1}{z} + \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+1)}{\Gamma(k+\beta+\alpha+1)} a_k z^k \quad (\alpha > 0;\ \beta > 0;\ z \in \mathbb{U}^*),$$

By setting

$$f_{\alpha,\beta}(z) := \frac{1}{z} + \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} \sum_{k=1}^{\infty} \frac{\Gamma(k+\beta+\alpha+1)}{\Gamma(k+\beta+1)} a_k z^k \quad (\alpha > 0;\ \beta > 0;\ z \in \mathbb{U}^*), \tag{1.5}$$

Wang et al. [8] defined and studied an integral operator $Q^{\lambda}_{\alpha,\beta} : \Sigma \to \Sigma$ which is defined as follows:

Let $f^{\lambda}_{\alpha,\beta}(z)$ be defined such that

$$f_{\alpha,\beta}(z) * f^{\lambda}_{\alpha,\beta}(z) = \frac{1}{z(1-z)^{\lambda}} \ (\alpha > 0;\ \beta > 0;\ \lambda > 0;\ z \in \mathbb{U}^*). \tag{1.6}$$

Then

$$Q^{\lambda}_{\alpha,\beta} f(z) := f^{\lambda}_{\alpha,\beta}(z) * f(z) \ (z \in \mathbb{U}^*,\ f \ \in \Sigma. \tag{1.7}$$

From (1.5), (1.6) and (1.7) it follows that

$$Q^{\lambda}_{\alpha,\beta} f(z) = \frac{1}{z} + \frac{\Gamma(\beta+\alpha)}{\Gamma(\beta)} \sum_{k=1}^{\infty} \frac{(\lambda)_{k+1}\Gamma(k+\beta+1)}{(k+1)!\Gamma(k+\beta+\alpha+1)} a_k z^k \quad (\ z \in \mathbb{U}^*), \tag{1.8}$$

where $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k = \left\{ \begin{smallmatrix} 1, & k=0 \\ \lambda(\lambda+1)\ldots(\lambda+k-1), & (k \in \mathbb{N} := \{1,2,\ldots\}) \end{smallmatrix} \right\}. \tag{1.9}$$

Clearly, we know that

$$Q^{1}_{\alpha,\beta} = Q_{\alpha,\beta}.$$

It is readily verified from (1.8) that

$$z(Q^{\lambda}_{\alpha,\beta} f)'(z) = \lambda Q^{\lambda+1}_{\alpha,\beta} f(z) - (\lambda+1) Q^{\lambda}_{\alpha,\beta} f(z) \tag{1.10}$$

$$z(Q^{\lambda}_{\alpha,\beta} f)'(z) = (\beta+\alpha-1) Q^{\lambda}_{\alpha-1,\beta} f(z) - (\beta+\alpha) Q^{\lambda}_{\alpha,\beta} f(z). \tag{1.11}$$

2. A Set of Lemmas

The following lemmas will be required in our present investigation.

Lemma 1 [7]**.** The function $L(z,t) : \mathbb{U} \times [0,1) \longrightarrow \mathbb{C}$ of the form

$L(z,t) = a_1(t)z + a_2(t)z^2 + \ldots$ with $(a_1(t) \neq 0,\ t \geq 0\ )$ and $\lim_{t\to\infty} |a_1(t)| = \infty$ is a subordination chain if and only if

$$\Re\left\{ \frac{\frac{z\partial L(z,t)}{\partial z}}{\frac{\partial L(z,t)}{\partial t}} \right\} > 0 \quad (z \in \mathbb{U};\ 0 \leq t < \infty).$$

