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DYNAMICAL BEHAVIORS OF A NEW HYPERCHAOTIC SYSTEM WITH ONE NONLINEAR TERM

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Abstract. This paper is devoted to introduce a novel hyperchaotic system with only one nonlinear term. Existence and uniqueness of the solution of the proposed system are studied. Continuous dependence on initial conditions of the system's solution and stability of system's equilibrium points are investigated. Dynamical behaviors are explored via theoretical analysis, bifurcation diagrams, and phase portraits of the new system. Finally, a circuit implementation of the novel hyperchaotic system is proposed.

1. Introduction

In various disciplines of engineering, physics, biology, chemistry, and economy, we encounter systems that undergo spatial and temporal evolution [1] and [2]. To model, understand, and analyze these phenomena, the study of dynamical systems is a useful tool that helps in achieving these aims. One important behavior that exists in some dynamical systems is chaos.

Chaotic dynamical system is characterized by its sensitive dependence on initial conditions and by having positive Lyapunov exponents for its attractor. When the system's attractor has more than one positive Lyapunov exponents, it is called a hyperchaotic system. Any hyperchaotic system has the following properties: (i) an autonomous system with a phase space of dimension at least four, (ii) dissipative, and (iii) has at least two unstable directions. Thus, the hyperchaotic systems have higher unpredictability and more randomness than simple chaotic systems. Some of the applications of dynamical systems and chaos involve mathematical biology, financial systems, chaos control, synchronization, electronic circuits, and neuroscience research, while hyperchaotic systems are preferred in many applications including secure communications, chaos based image encryption, and cryptography [3]-[23].

Various new hyperchaotic systems are introduced and studied in literatures, see for example [24]-[29] and references therein. However, the most of the presented hyperchaotic system in literatures have multiterms with quadratic nonlinearity or linear piecewise terms. The other hyperchaotic systems such as [24] contain cubic

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nonlinearity in one of the state variables of the systems. Based on modifying the system introduced in [6], we propose a new hyperchaotic system with one cubic nonlinear term contains two of system's state variables. The dynamics of the new hyperchaotic system are explored via theoretical and numerical techniques which include center manifold theorem [30]-[33], first Lyapunov coefficients [32], Lyapunov exponents [34], bifurcation diagrams, and phase portraits. It is shown that the proposed system has rich dynamics and chaotic behavior exists over a wide range of parameters.

The rest of the paper is organized as follows: The proposed system is introduced in section 2 along with examination of existence, uniqueness, and continuous dependence on initial conditions of solution. The Hopf bifurcation and stability of equilibrium points of the system in hyperbolic and nonhyperbolic cases are studied in section 3. Numerical simulations are performed in section 4 to verify hyperchaos existence. Circuit implementation of the model is presented in section 5. Section 6 contains the conclusion and the general discussions of this work.

2. The proposed hyperchaotic system

We modify the memristor based chaotic system presented in [6] to get following hyperchaotic system

$$\frac{dx(t)}{dt} = a_1x + b_1y + c_1xw^2, \tag{1}$$

$$\frac{dy(t)}{dt} = a_2x + b_2y + h_1z + c_2w, \tag{2}$$

$$\frac{dz(t)}{dt} = b_3y, \tag{3}$$

$$\frac{dw(t)}{dt} = a_3x + h_2z, \tag{4}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, h_1, h_2$ are the parameters of system (1)-(4). The proposed system has three equilibrium points given by $E_0 = (0,0,0,0)$ and $E_\pm = (-\frac{h_2}{a_3}\bar{z}, 0, \bar{z}, (\frac{a_2h_2}{a_3c_2} - \frac{h_1}{c_2})\bar{z})$, where $\bar{z} = \pm\sqrt{\frac{-a_1}{c_1(\frac{a_2h_2}{a_3c_2} - \frac{h_1}{c_2})^2}}$ and $c_1(a_2h_2 - a_3h_1) \neq 0$.

2.1. **Existence and uniqueness of the solution.**

System (1)-(4) can be written in the following form:

$$\begin{aligned} D\mathbf{X}(\mathbf{t}) &= \mathbf{F}(\mathbf{X}(\mathbf{t})), \\ t &\in (0,T], \end{aligned} \tag{5}$$

with initial conditions of the system given by

$$\mathbf{X}(\mathbf{0}) = \mathbf{X}_0, \tag{6}$$

where

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \ \mathbf{X}_0 = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{bmatrix}, \text{and } \mathbf{F}(\mathbf{X}) = \begin{bmatrix} a_1x + b_1y + c_1xw^2 \\ a_2x + b_2y + h_1z + c_2w \\ b_3y \\ a_3x + h_2z \end{bmatrix}. \tag{7}$$

The supremum norm utilized in the following analysis is defined for the class of continuous function $C[0,T]$ by

$$\|\Psi\| = \sup_{t\in(0,T]} |\Psi(t)|\,,\ \ \Psi(t) \in C[0,T], \tag{8}$$

and for a matrix M =$[m_{ij}[t]]$, it is defined by

$$\|M\| = \sum_{i,j} \sup_{t\in(0,T]} |m_{ij}[t]|\,. \tag{9}$$

The existence and uniqueness of the solution is studied in the region $\Omega\times J$ where $J = (0,T]$ and

$$\Omega = \{(x,y,z,w) : \max\{|x|\,,|y|\,,|z|\,,\text{and }\ |w|\} \leq A\}. \tag{10}$$

The solution of (5) and (6) is given by:

$$\mathbf{X} = \mathbf{X}_0 + \int_0^t \mathbf{F}(\mathbf{X}(\mathbf{s}))\mathbf{ds}. \tag{11}$$

From the equivalence of the integral equation (11) and the system (5)-(6), denoting the right hand side of (11) by $\mathbf{G}(\mathbf{X})$, then

$$\mathbf{G}(\mathbf{X}_1) - \mathbf{G}(\mathbf{X}_2) = \int_0^t (\mathbf{F}(\mathbf{X}_1(\mathbf{s})) - \mathbf{F}(\mathbf{X}_2(\mathbf{s})))\mathbf{ds}, \tag{12}$$

and therefore

$$|\mathbf{G}(\mathbf{X}_1) - \mathbf{G}(\mathbf{X}_2)| \leq \int_0^t |(\mathbf{F}(\mathbf{X}_1(\mathbf{s})) - \mathbf{F}(\mathbf{X}_2(\mathbf{s})))|\,\mathbf{ds}. \tag{13}$$

After some calculations, the following inequality is obtained

$$\|\mathbf{G}(\mathbf{X}_1) - \mathbf{G}(\mathbf{X}_2)\| \leq K\,\|\mathbf{X}_1 - \mathbf{X}_2\|\,, \tag{14}$$

where

$$K = T\max\{(\sum_{i=1}^{3}|a_i| + |c_1|\,A^2), (\sum_{i=1}^{3}|b_i|), (2\,|c_1|\,A^2 + |c_2|), (\sum_{i=1}^{2}|h_i|)\}. \tag{15}$$

Then if $K < 1$, the mapping $\mathbf{X} = \mathbf{G}(\mathbf{X})$ is a contraction mapping and the following theorem gives the sufficient condition for existence and uniqueness of the solution of system (1)-(4).

