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SUBORDINATION PROPERTIES FOR NEW CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS

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Abstract. In this paper, we obtain some subordination results for certian subclasses of univalent functions defined by convolution.

1. Introduction

Let $\mathcal{A}$ denote the class of the functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $f \in \mathcal{A}$ be given by (1.1) and g be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k. \tag{2}$$

Definition 1 Hadamard product or convolution). Let a function f defined by (1) and g defined by (1.2) the Hadamard product (or convolution) $(f * g)$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

Also, we denote by Ω the class of analytic functions $\omega(z)$ in U, normalized by $\omega(0) = 0$ and satisfying the condition $|\omega(z)| < 1$ for all $z \in U$ (see [9]).
Further let S denote the subclass of $\mathcal{A}$ consisting of analytic and univalent functions f in U. A function f in S is said to be starlike of order α if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \leq \alpha < 1; z \in U). \tag{4}$$

We denote by $S^*(\alpha)$ the class of all starlike functions of order α. Also a function f in S is said to be convex of order α if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \leq \alpha < 1; z \in U). \tag{5}$$

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We denote by $K(\alpha)$ the class of all convex functions of order α.
We note that

$$f(z)\in K(\alpha) \iff zf'(z)\in \mathcal{S}^*(\alpha), \tag{6}$$
$$\mathcal{S}^*(\alpha)\subseteq \mathcal{S}^*(0)\equiv \mathcal{S}^* \text{ and } K(\alpha)\subseteq K(0)\equiv K.$$

The classes $\mathcal{S}^*, K$, $\mathcal{S}^*(\alpha)$ and $K(\alpha)$ were first introduced by Robertson [13] and the classes $S^*(\alpha)$ and $K(\alpha)$ were studied subsequently by MacGregor [10] Schild [16], Pinchuk [12], Jack [9] and others.

Definition 2 [11] (Subordination Principle). For two functions $f(z)$ and $F(z)$, analytic in U, we say that $f(z)$ is subordiate to $F(z)$, written symbolically as follows:

$$f\prec F \text{ in } U \text{ or } f(z)\prec F(z)(z\in U),$$

if there exists a Schwarz function $\omega(z)\in\Omega$, which (by definition) is analytic in U with

$$\omega(0)=0 \text{ and } |\omega(z)|<1(z\in U)$$

such that

$$f(z)=F(\omega(z))(z\in U).$$

Indeed it is known that

$$f(z)\prec F(z)(z\in U) \implies f(0)=F(0) \text{ and } f(U)\subset F(U).$$

In particular, if the function $F(z)$ is univalent in U, we have the following equivalence

$$f(z)\prec F(z)(z\in U) \iff f(0)=F(0) \text{ and } f(U)\subset F(U).$$

For positive real values of $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ $(\beta_j\notin \mathbb{Z}_0^-=\{0,-1,-2,...\};j=1,2,...,s)$, we now define the generalized hypergeometric function ${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ by

$${}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k}.\frac{z^k}{k!}$$
$$(q\le s+1;q,s\in\mathbb{N}_0=\mathbb{N}\cup\{0\},\mathbb{N}=\{1,2,...\};z\in U),$$

where $(a)_m$ is the Pochhammer symbol defined by

$$(a)_m=\frac{\Gamma(a+m)}{\Gamma(a)}=\begin{cases}1 & (m=0),\\ a(a+1)....(a+m-1) & (m\in\mathbb{N}).\end{cases}$$

Corresponding to the function $h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)$ defined by

$$h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)=z\,{}_qF_s(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z),$$

we consider a linear operator $H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z):\mathcal{A}\longrightarrow\mathcal{A}$ which is defined by following Hadamard product (or convolution):

$$H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=h(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s;z)*f(z).$$

We observe that for function $f(z)$ of the form (1) we have

$$H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s)f(z)=z+\sum_{k=2}^{\infty}\Gamma_k(\alpha_1)a_kz^k,$$

where

$$\Gamma_k(\alpha_1)=\frac{(\alpha_1)_{k-1}...(\alpha_q)_{k-1}}{(\beta_1)_{k-1}....(\beta_s)_{k-1}}\cdot\frac{1}{(1)_{k-1}}\,(k\ge 2). \tag{7}$$