Lemma 2 [3]**.** Suppose that the function $H : \mathbb{C}^2 \to \mathbb{C}$ satisfies the following condition:

$$\Re\{H(is,t)\} \leq 0$$

for all real s and

$$t \leq -n(1+s^2)/2, \quad (n \in \mathbb{N}).$$

If the function $p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} \ldots$ is analytic in $\mathbb{U}$ and

$$\Re\{H(p(z), zp'(z))\} > 0 \ \ (z \in \mathbb{U}),$$

then,

$$\Re\{p(z)\} > 0 \ \ (z \in \mathbb{U}).$$

Lemma 3 [4]**.** Let $k,\ \gamma \in \mathbb{C}$ with $k \neq 0$ and $h \in \mathcal{H}(\mathbb{U})$ with $h(0) = c$. If

$$\Re\{kh(z) + \gamma\} > 0 \ \ (z \in \mathbb{U}),$$

then, the solution of the following differential equation

$$q(z) + \frac{zq'(z)}{kq(z)+\gamma} = h(z) \quad (z \in \mathbb{U};\ q(0) = c)$$

is analytic in $\mathbb{U}$ and satisfies the inequality

$$\Re\{kq(z)+\gamma\} > 0 \ \ (z \in \mathbb{U}).$$

Lemma 4 [5]**.** Let $p \in \mathcal{F}(a)$ and let

$$q(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$$

be analytic in $\mathbb{U}$ with

$$q(z) \neq a \ \ \text{and} \ \ n \geq 1.$$

If q is not subordinate to p, then there exist two points

$$z_0 = r_0 e^{i\theta} \in \mathbb{U} \ \ \text{and} \ \zeta_0 \in \partial\mathbb{U} \backslash E(q),$$

such that

$$q(\mathbb{U}_{r_0}) \subset p(\mathbb{U}),\ q(z_0) = p(\zeta_0) \ \ \text{and} \ \ z_0 q'(z_0) = m\zeta_0 p'(\zeta_0) \quad (m \geq n).$$

Lemma 5 [6]**.** Let $q \in \mathcal{H}[a,1]$ and $\varphi : \mathbb{C}^2 \to \mathbb{C}$. Also set

$$\varphi(q(z), zq'(z)) \equiv h(z) \qquad (z \in \mathbb{U}).$$

If $L(z,t) = \varphi(q(z), tzq'(z))$ is a subordination chain and $p \in \mathcal{H}[a,1] \cap \mathcal{F}(a)$, then.

$$h(z) \prec \varphi(q(z), zq'(z)) \qquad (z \in \mathbb{U}).$$

implies that

$$q(z) \prec p(z) \qquad (z \in \mathbb{U}).$$

Furthermore, if $\varphi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in \mathcal{F}(a)$, then q is the best subordinant.

In this article, we investigate the subordination and superordination preserving properties of the integral operator $Q^{\lambda}_{\alpha,\beta}$ with the Sandwich-type Theorems.

2. Main Results

We begin with proving the following subordination theorem involving the operator $Q^{\lambda}_{\alpha,\beta} f$ defined by (1.8).

Theorem 1. Let $f,\ g \in \Sigma$ and

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \left(\phi(z) = \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu};\ z \in \mathbb{U}\right), \tag{3.1}$$

$$(\lambda > 0;\ \alpha > 1;\ \beta > 0;\ \mu > 0),$$

where δ is given by

$$\delta = \frac{1 + \mu^2(\beta+\alpha-1)^2 - \left|1 - \mu^2(\beta+\alpha-1)^2\right|}{4\mu(\beta+\alpha-1)} \qquad (z \in \mathbb{U}). \tag{3.2}$$

Then the subordination condition

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}, \tag{3.3}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}, \tag{3.4}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}$ is the best dominant.

Proof. Let us define the functions $F(z)$ and $G(z)$ in $\mathbb{U}$ by

$$F(z) := \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \text{ and } \quad G(z) := \left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \ (z \in \mathbb{U}). \tag{3.5}$$

We first show that if the function q is defined by

$$q(z) := 1 + \frac{zG''(z)}{G'(z)} \quad (z \in \mathbb{U}), \tag{3.6}$$

then,

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

From (1.11) and the definition of functions G and ϕ, we obtain that

$$\phi(z) = G(z) + \frac{zG^{'}(z)}{\mu(\beta+\alpha-1)}. \tag{3.7}$$

Differentiating both sides of (3.7) with respect to z yields

$$\phi'(z) = \left(1 + \frac{1}{\mu(\beta+\alpha-1)}\right) G'(z) + \frac{zG^{''}(z)}{\mu(\beta+\alpha-1)}. \tag{3.8}$$