Theorem 1. *The sufficient condition for existence and uniqueness of the solution of system (1)-(4) with initial conditions* $\mathbf{X}(\mathbf{0}) = \mathbf{X}_0$ *in the region* $\Omega\times J$ *is:* $T\max\{(\sum_{i=1}^{3}|a_i| + |c_1|\,A^2), (\sum_{i=1}^{3}|b_i|), (2\,|c_1|\,A^2 + |c_2|), (\sum_{i=1}^{2}|h_i|)\} < 1.$

2.2. **Continuous dependence on initial conditions.** In this subsection we determine the range of values for the parameters where the solution of system (1)-(4) exhibits continuous dependence on initial conditions. As continuous dependence on initial conditions opposes sensitive dependence on initial conditions that characterizes the chaotic behavior. The main benefit from knowing this range of parameters is that it theoretically enables researchers to determine the range of values of the system's parameters and time T where the system do not exhibit chaotic behavior.

Assume that there are two sets of initial conditions to system (5), $\mathbf{X}_{01}$ and $\mathbf{X}_{02}$, which satisfy

$$\|\mathbf{X}_{01} - \mathbf{X}_{02}\| \leq \boldsymbol{\delta}. \tag{16}$$

Also, assume that the condition of Theorem (1) is satisfied. Then

$$\mathbf{X}_1 = \mathbf{X}_{01} + \int_0^t \mathbf{F}(\mathbf{X}_1(\mathbf{s}))\mathbf{ds}, \quad (17)$$

$$\mathbf{X}_2 = \mathbf{X}_{02} + \int_0^t \mathbf{F}(\mathbf{X}_2(\mathbf{s}))\mathbf{ds}, \quad (18)$$

and the following inequality is obtained

$$\|\mathbf{X}_1 - \mathbf{X}_2\| \leq \|\mathbf{X}_{01} - \mathbf{X}_{02}\| + K\|\mathbf{X}_1 - \mathbf{X}_2\|, \quad (19)$$

and therefore

$$(1-K)\|\mathbf{X}_1 - \mathbf{X}_2\| \leq \|\mathbf{X}_{01} - \mathbf{X}_{02}\|, \quad (20)$$

where

$$K < 1, \quad (21)$$

is defined by (15).

Denoting $\frac{\delta}{(1-K)}$ by ϵ, then the following relation holds

$$\|\mathbf{X}_1 - \mathbf{X}_2\| \leq \epsilon. \quad (22)$$

Theorem 2. *For system* (1) − (4) *satisfying the condition of Theorem (1) and* K *defined by* (15). *Then,* $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) = (1-K)\epsilon > 0$ *such that* $\|\mathbf{X}_{01} - \mathbf{X}_{02}\| \leq \boldsymbol{\delta}$ *implies that* $\|\mathbf{X}_1 - \mathbf{X}_2\| \leq \epsilon$ *i.e. the solution exhibits continuous dependence on initial conditions.*

It is important to notice that the condition of preserving continuous dependence on initial conditions for the proposed system implies the existence of small range of time and values of system's parameters that are corresponding to regular behavior. Therefore, it can be estimated that the nonregular and chaotic behavior will exist for a wide range of values. This result is verified by numerical simulations in section 4.

3. Dynamical analysis of the proposed hyperchaotic system

To simplify study of dynamical behavior of system (1)-(4) in the following subsections, the values of some system's parameters are fixed as follows: $a_2 = 7.82, a_3 = 11.6, b_2 = -1.5, b_3 = 7.8125, c_1 = -19, h_1 = -8.5$, and $h_2 = -5.731$, whereas, the remaining parameters are allowed to vary.

3.1. Stability of E_0.

Theorem 3. *The equilibrium point* E_0 *of system (1)-(4) is local asymptotically stable if the following conditions are satisfied: (1):* $a_1 < 1.5$, *(2):* $c_2(44.7734 - 11.6b_1) > 66.4063a_1$, *(3):* $a_1c_2 < 0$, *and (4):* $44.773a_1^3c_2 - 134.56b_1^2c_2^2 - 17.4a_1^2b_1c_2 - 90.712a_1b_1^2c_2 - 99.609a_1^3 - 519.29a_1^2b_1 - 67.160a_1^2c_2 + 136.07b_1^2c_2 + 1038.7b_1c_2^2 - 394.08a_1b_1c_2 + 149.41a_1^2 - 2004.6c_2^2 + 778.94a_1b_1 + 2973.2a_1c_2 - 1680.7b_1c_2 - 6614.7a_1 + 4459.8c_2 > 0$.

Proof. The Jacobian of system (1)-(4) evaluated at E_0 is given by

$$J = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ 7.82 & -1.5 & -8.5 & c_2 \\ 0 & 7.8125 & 0 & 0 \\ 11.6 & 0 & -5.731 & 0 \end{bmatrix}, \quad (23)$$

where the characteristic equation is

$$\lambda^4 + (1.5 - a_1)\lambda^3 + (66.406 - 1.5a_1 - 7.82b_1)\lambda^2 + (44.773c_2 - 66.406a_1 - 11.6b_1c_2)\lambda - 44.773a_1c_2 = 0. \tag{24}$$

Let $p_1 = 1.5 - a_1, p_2 = 66.406 - 1.5a_1 - 7.82b_1, p_3 = 44.773c_2 - 66.406a_1 - 11.6b_1c_2$, and $p_4 = -44.773a_1c_2$ then all the roots of equation (24) have negative real parts if the conditions of Routh Hurwitz criterion are satisfied i.e.

$$p_i > 0, i = 1, 3, 4 \text{ and } p_1p_2p_3 > p_1^2p_4 + p_3^2, \tag{25}$$

and therefore the conditions of the theorem are obtained. □

Fig.1 shows the results of numerical simulations for $a_1 = 0.6$, $b_1 = 5$ and $c_2 = -4$ which satisfy the stability conditions of E_0 and the corresponding Jacobian matrix has the following eigenvalues: $-0.185463 \pm 4.59221i$ and $-0.264537 \pm 2.23992i$.