For convenience, we write

$$H_{q,s}(\alpha_1)=H(\alpha_1,...,\alpha_q;\beta_1,...,\beta_s).$$

The linear operator $H_{q,s}(\alpha_1)$ was introduced and studied by Dziok and Srivastava [6], and includes (as its special cases) various other linear operators for example Carlson and Shaffer [3], Ruscheweyh [14] and others.
For A, B fixed $-1 \leq B < A \leq 1$ and $0 \leq \gamma \leq 1$, we define the subclass $S_\gamma(f, g; A, B)$ of $\mathcal{A}$ consisting of functions f of the form (1.1) and functions g given by (1.2) with $b_k \geq 0$,as follows:

$$\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} \prec \frac{1+Az}{1+Bz}, \tag{8}$$

where

$$zF'_\gamma(f,g)(z) = z(f*g)'(z) + \gamma z^2 (f*g)''(z),$$

and

$$F_\gamma(f,g)(z) = (1-\gamma)(f*g)(z) + \gamma z (f*g)'(z)$$

From (8) and the definition of subordination we obtain

$$\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \omega(z) \in \Omega$$

and hence

$$\left| \frac{\frac{zF'_\gamma(f,g)(z)}{F\gamma(f,g)(z)} - 1}{B\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} - A} \right| < 1. \tag{9}$$

We note that for suitable choices of g, γ, A and B, we obtain the following subclasses:
(i)Putting $g(z) = \frac{z}{1-z}, \gamma = 0, A = 1-2\alpha\,(0 \leq \alpha < 1)$ and $B = -1$, we have $S_0(f, \frac{z}{1-z}; 1-2\alpha, -1) = S^*(\alpha)$ and $g(z) = \frac{z}{1-z}, \gamma = 1, A = 1-2\alpha\,(0 \leq \alpha < 1)$ and $B = -1$, we have $S_1(f, \frac{z}{1-z}; 1-2\alpha, -1) = K(\alpha)$
(ii) Putting $g(z) = \frac{z}{1-z}, \gamma = 0, A = (1-2\alpha)\beta$ and $B = -\beta\,(0 \leq \alpha < 1, 0 < \beta \leq 1)$, we have $S_0(f, \frac{z}{1-z}; (1-2\alpha)\beta, -\beta) = S(\alpha, \beta)$ and $g(z) = \frac{z}{1-z}, \gamma = 1, A = (1-2\alpha)\beta$ and $B = -\beta\,(0 \leq \alpha < 1, 0 < \beta \leq 1)$, we have $S_1(f, \frac{z}{1-z}; (1-2\alpha)\beta, -\beta) = C(\alpha, \beta)$ (see Gupta and Jain [8]).
Also we note that
(i) Putting $g(z) = z + \sum\limits_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k$, where $\Gamma_k(\alpha_1)$ is given by (1.7), we have $S_\gamma(f, z + \sum\limits_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k; A, B) = S_\gamma(f, H_{q,s}(\alpha_1); A, B)$

$$= \left\{ f \in \mathcal{A} : \frac{z(H_{q,s}(\alpha_1) f(z))' + \gamma z^2 (H_{q,s}(\alpha_1) f(z))''}{(1-\gamma)(H_{q,s}(\alpha_1) f(z)) + \gamma z (H_{q,s}(\alpha_1) f(z))'} \prec \frac{1+Az}{1+Bz}, z \in U \right\};$$

(ii) Putting $g(z) = z + \sum_{k=2}^{\infty} \left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k$, where $\lambda \geq 0; \ell \geq 0$ and $m \in \mathbb{N}_0$, we have $S_\gamma(f, z + \sum_{k=2}^{\infty} \left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k; A, B) = S_\gamma(f, I^m_{\lambda,\ell}; A, B)$

$$= \left\{ f \in \mathcal{A} : \frac{z\left(I^m_{\lambda,\ell} f(z)\right)' + \gamma z^2 \left(I^m_{\lambda,\ell} f(z)\right)''}{(1-\gamma)\left(I^m_{\lambda,\ell} f(z)\right) + \gamma z \left(I^m_{\lambda,\ell}(\alpha_1) f(z)\right)'} \prec \frac{1+Az}{1+Bz}, z \in U \right\},$$

where $I^m_{\lambda,\ell}$ is Catas operator (see [4])
(iii) Putting $g(z) = z + \sum_{k=2}^{\infty} \binom{k+\lambda-1}{\lambda} z^k$, where $\lambda > -1$, we have $S_\gamma(f, z +$