Combining (3.6) and (3.8), we easily get

$$1 + \frac{z\phi''(z)}{\phi'(z)} = q(z) + \frac{zq'(z)}{\mu(\beta+\alpha-1)+q(z)} = h(z) \ (z \in \mathbb{U}). \tag{3.9}$$

It follows from (3.1) and (3.9) that

$$\Re\left\{h(z) + \mu(\beta+\alpha-1)\right\} > 0 \quad (z \in \mathbb{U}). \tag{3.10}$$

Moreover, by using Lemma 3, we conclude the differential equation (3.9) has a solution $q(z) \in \mathcal{H}(\mathbb{U})$ with $h(0) = q(0) = 1$. Let

$$H(u,v) = u + \frac{v}{u+\mu(\beta+\alpha-1)} + \delta, \tag{3.11}$$

where δ is given by (3.2). From (3.9), and (3.10), we obtain

$$\Re\{H(q(z), zq'(z))\} > 0 \quad (z \in \mathbb{U}).$$

To verify the condition

$$\Re\{H(iv,t)\} \leq 0 \quad \left(v \in \mathbb{R};\ t \leq -\frac{1}{2}(1+v^2)\right), \tag{3.12}$$

we proceed as follows:

$$\Re\{H(iv,t)\} = \Re\left\{iv + \frac{t}{\mu(\beta+\alpha-1)+iv} + \delta\right\}$$

$$= \frac{t\mu(\beta+\alpha-1)}{|\mu(\beta+\alpha-1)+iv|^2} + \delta \leq -\frac{E_{\delta}(v)}{2\,|\mu(\beta+\alpha-1)+iv|^2},$$

where

$$E_{\delta}(v) := \left[\mu(\beta+\alpha-1) - 2\delta\right] v^2 - \mu(\beta+\alpha-1)\left[2\delta\mu(\beta+\alpha-1) - 1\right]. \tag{3.13}$$

For δ given by (3.2), we can prove easily that the expression $E_\delta(v)$ given by (3.13) is greater than or equal to zero. Hence, from (3.11), we see that (3.12) holds true. Thus, using Lemma 2, we conclude that

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Moreover, we see that the condition:

$$G'(0) \neq 0$$

is satisfied. Hence, the function G defined by (3.5) is convex (univalent) in $\mathbb{U}$.

Next, we prove that the subordination condition (3.3) implies that

$$F(z) \prec G(z) \qquad (z \in \mathbb{U}),$$

for the functions F and G defined by (3.5). Without loss of generality, we can assume that G is analytic and univalent on $\overline{\mathbb{U}}$ and

$$G'(\zeta) \neq 0 \ \ (\zeta \in \partial\mathbb{U}).$$

For this purpose, we consider the function $L(z,t)$ given by

$$L(z,t) := G(z) + \frac{(1+t)}{\mu(\beta+\alpha-1)} zG'(z), \tag{3.14}$$

$$(0 \leq t < \infty;\ z \in \mathbb{U};\ \alpha > 1;\ \beta > 0;\ \mu > 0).$$

We note that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left(1 + \frac{(1+t)}{\mu(\beta+\alpha-1)}\right) \neq 0,$$

$$(0 \leq t < \infty;\ z \in \mathbb{U};\ \alpha > 1;\ \beta > 0;\ \mu > 0).$$

This shows that the function

$$L(z,t) = a_1(t)z + ...$$

satisfies the condition $a_1(t) \neq 0$ $(0 \leq t < \infty)$. Furthermore, we have

$$\Re\left\{\frac{z\partial L(z,t)/\partial z}{\partial L(z,t)/\partial t}\right\} = \Re\left\{\mu(\beta+\alpha-1) + (1+t)(1 + \frac{zG''(z)}{G'(z)})\right\} > 0.$$