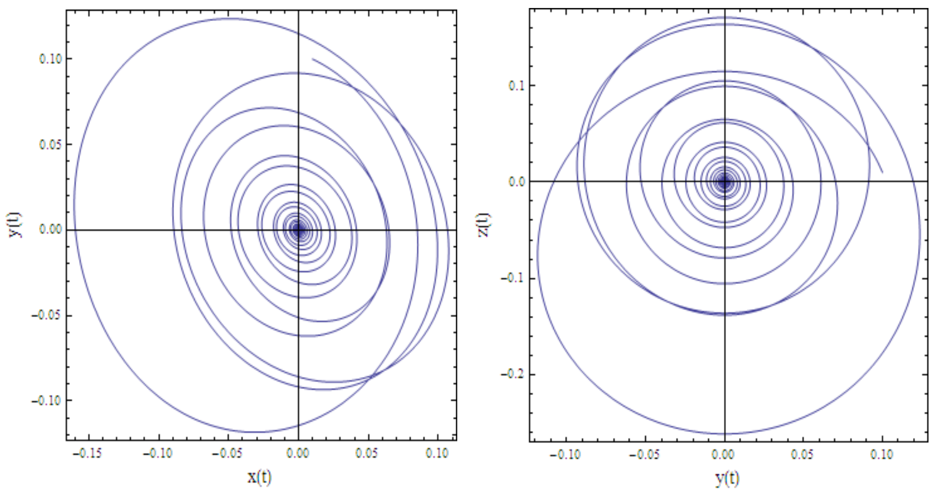

Fig.1 (a): $x - y$ plane Fig.1 (b): $y - z$ plane

Fig.1: Stable equilibrium point E_0 of system (1)-(4) at $a_1 = 0.6$, $b_1 = 5$ and $c_2 = -4$.

3.2. **Stability of E_+ and E_-.**

Theorem 4. *The equilibrium point E_+ (E_-) of system (1)-(4) is local asymptotically stable if the following conditions are satisfied: (1): $b_1 < \dfrac{3.708a_1 + 44.773}{11.6}$ for $c_2 > 0$ or $b_1 > \dfrac{3.708a_1 + 44.773}{11.6}$ for $c_2 < 0$, (2): $a_1c_2 > 0$, and (4): $43.013a_1b_1c_2^2 - 134.56b_1^2c_2^2 - 166.03a_1c_2^2 + 136.07b_1^2c_2 + 1038.7b_1c_2^2 - 43.495a_1b_1c_2 + 167.89a_1c_2 - 1680.7b_1c_2 - 2004.6c_2^2 + 4459.85c_2 > 0$.*

Proof. The Jacobian of system (1)-(4) evaluated at E_1 (E_2) is given by

$$J = \begin{bmatrix} 0 & b_1 & 0 & -0.2131a_1c_2 \\ 7.82 & -1.5 & -8.5 & c_2 \\ 0 & 7.8125 & 0 & 0 \\ 11.6 & 0 & -5.731 & 0 \end{bmatrix}, \tag{26}$$

where the characteristic equation is

$$\lambda^4 + 1.5\lambda^3 + (2.472a_1c_2 - 7.82b_1 + 66.406)\lambda^2 + (3.708a_1c_2 - 11.6b_1c_2 + 44.773c_2)\lambda + 89.547a_1c_2. \tag{27}$$

Let $p_1 = 1.5, p_2 = 2.472a_1c_2 - 7.82b_1 + 66.406, p_3 = 3.708a_1c_2 - 11.6b_1c_2 + 44.773c_2$, and $p_4 = 89.547a_1c_2$ then all the roots of equation (27) have negative real parts if the conditions of Routh Hurwitz criterion (25) are satisfied. From (25), the conditions of local asymptotic stability of equilibrium point E_+ (E_-) can be obtained. □

Fig.2 shows the results of numerical simulations for $a_1 = 0.5$, $b_1 = 2$ and $c_2 = 2$ at which $E_+ = (0.0346, 0, 0.07, 0.162)$ is local asymptotically stable where $-0.300466 \pm 7.13091i$ and $-0.449534 \pm 1.24732i$ are the eigenvalues of corresponding Jacobian matrix .

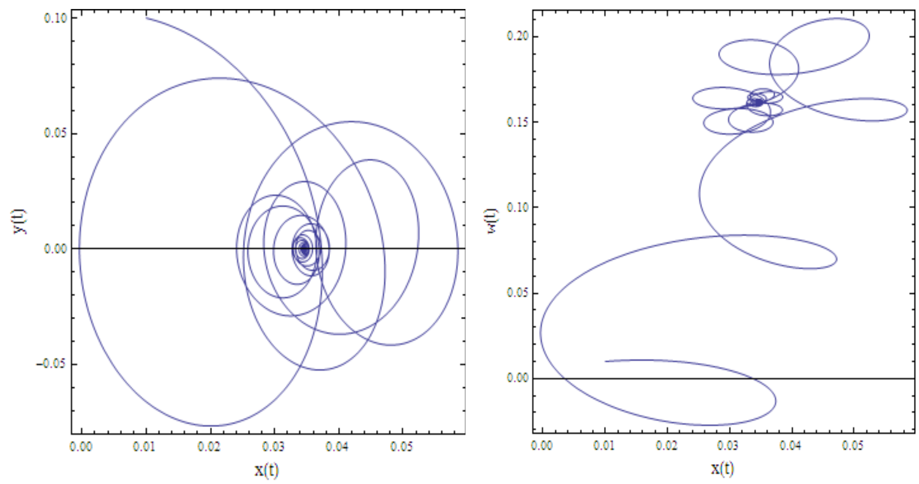

Fig.2 (a): $x - y$ plane Fig.2 (b): $x - w$ plane

Fig.2: Stable equilibrium point E_+ of system (1)-(4) at a_1= 0.5, b_1= 2 and c_2= 2.

3.3. **Hopf bifurcation about** E_0**.** Substitute $\lambda = i\omega$ in (24), where $\omega \in \mathbb{R}^+$ and i is the imaginary unit, yields

$$\omega^4 - i(1.5 - a_1)\omega^3 - (66.406 - 1.5a_1 - 7.82b_1)\omega^2 + i(44.773c_2 - 66.406a_1 - 11.6b_1c_2)\omega - 44.773a_1c_2 = 0, \tag{28}$$

then solving the following two equations

$$\omega^4 - (66.406 - 1.5a_1 - 7.82b_1)\omega^2 - 44.773a_1c_2 = 0, \tag{29}$$

$$(1.5 - a_1)\omega^3 + (44.773c_2 - 66.406a_1 - 11.6b_1c_2)\omega = 0, \tag{30}$$

we can obtain

$$\omega = \omega_0 = \sqrt{\frac{44.773c_2 - 66.406a_1 - 11.6b_1c_2}{1.5 - a_1}},\ b_1 = b^* \tag{31}$$

where b* is depending on a_1 and c_2.

We set $a_1 = 5.9$ and $c_2 = 2.7$ and take b_1 as a bifurcation parameter, then

$$b^* = 0.467049,\ \omega_0 = 8.0557. \tag{32}$$

From (24), let $\lambda = \alpha + i\omega$ and compute the value of $\frac{d\alpha}{db_1}$ at $b_1 = b^*$ and $\omega = \omega_0$ to obtain

$$\left(\frac{d\alpha}{db_1}\right)_{\substack{b_1=b^* \\ \omega=\omega_0}} = -0.328557 \neq 0 \tag{33}$$

and therefore the conditions of existence of Hopf bifurcation are satisfied.