$\sum_{k=2}^{\infty}\binom{k+\lambda-1}{\lambda}z^k; A, B) = S_\gamma(f, D^\lambda; A, B)$

$$= \left\{ f \in \mathcal{A} : \frac{z\left(D^\lambda f(z)\right)' + \gamma z^2 \left(D^\lambda f(z)\right)''}{(1-\gamma)\left(D^\lambda f(z)\right) + \gamma z \left(D^\lambda f(z)\right)'} \prec \frac{1+Az}{1+Bz}, z \in U \right\},$$

where D^λ is Ruscheweyh derivative [14], defined by

$$D^\lambda f(z) = \frac{z(z^{\lambda-1}f(z))^\lambda}{\lambda!} = \frac{z}{(1-z)^{\lambda+1}} * f(z);$$

(iv) Putting $g(z) = z+\sum_{k=2}^{\infty} k^n z^k$, where $n \in \mathbb{N}_0$, we have $S_\gamma(f, z+\sum_{k=2}^{\infty} k^n z^k; A, B) = S_\gamma(f, D^n; A, B)$

$$= \left\{ f \in \mathcal{A} : \frac{z\left(D^n f(z)\right)' + \gamma z^2 \left(D^n f(z)\right)''}{(1-\gamma)\left(D^n f(z)\right) + \gamma z \left(D^n f(z)\right)'} \prec \frac{1+Az}{1+Bz}, z \in U \right\},$$

where D^n is Salagean operator [15], defined by

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k;$$

(v) Putting $g(z) = z+\sum_{k=2}^{\infty}\left(\frac{k+\ell}{1+\ell}\right)^m z^k$, where $m \in \mathbb{N}_0$ and $\ell \geq 0$ we have $S_\gamma(f, z+\sum_{k=2}^{\infty}\left(\frac{k+\ell}{1+\ell}\right)^m z^k; A, B) = S_\gamma(f, I_\ell^m; A, B)$

$$= \left\{ f \in \mathcal{A} : \frac{z\left(I_\ell^m f(z)\right)' + \gamma z^2 \left(I_\ell^m f(z)\right)''}{(1-\gamma)\left(I_\ell^m f(z)\right) + \gamma z \left(I_\ell^m f(z)\right)'} \prec \frac{1+Az}{1+Bz}, z \in U \right\},$$

where I_ℓ^m is Cho and Kim operator [5], defined by

$$I_\ell^m f(z) = z + \sum_{k=2}^{\infty}\left(\frac{k+\ell}{1+\ell}\right)^m a_k z^k.$$

Definition 3 [18] (Subordination Factor Sequence). A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be subordinating factor sequence if, whenever $f(z)$ of the form (1) is analytic, univalent and convex in U, we have the subordination given by

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z)(z \in U; a_1 = 1). \tag{10}$$

2. Main Results

To prove our main results we need the following lemmas.

Lemma 1 [18]. The sequence $\{c_k\}_{k=1}^{\infty}$ is subordinating factor sequence if and only if

$$Re\left\{1 + 2\sum_{k=1}^{\infty} c_k z^k\right\} > 0 \qquad (z \in U).$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $S_\gamma(f, g; A, B)$.