Therefore, by using of Lemma 1, we deduce that $L(z,t)$ is a subordination chain. It follows from the definition of subordination chain that

$$\phi(z) = G(z) + \frac{zG^{'}(z)}{\mu(\beta+\alpha-1)} = L(z,0)$$

and

$$L(z,0) \prec L(z,t) \ (0 \leq t < \infty),$$

which implies that

$$L(\zeta,t) \notin L(\mathbb{U},0) = \phi(\mathbb{U}) \ (\zeta \in \partial\mathbb{U};\ 0 \leq t < \infty), \tag{3.15}$$

if F is not subordinate to G, by using Lemma 4, we know that there exists two points $z_0 \in \mathbb{U}$ and $\zeta_0 \in \partial\mathbb{U}$, such that

$$F(z_0) = G(\zeta_0) \ \text{ and } \ z_0F'(z_0) = (1+t)\zeta_0G'(\zeta_0) \ \ (0 \leq t < \infty). \tag{3.16}$$

Hence, by using (3.5), (3.14), (3.16) and (3.3), we have

$$L(\zeta_0,t) = G(\zeta_0) + \frac{(1+t)}{\mu(\beta+\alpha-1)}\zeta_0G'(\zeta_0) = F(z_0) + \frac{1}{\mu(\beta+\alpha-1)}z_0F'(z_0)$$

$$= \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z_0)}{Q^{\lambda}_{\alpha,\beta}(f)(z_0)}\right)\left(z_0 Q^{\lambda}_{\alpha,\beta}(f)(z_0)\right)^{\mu} \in \phi(\mathbb{U}).$$

This contradicts (3.15). Thus, we deduce that $F \prec G$. Considering $F = G$, we see that the function G is the best dominant. This completes the proof of Theorem 1.

Theorem 2. Let $f,\ g \in \Sigma$ and

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta\left(\phi(z) = \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}; z \in \mathbb{U}\right), \quad (3.17)$$

$$(\lambda > 0;\ \alpha > 0;\ \beta > 0;\ \mu > 0),$$

where δ is given by

$$\delta = \frac{1 + \lambda^2\mu^2 - \left|1 - \lambda^2\mu^2\right|}{4\mu\lambda\mu} \qquad (z \in \mathbb{U}). \quad (3.18)$$

Then the subordination condition

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}, \quad (3.19)$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}, \quad (3.20)$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}$ is the best dominant.

Proof. Let us define the functions $F(z)$ and $G(z)$ in $\mathbb{U}$ by

$$F(z) := \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \text{ and } \quad G(z) := \left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \ (z \in \mathbb{U}). \quad (3.21)$$

Taking the logarithmic differentiation on both sides of the second equation in (3.21) and using the equation (1.10). The proof is similar to that of Theorem 1.

We now derive the following superordination result.

Theorem 3. Let $f.g \in \Sigma$. and

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta\left(\phi(z) = \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}; z \in \mathbb{U}\right), \quad (3.22)$$

$$(\lambda > 0;\ \alpha > 1;\ \beta > 0;\ \mu > 0),$$

where δ is given by (3.2). If the function

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$

is univalent in $\mathbb{U}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \in \mathcal{F}$, Then the superordination condition

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}, \quad (3.23)$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}, \quad (3.24)$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$ is the best subordinant.

Proof. Suppose that the function F, G and q are defined by (3.5)and (3.6), respectively. By applying similar method as in the proof of Theorem 1, we get

$$\Re\{q(z)\} > 0 \quad (z \in \mathbb{U}).$$

Next to arrive at our desired result, we show that $G \prec F$. For this, we suppose that the function $L(z,t)$ be define by (3.14). Since G is convex, by applying a similar method as in Theorem 1, we deduce that $L(z,t)$ is subordination chain. Therefore, by using Lemma 5, we conclude that $G \prec F$. Moreover, since the differential equation

$$\phi(z) = G(z) + \frac{zG^{'}(z)}{\mu(\beta+\alpha-1)} = \phi(G(z), G'(z))$$

has a univalent solution G, it is the best subordinant. This completes the proof of Theorem 3.

Theorem 4. Let $f,\ g \in \Sigma$ and

$$\Re\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta\left(\phi(z) = \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu}; z \in \mathbb{U}\right), \quad (3.25)$$

$$(\lambda > 0;\ \alpha > 0;\ \beta > 0;\ \mu > 0),$$

where δ is given by (3.18). If the function

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$

is univalent in $\mathbb{U}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \in \mathcal{F}$, Then the superordination condition

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g)(z)}{Q^{\lambda}_{\alpha,\beta}(g)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}, \quad (3.26)$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(g)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \quad (3.27)$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$ is the best subordinant.