The stability of limit cycle is determined using the first Lyapunov coefficient as follows:

At the critical parameter value the Jacobian matrix evaluated at critical point b^* has the form

$$J = \begin{bmatrix} 5.9 & 0.467049 & 0 & 0 \\ 7.82 & -1.5 & -8.5 & 2.7 \\ 0 & 7.8125 & 0 & 0 \\ 11.6 & 0 & -5.731 & 0 \end{bmatrix}, \tag{34}$$

and the vectors

$$q = \begin{bmatrix} -0.0180353 - i0.024625 \\ 0.652566 \\ -i0.632863 \\ 0.414772 + i0.0259703 \end{bmatrix}, p = \begin{bmatrix} -0.503485 + i0.400889 \\ 0.72189 - i0.142706 \\ 0.0215536 - i0.79573 \\ 0.0478301 + i0.241953 \end{bmatrix} \tag{35}$$

are eigenvectors of J and J^T that are corresponding to the eigenvalues $i8.\,0557$ and $-i8.\,0557$, respectively, i.e.

$$Jq = i8.\,0557q, \ J^T p = -i8.\,0557p \text{ and } \langle p, q\rangle = 1 \tag{36}$$

where $\langle p, q\rangle = \sum_{k=1}^{n} \bar{p}_k q_k$.

Suppose that the system (1)-(4) is written as

$$\dot{\mathbf{X}}(\mathbf{t}) = A(\eta) + \mathbf{F}_1(\mathbf{X}, \eta), \tag{37}$$

where $F_1 = O(\|X\|^2)$ and $\eta = b_1$ is the bifurcation parameter. Then $\mathbf{F}_1(\mathbf{X}, \eta_0)$ can be represented by

$$\mathbf{F}_1(\mathbf{X}, \eta) = \frac{1}{2}B(X, X) + \frac{1}{6}C(X, X, X) + O(\|X\|^4), \tag{38}$$

in which $B(X, Y)$ and $C(X, Y, U)$ are bilinear and trilinear vector functions of X, Y, and $U \in \mathbb{R}^n$, $n = 4$, and they can be obtained by

$$B_i(X, Y) = \sum_{j,k=1}^{n} \frac{\partial^2 F_i(\boldsymbol{\zeta}, b^*)}{\partial\zeta_j \partial\zeta_k}|_{\boldsymbol{\zeta}=\mathbf{o}} x_j y_k, i = 1, 2, ..., n \tag{39}$$

$$C_i(X, Y, U) = \sum_{j,k,m=1}^{n} \frac{\partial^3 F_i(\boldsymbol{\zeta}, b^*)}{\partial\zeta_j \partial\zeta_k \partial\zeta_m}|_{\boldsymbol{\zeta}=\mathbf{o}} x_j y_k u_m, i = 1, 2, ..., n. \tag{40}$$

After some calculations, we obtain the forms of $B(X_1, X_2)$ and $C(X_1, X_2, X_3)$ as

$$B(X_1, X_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ C(X_1, X_2, X_3) = \begin{bmatrix} -38(x_1 w_2 w_3 - x_2 w_1 w_3 - x_3 w_1 w_2) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{41}$$

for $X_i = [x_i, y_i, z_i, w_i]^T$. Then the the first Lyapunov coefficient l_1 is given by [32]

$$\begin{aligned} l_1 &= \frac{1}{2\omega_0} \operatorname{Re}[\langle p, C(q, q, \bar{q})\rangle] \\ &\approx -0.05 < 0. \end{aligned} \tag{42}$$

The Lyapunov coefficient is negative, thus the Hopf bifurcation is nondegenerate and supercritical such that a stable limit cycle is generated for $b_1 > b^*$, see Fig.3.

Now, we fix $b_1 = 0.467049$, find the other values of a_1 and c_2 at which Hopf bifurcation occurs, and represent these values by a curve in the plane of parameters a_1 and c_2 as shown in Fig.4. Generalized Hopf (GH) bifurcation or Bautin bifurcation occurs when the equilibrium point in two parameter family of autonomous system of differential equations has a pair of purely imaginary eigenvalues and the first Lyapunov coefficient for the Andronov-Hopf bifurcation vanishes. Zero-Hopf (ZH) bifurcation or saddle-node Hopf bifurcation occurs when the equilibrium point in two parameter family of autonomous system of differential equations has a zero eigenvalue and a pair of purely imaginary eigenvalues [32].

Choosing different set of values of parameters, $b_1 = 5$ and $c_2 = -4$, and varying parameter a_1, we illustrate the dynamics examined from computations such as Pitchfork bifurcation (or Branch Point (BP)) and Hopf bifurcation (H) in bifurcation diagram of Fig.5.

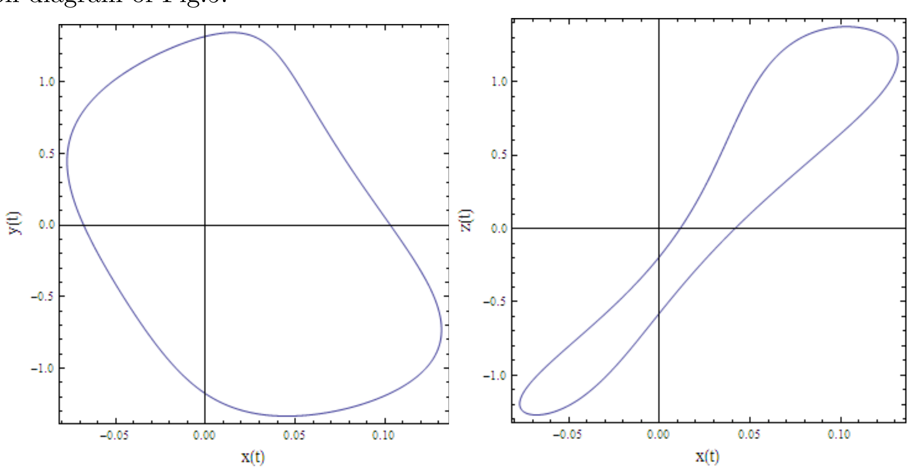

Fig.3 (a) Fig.9 (b)

Fig.3: A stable limit cycle around E_0 exists at $b_1 = 0.51$.

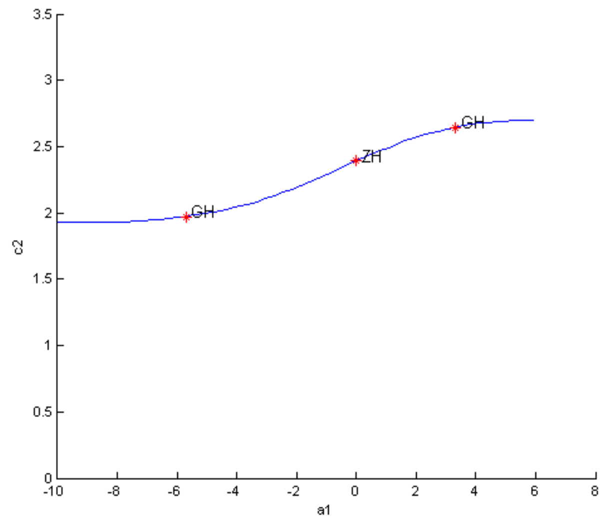

Fig.4: The double parameters Hopf bifurcation diagram in $a_1 - c_2$ plane.