Lemma 2. A function $f(z)$ of the form (1.1) is in the class $S_\gamma(f, g; A, B)$ if

$$\sum_{k=2}^{\infty}\left[k(1-B) + (A-1)\right]\left[1 + \gamma(k-1)\right]|a_k|\, b_k \leq (A-B), \tag{11}$$

where $-1 \leq B < A \leq 1, 0 \leq \gamma \leq 1$ and $b_k \geq b_2\,(k \geq 2)$.
Proof. From (8) we obtain

$$\frac{zF'_\gamma(f,g)(z)}{F_\gamma(f,g)(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \omega(z) \in \Omega,$$

$$\left|\frac{zF'_\gamma(f,g)(z) - F_\gamma(f,g)(z)}{BzF'_\gamma(f,g)(z) - AF_\gamma(f,g)(z)}\right| < 1,$$

we have

$$\left|zF'_\gamma(f,g)(z) - F_\gamma(f,g)(z)\right| < \left|BzF'_\gamma(f,g)(z) - AF_\gamma(f,g)(z)\right|$$

$$= \left|\sum_{k=2}^{\infty}(k-1)\left[1+\gamma(k-1)\right]a_k b_k z^k\right|$$

$$- \left|(A-B)z + \sum_{k=2}^{\infty}(A-Bk)\left[1+\gamma(k-1)\right]a_k b_k z^k\right|$$

$$\leq \sum_{k=2}^{\infty}(k-1)\left[1+\gamma(k-1)\right]|a_k|\,b_k$$

$$-(A-B) + \sum_{k=2}^{\infty}(A-Bk)\left[1+\gamma(k-1)\right]|a_k|\,b_k < 0$$

$$\sum_{k=2}^{\infty}\left[k(1-B)+(A-1)\right]\left[1+\gamma(k-1)\right]|a_k|\,b_k \leq (A-B) \tag{12}$$

and hence the proof of Lemma 2 is completed.
Remark 1. Putting $A = 1-2\alpha (0 \leq \alpha < 1)$ and $B = -1$ in Lemma 2, we obtain the result obtained by Aouf et al. [1, Lemma 2, with $\beta = 0$].
Let $S^*_\gamma(f,g;A,B)$ denote the class of $f(z) \in \mathcal{A}$ whose coefficients satisfy the condition (11). We note that $S^*_\gamma(f,g;A,B) \subset S_\gamma(f,g;A,B)$.
Employing the technique used earlier by Attiya [2] and Srivastava and Attiya [17], we prove
Theorem 1. Let $f(z) \in S^*_\gamma(f,g;A,B)$. Then

$$\frac{(1-2B+A)(1+\gamma)b_2}{2\left[(1-2B+A)(1+\gamma)b_2 + (A-B)\right]}(f*h)(z) \prec h(z)\,(z \in U), \tag{13}$$

for every function h in K, and

$$Re\,(f(z)) > -\frac{\left[(1-2B+A)(1+\gamma)b_2 + (A-B)\right]}{(1-2B+A)(1+\gamma)b_2}, \qquad (z \in U). \tag{14}$$

The constant factor $\frac{(1-2B+A)(1+\gamma)b_2}{2[(1-2B+A)(1+\gamma)b_2+(A-B)]}$ in the subordination result (13) can not be replaced by a larger one.

Proof. Let $f(z) \in S^*_\gamma(f,g;A,B)$ and let $h(z) = z + \sum\limits_{k=2}^{\infty} d_k z^k \in K$. Then we have

$$\frac{(1-2B+A)(1+\gamma)b_2}{2\left[(1-2B+A)(1+\gamma)b_2 + (A-B)\right]}(f*h)(z) =$$

$$\frac{(1-2B+A)(1+\gamma)b_2}{2\left[(1-2B+A)(1+\gamma)b_2 + (A-B)\right]}\left(z + \sum_{k=2}^{\infty} a_k d_k z^k\right). \tag{15}$$

Thus, by Definition 2, the subordintion result (13) will hold true if the sequence

$$\left\{ \frac{(1-2B+A)(1+\gamma)b_2}{2\left[(1-2B+A)(1+\gamma)b_2+(A-B)\right]} a_k \right\}_{k=1}^{\infty} \tag{16}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1, this is equivalence to the following inequality:

$$Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{(1-2B+A)(1+\gamma)b_2}{[(1-2B+A)(1+\gamma)b_2+(A-B)]} a_k z^k \right\} > 0 \, (z \in U) \, . \tag{17}$$