Proof. the proof is similar to that of Theorem 3.

Combining the above - mentioned subordination and superordination results involving the operator $Q^{\lambda}_{\alpha,\beta}$ the following "Sandwich-type result " is derived.

Theorem 5. Let $f,\ g_j \in \Sigma$ $(j = 1, 2)$. and

$$\Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta\left(\phi_j(z) = \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g_j)(z)}{Q^{\lambda}_{\alpha,\beta}(g_j)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_j)(z)\right)^{\mu}\right),$$

$$(\ j = 1, 2;\ \lambda > 0;\ \alpha > 1;\ \beta > 0;\ \mu > 0;\ z \in \mathbb{U}),$$

where δ is given by (3.2). If the function

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$

is univalent in $\mathbb{U}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \in \mathcal{F}$, then the condition

$$\left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g_1)(z)}{Q^{\lambda}_{\alpha,\beta}(g_1)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$
$$\prec \left(\frac{Q^{\lambda}_{\alpha-1,\beta}(g_2)(z)}{Q^{\lambda}_{\alpha,\beta}(g_2)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}, \tag{3.28}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}, \tag{3.29}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}$ are respectively, the best subordinant and the best dominant.

Theorem 6. Let $f,\ g_j \in \Sigma(j = 1, 2)$ and

$$\Re\left\{1 + \frac{z\phi_j''(z)}{\phi_j'(z)}\right\} > -\delta\left(\phi_j(z) = \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g_j)(z)}{Q^{\lambda}_{\alpha,\beta}(g_j)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_j)(z)\right)^{\mu}; z \in \mathbb{U}\right),$$
$$(\lambda > 0;\ \alpha > 0;\ \beta > 0;\ \mu > 0),$$

where δ is given by (3.18). If the function

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu},$$

is univalent in $\mathbb{U}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \in \mathcal{F}$, then the condition

$$\left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g_1)(z)}{Q^{\lambda}_{\alpha,\beta}(g_1)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu} \prec \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(f)(z)}{Q^{\lambda}_{\alpha,\beta}(f)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu}$$
$$\prec \left(\frac{Q^{\lambda+1}_{\alpha,\beta}(g_2)(z)}{Q^{\lambda}_{\alpha,\beta}(g_2)(z)}\right)\left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}, \tag{3.30}$$

implies that

$$\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(f)(z)\right)^{\mu} \prec \left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}, \tag{3.31}$$

where $\left(zQ^{\lambda}_{\alpha,\beta}(g_1)(z)\right)^{\mu}$ and $\left(zQ^{\lambda}_{\alpha,\beta}(g_2)(z)\right)^{\mu}$ are respectively, the best subordinant and the best dominant.

References

[1] I. B. Jung, Y. C. kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176(1)(1993), 138-147.

[2] A. Y. Lashin, On certain subclases of meromorphic functions associated with certain integral operators, Comput. Math. Appl., 59(2009), 524-531.

[3] S. S. Miller and P. T. Mocanu, Differential subordination and univalent functions., Michigan. Math. J., 28(2)(1981), 157-171.

[4] S. S. Miller and P. T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, J. Different. Eq., 56(3)(1985) 297-309.

[5] S. S. Miller and P. T. Mocanu, Differential subordinations: Theory and Applications, Series on Monographs and Textbooks in Pure and Applied mathematics (No. 225), Marcel Dekker, New York and Basel, 2000.

[6] S. S. Miller and P. T. Mocanu, Subordination of differential superordinations., Complex Var. Theory Appl., 48(10)(2003) 815-826.
[7] Ch. Pommerenke, Univalent Functions, Vanderhoeck and Ruprecht; Gottingen, 1975.
[8] Z- G. Wang, Z- H. Liu and Y. Sun, Some subclasses of meromorphic functions associated with a family of integral operators, J. Ineq. Appl., 18(2009), 1-18.

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