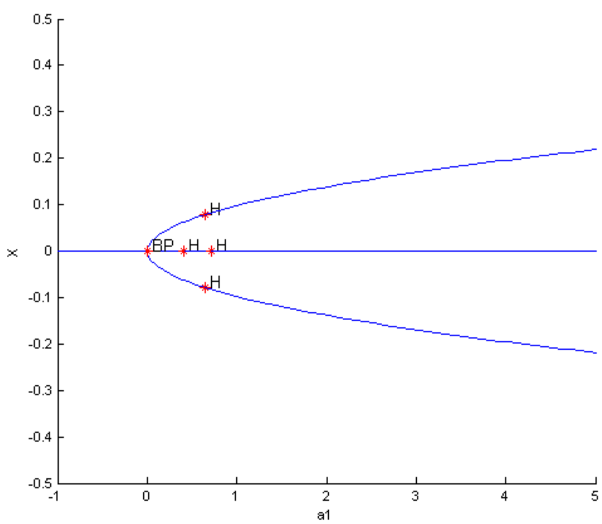

Fig.5: Different dynamics of system (1)-(4) exist at $b_1 = 5$ and $c_2 = -4$.

3.4. **Nonhyperbolic case of E_0.** Let $a_1 = a_c$ where the critical value $a_c = 0$, then one of the roots of equation (24) equals zero and the equilibrium point E_0 is nonhyperbolic. We use the center manifold theorem to investigate the stability of E_0 when one root of equation (24) equals zero and the other roots are negative reals as follows:

Proposition 5. *Suppose that the following relations hold*

$$
\begin{aligned}
b_1 &= 0.127\,88(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2)-0.191\,82(\lambda_2+\lambda_3)+8.\,491\,9, & (43)\\
c_2 &= \frac{\lambda_2\lambda_3(50048(\lambda_2+\lambda_3)-75072)}{74240(\lambda_2^2+\lambda_2\lambda_3+\lambda_3^2)-111360(\lambda_2-\lambda_3)+2.68918\times 10^6}, & (44)\\
\lambda_1 &= 0.5(3-2(\lambda_2+\lambda_3)), & (45)
\end{aligned}
$$

where $\lambda_i \in \mathbb{R}^-$, then the characteristic equation corresponding to E_0 is given by

$$\lambda(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)=0. \quad (46)$$

For example, assume that $b_1 = 8.39722$ and $c_2 = -0.00228$, then the eigenvalues of Jacobian matrix corresponding to E_0 are $0, -0.4, -0.5$, and -0.6. To study the stability of E_0, the following transformations are defined

$$
\begin{pmatrix} x\\ y\\ z\\ w \end{pmatrix} = \begin{pmatrix} -0.000243 & -0.0793 & 0.0949 & 0.06357\\ 0 & 0.00472 & -0.00678 & -0.003028\\ -0.000492 & -0.0738 & 0.0883 & 0.05915\\ 1 & 0.9941 & -0.9915 & -0.99622 \end{pmatrix} \begin{pmatrix} x_1\\ y_1\\ z_1\\ w_1 \end{pmatrix}, \quad (47)
$$

$$\mu = a_1 - a_c, \quad (48)$$

then the system (1)-(4) is transformed to the following standard form

$$
\begin{aligned}
\begin{pmatrix} \dot{x}_1\\ \dot{y}_1\\ \dot{z}_1\\ \dot{w}_1 \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & -0.5 & 0 & 0\\ 0 & 0 & -0.6 & 0\\ 0 & 0 & 0 & -0.4 \end{pmatrix} \begin{pmatrix} x_1\\ y_1\\ z_1\\ w_1 \end{pmatrix} + \begin{pmatrix} g_1\\ g_2\\ g_3\\ g_4 \end{pmatrix}, \quad (49)\\
\dot{\mu} &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
g_1 = & -0.85\mu x_1 - 277.7\mu y_1 + 332.3\mu z_1 + 222.6\mu w_1 + 16.2x_1^3 + 5213.7y_1^3 \\
& - 6208.13z_1^3 - 4197.62w_1^3 - 4261.7x_1^2 w_1 + 5307.91x_1^2 y_1 - 6346.6x_1^2 z_1 \\
& - 16640.81y_1^2 z_1 - 14629.42y_1^2 w_1 + 10505.21x_1 y_1^2 + 12538.06x_1 z_1^2 \\
& + 8443.14x_1 w_1^2 + 17635.21y_1 z_1^2 + 13613.32y_1 w_1^2 - 16633.104w_1 z_1^2 \\
& - 14622.6z_1 w_1^2 - 23048.6x_1 y_1 z_1 - 18952.8x_1 y_1 w_1 + 21000.7x_1 \\
& z_1 w_1 + 31267.7y_1 z_1 w_1, \\
g_2 = & -20.3\mu x_1 - 6611\mu y_1 + 7912.8\mu z_1 + 5300\mu w_1 + 385x_1^3 + 124133.4y_1^3 \\
& - 147808.8z_1^3 - 99941.11w_1^3 + 126374.92x_1^2 y_1 - 151105.5x_1^2 z_1 \\
& - 101468.1x_1^2 w_1 - 396200.8y_1^2 z_1 - 396017z_1^2 w_1 + 250119x_1 y_1^2 \\
& + 298518.8x_1 z_1^2 + 201023x_1 w_1^2 + 419876.1y_1 z_1^2 + 324120y_1 w_1^2 - \\
& 451247.3x_1 y_1 w_1 - 348312y_1^2 w_1 - 348150.7z_1 w_1^2 - 548765x_1 y_1 z_1 \\
& + 500006.4x_1 z_1 w_1 + 744455.2y_1 z_1 w_1, \\
g_3 = & -8.5\mu x_1 - 2758.6\mu y_1 + 3302\mu z_1 + 2211.6\mu w_1 + 1601x_1^3 + 51798y_1^3 \\
& - 61677.4z_1^3 - 41703.1w_1^3 + 52733.4x_1^2 y_1 - 63053x_1^2 z_1 - 42340.4x_1^2 w_1 \\
& - 165326y_1^2 z_1 - 145342.9y_1^2 w_1 - 165249.3z_1^2 w_1 + 104369.2x_1 y_1^2 \\
& + 124565.3x_1 z_1^2 + 83882.4x_1 w_1^2 + 175205y_1 z_1^2 + 135248.1y_1 w_1^2 \\
& - 145276z_1 w_1^2 - 228987x_1 y_1 z_1 - 188295.5x_1 y_1 w_1 + 208642x_1 z_1 w_1 \\
& + 310644.7y_1 z_1 w_1. \\
g_4 = & -12.7\mu x_1 - 4130\mu y_1 + 4943.28\mu z_1 + 3311\mu w_1 + 240.4x_1^3 + 77548.9y_1^3 \\
& - 92339.4z_1^3 - 62435.4w_1^3 + 78949.2x_1^2 y_1 - 94399x_1^2 z_1 - 63389.4x_1^2 w_1 \\
& - 247515.5y_1^2 z_1 - 217598y_1^2 w_1 - 247401z_1^2 w_1 + 156255x_1 y_1^2 + 186491x_1 z_1^2 \\
& + 125584x_1 w_1^2 + 262306y_1 z_1^2 + 202485y_1 w_1^2 - 217497z_1 w_1^2 - 342826x_1 y_1 z_1 \\
& - 281904x_1 y_1 w_1 + 312365x_1 z_1 w_1 + 465078y_1 z_1 w_1.
\end{aligned}
$$