Now, since

$$\Psi(k) = [k\,(1-B)+(A-1)]\,[1+\gamma\,(k-1)]\,b_k$$

is an increasing function of $k\,(k \geq 2)$, we have

$$\begin{aligned} & Re \left\{ 1 + \sum_{k=1}^{\infty} \frac{(1-2B+A)(1+\gamma)b_2}{[(1-2B+A)(1+\gamma)b_2+(A-B)]} a_k z^k \right\} \\ = \quad & Re \left\{ 1 + \frac{(1-2B+A)(1+\gamma)b_2}{[(1-2B+A)(1+\gamma)b_2+(A-B)]} z + \right. \\ & \left. \tfrac{1}{[(1-2B+A)(1+\gamma)b_2+(A-B)]} \sum_{k=2}^{\infty} (1-2B+A)(1+\gamma)b_2 a_k z^k \right\} \\ & \geq 1 - \frac{(1-2B+A)(1+\gamma)b_2}{[(1-2B+A)(1+\gamma)b_2+(A-B)]} r \\ & - \tfrac{1}{[(1-2B+A)(1+\gamma)b_2+(A-B)]} \sum_{k=2}^{\infty} [k\,(1-B)+(A-1)]\,[1+\gamma\,(k-1)]\,b_k\,|a_k|\,r^k \\ & > 1 - \tfrac{(1-2B+A)(1+\gamma)b_2}{[(1-2B+A)(1+\gamma)b_2+(A-B)]} r - \tfrac{(A-B)}{[(1-2B+A)(1+\gamma)b_2+(A-B)]} r \\ & > 0 \, (|z| = r < 1) \, , \end{aligned}$$

where we have also made use of assertion (12) of Lemma 2. Thus (7) holds true in U. This proves the inequality (13). The inequality (2.4) follows from (2.3) by taking the convex function $h(z) = \frac{z}{1-z} = z + \sum\limits_{k=2}^{\infty} z^k$. To prove the sharpness of the constant $\frac{(1-2B+A)(1+\gamma)b_2}{2[(1-2B+A)(1+\gamma)b_2+(A-B)]}$, we consider the function $f_0(z) \in S_\gamma^*(f, g; A, B)$ given by

$$f_0(z) = z - \tfrac{(A-B)}{(1-2B+A)(1+\gamma)b_2} z^2. \tag{18}$$

Thus from(13) we have

$$\frac{(1-2B+A)(1+\gamma)b_2}{2\left[(1-2B+A)(1+\gamma)b_2+(A-B)\right]} \left(f_0\right)(z) \prec \frac{z}{1-z} \, (z \in U) \, . \tag{19}$$

Moreover, it can easily be verified for the function given by (18) that

$$\min_{|z|<r} Re \left\{ \frac{(1-2B+A)(1+\gamma)b_2}{2\left[(1-2B+A)(1+\gamma)b_2+(A-B)\right]} \left(f_0\right)(z) \right\} = -\frac{1}{2}. \tag{20}$$

This shows that the constant $\frac{(1-2B+A)(1+\gamma)b_2}{2[(1-2B+A)(1+\gamma)b_2+(A-B)]}$ is the best possible. This completes the proof of Theorem 1.

Remark 2.

(i) Putting $A = 1 - 2\alpha (0 \leq \alpha < 1)$ and $B = -1$ in Theorem 1, we obtain the result

obtained by Aouf et al. [1, Theorem 1, with $\beta = 0$]
(ii) Putting $g(z) = \frac{z}{1-z}, \gamma = 0, A = 1 - 2\alpha (0 \le \alpha < 1)$ and $B = -1$ in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.3]
(iii) Putting $g(z) = \frac{z}{1-z}, \gamma = 0, A = 1$ and $B = -1$ in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.4];
(iv) Putting $g(z) = \frac{z}{1-z}, \gamma = A = 1$ and $B = -1$ in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.7].
(v) Putting $g(z) = \frac{z}{1-z}, \gamma = 1, A = 1 - 2\alpha (0 \le \alpha < 1)$ and $B = -1$ in Theorem 1, we obtain the result obtained by Frasin [7, Corollary 2.6].
Also, we establish subordination results for the associated subclasses, $S^*(\alpha, \beta)$, $C^*(\alpha, \beta)$, $S^*_\gamma(f, H_{q,s}(\alpha_1); A, B)$, $S^*_\gamma(f, I^m_{\lambda,\ell}; A, B)$ $S^*_\gamma(f, D^\lambda; A, B)$ $S^*_\gamma(f, D^n; A, B)$ $S^*_\gamma(f, I^m_\ell; A, B)$.
Putting $g(z) = \frac{z}{1-z}, \gamma = 0, A = (1-2\alpha)\beta\,(0 \le \alpha < 1), (0 < \beta \le 1)$ and $B = -\beta$ in Theorem 1, we obtain the following corollary

Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $S^*(\alpha, \beta)$ and suppose that $h(z) \in K$. Then

$$\frac{1 + \beta(3 - 2\alpha)}{2[1 + \beta(5 - 4\alpha)]} (f * h)(z) \prec h(z)\,(z \in U), \tag{21}$$

and

$$Re\,(f(z)) > -\frac{1 + \beta(5 - 4\alpha)}{1 + \beta(3 - 2\alpha)}, \qquad (z \in U).$$

The constant factor $\frac{1+\beta(3-2\alpha)}{2[1+\beta(5-4\alpha)]}$ in the subordination result (12) can not be replaced by a larger one.
Putting $g(z) = \frac{z}{1-z}, \gamma = 1, A = (1 - 2\alpha)\beta$ and $B = -\beta$ $(0 \le \alpha < 1, 0 < \beta \le 1)$ in Theorem 1, we obtain the following corollary.

Corollary 2. Let the function $f(z)$ defined by (1) be in the class $C^*(\alpha, \beta)$ and suppose that $h(z) \in K$. Then

$$\frac{1 + \beta(3 - 2\alpha)}{2[1 + \beta(4 - 3\alpha)]} (f * h)(z) \prec h(z)\,(z \in U), \tag{22}$$

and

$$Re\,(f(z)) > -\frac{1 + \beta(4 - 3\alpha)}{1 + \beta(3 - 2\alpha)}, \qquad (z \in U).$$

The constant factor $\frac{1+\beta(3-2\alpha)}{2[1+\beta(4-3\alpha)]}$ in the subordination result (21) can not be replaced by a larger one.
Putting $g(z) = z + \sum\limits_{k=2}^{\infty} \Gamma_k(\alpha_1) z^k$, where $\Gamma_k(\alpha_1)$ is given by (7) in Theorem 1, we obtain the following corollary.

Corollary 3. Let the function $f(z)$ defined by (1.1) be in the class $S^*_\gamma(f, H_{q,s}(\alpha_1); A, B)$ and suppose that $h(z) \in K$. Then

$$\frac{(1 - 2B + A)(\gamma + 1)\Gamma_2(\alpha_1)}{2[(1 - 2B + A)(\gamma + 1)\Gamma_2(\alpha_1) + (A - B)]} (f * h)(z) \prec h(z)\,(z \in U), \tag{23}$$

and

$$Re\,(f(z)) > -\frac{(1 - 2B + A)(\gamma + 1)\Gamma_2(\alpha_1) + (A - B)}{(1 - 2B + A)(\gamma + 1)\Gamma_2(\alpha_1)}, \qquad (z \in U).$$

The constant factor $\frac{(1-2B+A)(\gamma+1)\Gamma_2(\alpha_1)}{2[(1-2B+A)(\gamma+1)\Gamma_2(\alpha_1)+(A-B)]}$ in the subordination result (23) can not be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{1+\ell+\lambda(k-1)}{1+\ell}\right)^m z^k$ ($\lambda \geq 0, \ell \geq 0, m \in \mathbb{N}_0$) in Theorem 1, we obtain the following corollary.