We take the parameter μ as the bifurcation parameter of system (49) and also as a new independent variable of system (49). Thus, from center manifold theory [30]-[33], there exists a center manifold for (49) that is given by:

$$
\begin{aligned}
\Psi^c(0) \quad &= \quad \{(x_1, y_1, z_1, w_1, \mu) \in \mathbb{R}^5 | y_1 = \phi_1(x_1, \mu), z_1 = \phi_2(x_1, \mu), w_1 = \phi_3(x_1, \mu), \\
|x_1| \quad &< \quad \epsilon, |\mu| < \delta, \phi_i(0,0) = 0, D\phi_i(0,0) = 0, i = 1, 2, 3\}, \qquad (50)
\end{aligned}
$$

for ε and δ sufficiently small. The center manifold $\Psi^c(0)$ is computed by assuming that $\phi_i(x_1, \mu)$ have the following forms

$$
\begin{aligned}
\phi_1(x_1, \mu) &= \alpha_1 x_1^2 + \alpha_2 \mu x_1 + \alpha_3 \mu^2 + ..., \\
\phi_2(x_1, \mu) &= \beta_1 x_1^2 + \beta_2 \mu x_1 + \beta_3 \mu^2 + ..., \qquad (51) \\
\phi_3(x_1, \mu) &= \gamma_1 x_1^2 + \gamma_2 \mu x_1 + \gamma_3 \mu^2 + ...,
\end{aligned}
$$

The center manifold must satisfy

$$
D_{x_1}\phi(x_1, \mu)[Ax_1 + g_1(x_1, \phi(x_1, \mu), \mu)] - B\phi(x_1, \mu) - \hat{g}(x_1, \phi(x_1, \mu), \mu) = 0, \quad (52)
$$

where $A = 0$, $B = \begin{pmatrix} -0.6 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.4 \end{pmatrix}$, $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}$,and $\hat{g} = \begin{pmatrix} g_2 \\ g_3 \\ g_4 \end{pmatrix}$. By substituting from Eq. (51) into Eq. (52) and then equating terms of like powers to zero, we get

$$\alpha_1 = \alpha_3 = 0, \alpha_2 = -40.51, \quad (53)$$

$$\beta_1 = \beta_3 = 0, \beta_2 = -14.085, \quad (54)$$

$$\gamma_1 = \gamma_3 = 0, \ \gamma_2 = -31.63, \quad (55)$$

and therefore the vector field reduces to the center manifold that is defined by

$$\dot{x}_1 = -0.851\mu x_1 + 16.16x_1^3 - 475.2\mu^2 x_1 + 9197.2\mu x_1^3 + 94527.2\mu^2 x_1^3 + 245122\mu^3 x_1^3, \quad (56)$$

$$\dot{\mu} = 0. \quad (57)$$

Lemma 6. *The dynamics of system (1)-(4) about E_0 at parameters values used in subsection 3.4 is restricted locally to dynamics of center manifold (56)-(57).*

4. Examination of chaotic behavior

In this section, we examine and illustrate the existence of chaotic behavior of the new system using different approaches.

First, the values of the system's parameters are chosen as follows: $a_1 = 5.9, c_1 = -19, a_2 = 7.82, b_2 = -1.5, h_1 = -8.5, c_2 = 2.7, b_3 = 7.8125, a_3 = 11.6$, and $h_2 = -5.731$. We vary the value of parameter b_1 to change the dynamical behavior of the system (1)-(4). Table 1 illustrates the values of Lyapunov exponents (LEs), calculated using the algorithm given in [34], for some selected values of parameter b_1.

Table 1: The values of Lyapunov exponents computed at different values of b_1.

b_1	LEs	Notes
0.99	$0.00863944, 0.00572583, -0.118931, -1.20856$	Torus
1.36	$0.76076, 0.1281503, -0.0883511, -2.84797$	Hyperchaotic
2.11	$1.015304, 0.027329, -0.9708, -2.4902$	Hyperchaotic
2.65	$0.0263368, -0.445012, -0.462173, -3.6269$	Periodic
5.1	$0.786302, -0.00914677, -1.19776, -7.17582$	Chaotic

The bifurcation diagrams are used to give a full view of the dynamics of the new system (1)-(4) and illustrate the complexity of system's behavior. Fig.6 shows the bifurcation diagrams that are obtained at different values of parameter b_1. Second, we fix $b_1 = 1.36$ and vary parameter a_1 of system (1)-(4) to obtain the bifurcation diagram in Fig.7 (a). Then we also fix $b_1 = 1.36$ but vary parameter b_3 to get the bifurcation diagrams in Fig.7 (b) to Fig.7 (d).

From the bifurcation diagrams, we deduce that the proposed system does not exhibit classical period doubling route to chaos for the selected values of parameters. But, the system shows the existence of tangent bifurcation that produces the periodic windows exist in Fig.6 and Fig.7 along with sudden changes in chaotic attractors with parameter variation and intermittency route to chaos. These sudden changes are called crises and there are three types of crises [35]. In the first type, the boundary crisis, a chaotic attractor is suddenly destroyed (or created) as the parameter passes through a critical value. In the second type, the interior crisis, the size of the attractor in phase space suddenly increases. In the third type, the attractor merging crisis, two or more chaotic attractors merge to form one chaotic attractor. So, the investigated behaviors include cascades of tangent bifurcations (and therefore periodic windows), intermittency route to chaos, and crises.

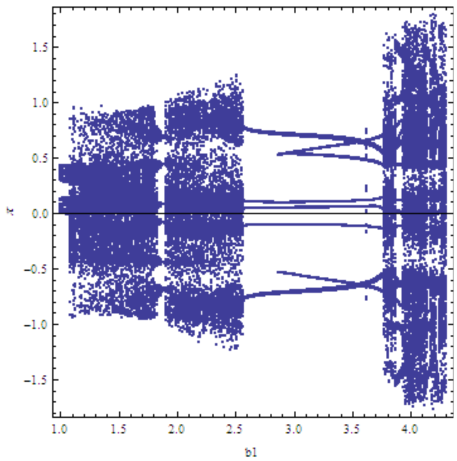

Fig.6 (a)

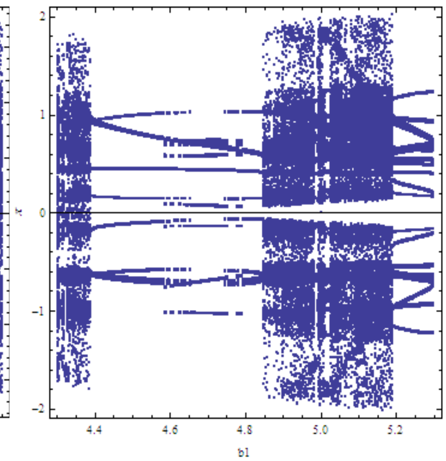

Fig.6 (b)

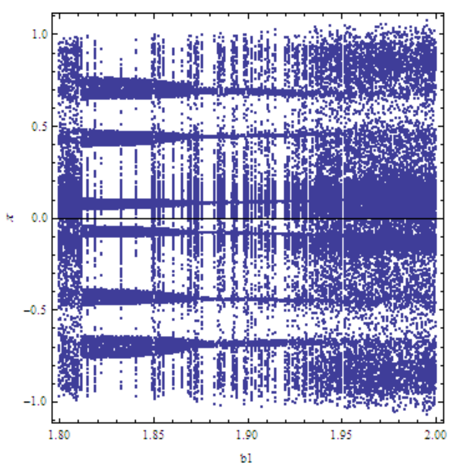

Fig.6 (c)

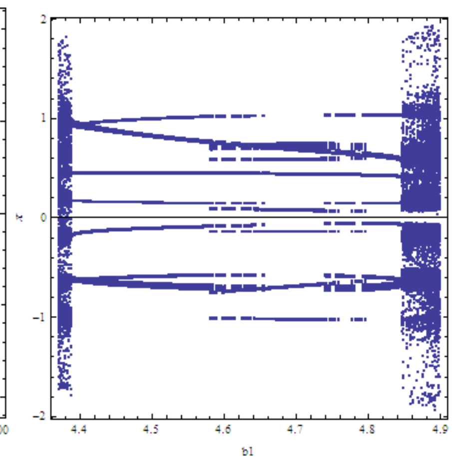

Fig.6 (d)

Fig.6: Bifurcation diagrams of the system (1)-(4) obtained via state variable x for (a) $b_1 \in [1, 4.3]$, (b) $b_1 \in [4.3, 5.3]$,(c) $b_1 \in [1.8, 2]$, and (d) $b_1 \in [4.37, 4.9]$

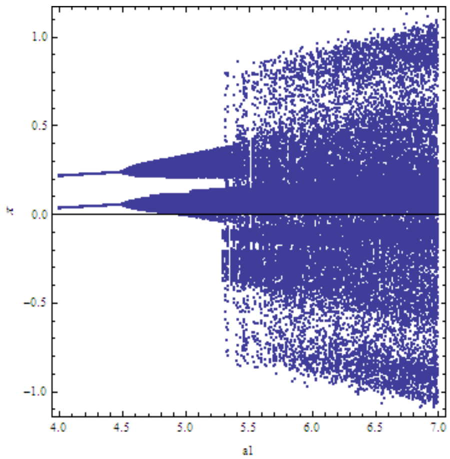

Fig.7 (a)

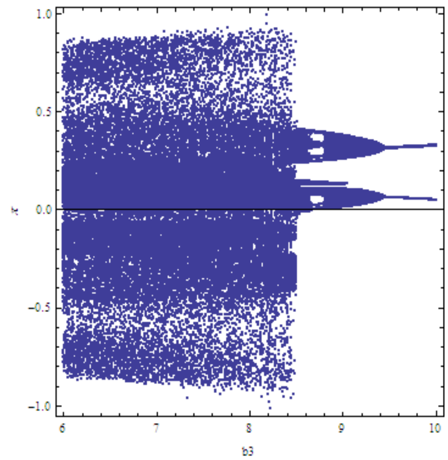

Fig.7 (b)

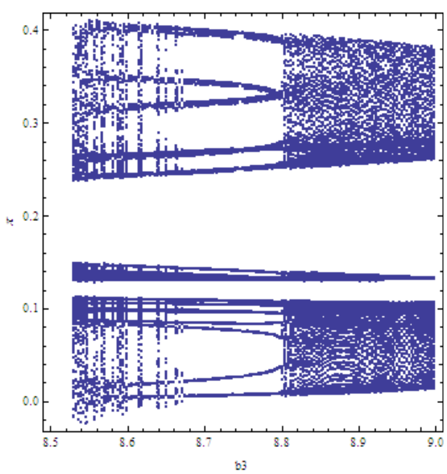

Fig.7 (c)

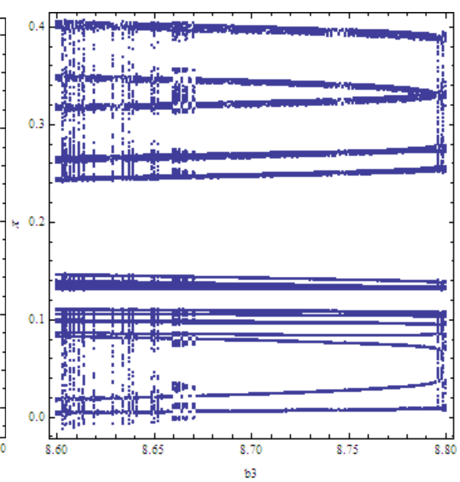

Fig.7 (d)

Fig.7: Bifurcation diagrams of the system (1)-(4) obtained via state variable x at b_1= 1.36 for (a) $4 \leq a_1 \leq 7$,(b) $6 \leq b_3 \leq 10$, (c) $8.5 \leq b_3 \leq 9$,and (d) $8.6 \leq b_3 \leq 8.8$.

Also, from Fig.6 and Fig.7, it is shown that numerical results agree with previous estimates in section 2 that there are wide ranges of values of systems's parameters at which nonregular behavior exists. Examples of phase portraits of state variables x and y obtained at different values of parameter b_1 of the system (1)-(4) are illustrated in Fig.8.

5. Circuit realization of the proposed system

Fig.9 illustrates the proposed circuit implementation of system (1)-(4) where the values of circuit elements used are illustrated in the figure. The value of parameter b_1 is related to resistor R in circuit schematic by $b_1 = \dfrac{1}{R}$ where R is measured in mega ohms.

Circuit simulations are carried out using *Multisim 11* and the outputs of the proposed circuit are illustrated on the oscilloscope. Fig.10 shows some examples of the results of circuit simulations where it is clear that these results agree with the results of numerical simulations, see Fig.8 (c) and Fig.8 (d).