Corollary 4. Let the function $f(z)$ defined by (1) be in the class $S_{\gamma}^{*}(f, D^{\lambda}; A, B)$ and suppose that $h(z) \in K$. Then

$$\frac{(1-2B+A)(\gamma+1)(1+\frac{\lambda}{1+\ell})^m}{2[(1-2B+A)(\gamma+1)(1+\frac{\lambda}{1+\ell})^m+(A-B)]}\left(f*h\right)(z) \prec h\left(z\right)(z\in U)\,, \qquad (24)$$

and

$$Re\left(f(z)\right) > -\frac{(1-2B+A)(\gamma+1)(1+\frac{\lambda}{1+\ell})^m+(A-B)}{(1-2B+A)(\gamma+1)(1+\frac{\lambda}{1+\ell})^m}, \qquad (z\in U)\,.$$

The constant factor $\frac{(1-2B+A)(\gamma+1)(1+\frac{\lambda}{1+\ell})^m}{2[(1-2B+A)(\gamma+1)(1+\frac{\lambda}{1+\ell})^m+(A-B)]}$ in the subordination result (2.14) can not be replaced by a larger one.

Putting $g(z) = g(z) = z + \sum_{k=2}^{\infty}\binom{k+\lambda-1}{\lambda}z^k(\lambda > -1)$, in Theorem 1, we obtain the following corollary.

Corollary 5. Let the function $f(z)$ defined by (1) be in the class $S_{\gamma}^{*}(f, I_{\lambda,\ell}^{m}; A, B)$ and suppose that $h(z) \in K$. Then

$$\frac{(1-2B+A)(\gamma+1)(1+\lambda)}{2[(1-2B+A)(\gamma+1)(1+\lambda)+(A-B)]}\left(f*h\right)(z) \prec h\left(z\right)(z\in U)\,, \qquad (25)$$

and

$$Re\left(f(z)\right) > -\frac{(1-2B+A)(\gamma+1)(1+\lambda)+(A-B)}{(1-2B+A)(\gamma+1)(1+\lambda)}, \qquad (z\in U)\,.$$

The constant factor $\frac{(1-2B+A)(\gamma+1)(1+\lambda)}{2[(1-2B+A)(\gamma+1)(1+\lambda)+(A-B)]}$ in the subordination result (25) can not be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty} k^n z^k (n \in \mathbb{N}_0)$, in Theorem 1, we obtain the following corollary.

Corollary 6. Let the function $f(z)$ defined by (1) be in the class $S_{\gamma}^{*}(f, D^n; A, B)$ and suppose that $h(z) \in K$. Then

$$\frac{2^n(1-2B+A)(\gamma+1)}{2[2^n(1-2B+A)(\gamma+1)+(A-B)]}\left(f*h\right)(z) \prec h\left(z\right)(z\in U)\,, \qquad (26)$$

and

$$Re\left(f(z)\right) > -\frac{2^n(1-2B+A)(\gamma+1)+(A-B)}{2^n(1-2B+A)(\gamma+1)}, \qquad (z\in U)\,.$$

The constant factor $\frac{2^n(1-2B+A)(\gamma+1)}{2[2^n(1-2B+A)(\gamma+1)+(A-B)]}$ in the subordination result (26) can not be replaced by a larger one.

Putting $g(z) = z + \sum_{k=2}^{\infty}\left(\frac{k+\ell}{1+\ell}\right)^m z^k$ ($\ell \geq 0$, $m \in \mathbb{N}_0$), in Theorem 1, we obtain the following corollary

Corollary 7. Let the function $f(z)$ defined by (1) be in the class $S_{\gamma}^{*}(f, I_{\ell}^{m}; A, B)$

and suppose that $h(z) \in K$. Then

$$\frac{\left(\frac{2+\ell}{1+\ell}\right)^m (1-2B+A)(\gamma+1)}{2[\left(\frac{2+\ell}{1+\ell}\right)^m (1-2B+A)(\gamma+1)+(A-B)]} (f*h)(z) \prec h(z)\,(z \in U), \qquad (27)$$

and

$$Re\,(f(z)) > -\frac{\left(\frac{2+\ell}{1+\ell}\right)^m (1-2B+A)(\gamma+1)+(A-B)}{\left(\frac{2+\ell}{1+\ell}\right)^m (1-2B+A)(\gamma+1)}, \qquad (z \in U).$$

The constant factor $\frac{\left(\frac{2+\ell}{1+\ell}\right)^m (1-2B+A)(\gamma+1)}{2[\left(\frac{2+\ell}{1+\ell}\right)^m (1-2B+A)(\gamma+1)+(A-B)]}$ in the subordination result (27) can not be replaced by a larger one.

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