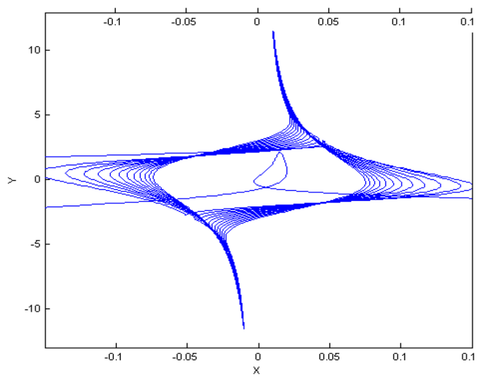

Fig.8 (a)

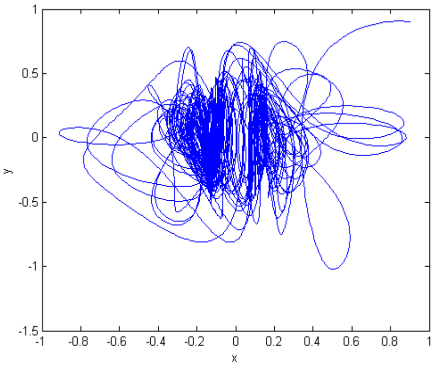

Fig.8 (b)

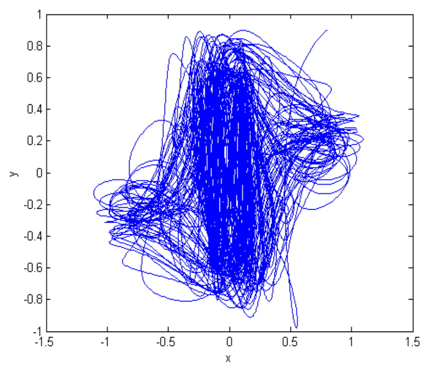

Fig.8 (c)

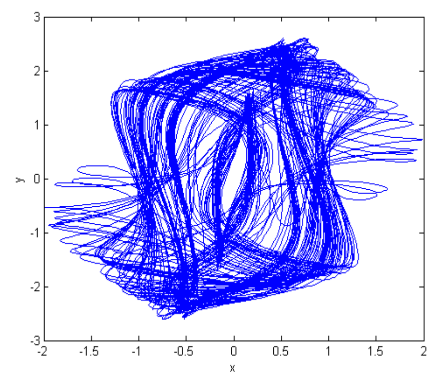

Fig.8 (d)

Fig.8: Phase portraits of state variables x and y of the system (1)-(4) obtained for (a) $b_1 = 0.99$, (b) $b_1 = 1.36$, (c) $b_1 = 2.11$, and (d) $b_1 = 5.1$.

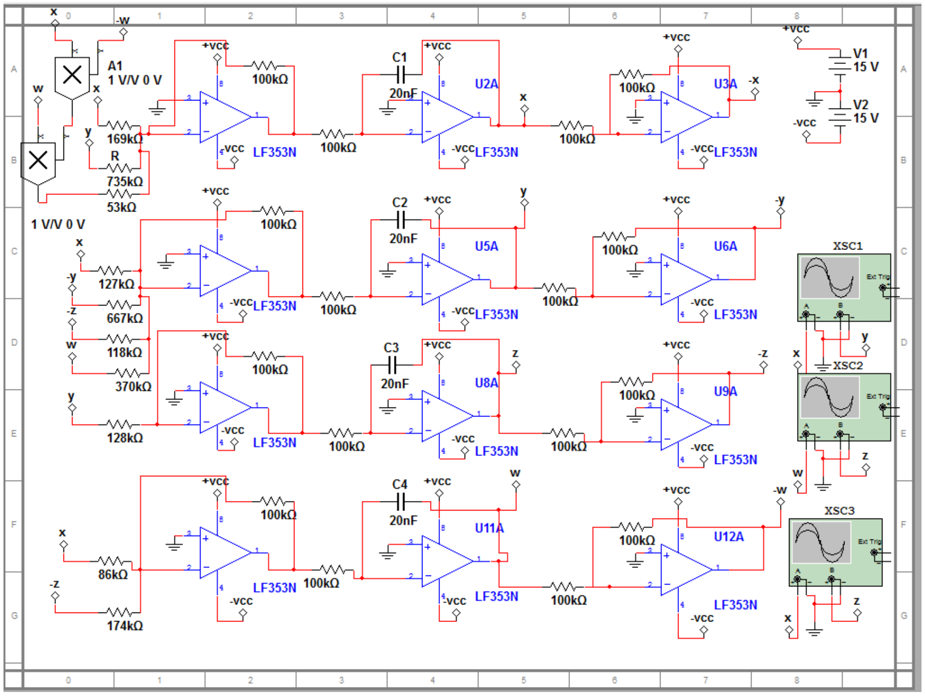

Fig.9: Circuit implementation of new hyperchaotic system (1)-(4) for $b_1 = 1.36$.

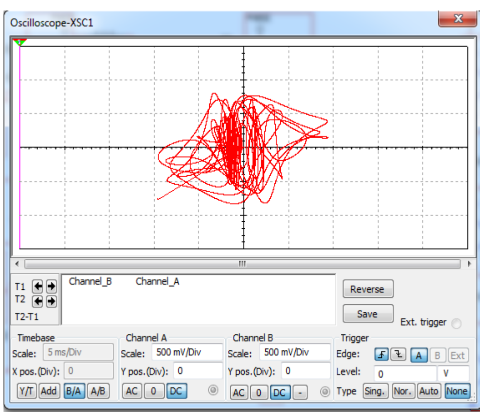

Fig.10 (a)

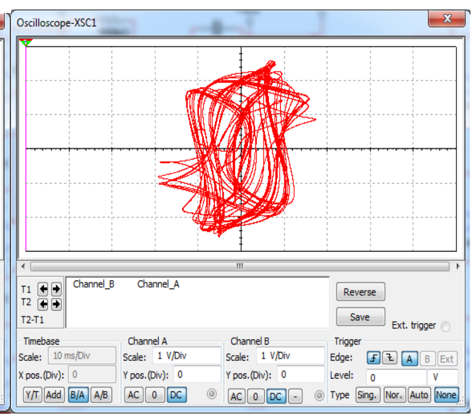

Fig.10 (b)

Fig.10: The outputs of circuit simulations represent phase portraits of state variables x and y in cases of (a) $b_1 = 2.11$ and (b) $b_1 = 5.1$

6. Conclusion

This paper is an attempt to introduce and investigate the dynamical behavior of a new 4D hyperchaotic system using theoretical and numerical methods, then realize these dynamics using an electronic circuit. It is shown that the proposed model has rich dynamics and chaos that exists over a wide range of parameters of the system and therefore has the advantage of possessing a large domain of secret keys which is more suitable for applications related to chaos based image and real-time video encryption. The future work can include the design and implementation of secure communication scheme based on the proposed new hyperchaotic system.

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