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ON A CLASS OF OPERATORS RELATED TO GENERALIZED PARANORMAL OPERATORS

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Abstract. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *generalized p-paranormal* if

$$\||T|^p U|T|^p x\| \, \|x\| \geq \frac{1}{M^p} \||T|^p x\|^2$$

for all $x \in \mathcal{H}$, $p > 0$, and $M > 0$, where U is the partial isometry appeared in the polar decomposition $T = U|T|$ of T. The aim of this note is to obtain some structure theorems for a class of generalized p-paranormal operators. Exactly we will give some conditions which are generalization of concepts of generalized paranormal operators.

1. Introduction

Let $\mathcal{H}$ be an infinite dimensional complex Hilbert and $\mathbf{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker(U) = \ker(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory ([6], [12], [16] and [21]). In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker(U) = \ker(|T|)$.

Recall that an operator $T \in \mathbf{B}(\mathcal{H})$ is *positive*, $T \geq 0$, if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathcal{H}$. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *hyponormal* if $T^*T \geq TT^*$. Hyponormal operators have been studied by many authors and it is known that hyponormal operators have many interesting properties similar to those of normal operators ([1], [4], [5], [9], [11] and [15]). An operator T is said to be *p-hyponormal* if$(T^*T)^p \geq (TT^*)^p$ for $p \in (0, 1]$ and an operator $T$is said to be log-*hyponormal* if T is invertible and $\log|T| \geq \log|T^*|$. p-hyponormal and log-hyponormal operators are defined as extension of hyponormal operator.

An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *paranormal* if it satisfies the following norm inequality

$$\left\|T^2 x\right\| \geq \|Tx\|^2$$

for every unit vector $x \in \mathcal{H}$. Ando [3] proved that every log-hyponormal operators is paranormal. It was originally introduced as an intermediate class between hyponormal operators and normaloid. It has been studied by many authors, so there are many to cite their references,for instance, [3, 9, 22]. We say that an operator T belong to *class A* if $|T^2| \geq |T|^2$. class A was first introduced by Furuta-Ito-Yamazaki [11] as a subclass of paranormal which include the class of p-hyponormal and log-hyponormal operators.

In [11], they showed that every log-hyponormal operator is a class A operator and every class A is paranormal operator. Moreover, in [11], they introduced new classes as follows: An operator

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T belong to class $A(k)$ for $k>0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geq |T|^2$, and also an operator T is *absolute-k-paranormal* for $k>0$ if $\||T|^kTx\| \geq \|Tx\|^{k+1}$ for every unit vector $x \in \mathcal{H}$. Particularly an operator T is a class A (resp. paranormal) operator if and only if T is a class $A(1)$ (resp. absolute-1-paranormal). On class $A(k)$ operators and absolute-k-paranormal operators, they proved the following result.

Theorem 1.1. [11, Theorem 2]

(i) *Every* log-*hyponormal is a class $A(k)$ operator for $k>0$.*
(ii) *For each $k>0$, every invertible class $A(k)$ operator is class $A(l)$ operator for $l \geq k$.*
(iii) *For each $k>0$, every absolute-k-paranormal operator is an absolute-l-paranormal operator for $l \geq k$.*
(iv) *For each $k>0$, every class $A(k)$ operator is absolute-k-paranormal operator.*

Theorem 1.1 states that invertible class $A(k)$ operators determined by operator inequalities and absolute-k-paranormal determined by norm inequalities have monotonicity on $k>0$, namely they constitute clearly parallel and increasing lines.

On the other hand, Fujii-Izumino-Nakamoto [6] introduced the p-paranormality for operators. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be *p-paranormal* if

$$\||T|^pU|T|^px\| \, \|x\| \geq \||T|^px\|^2$$

for all $x \in \mathcal{H}$ and $p>0$, where U is the partial isometry appeared in the polar decomposition $T=U|T|$ of T. And they proved that every p-paranormal operator is paranormal for $0<p<1$. It is easy that every 1-paranormal operator is paranormal. In addition, the p-paranormality is based on the fact that $T=U|T|$ is p-hyponormal if and only if $S=U|T|^p$ is hyponormal [7]. Actually, $T=U|T|$ is p-paranormal if and only if $S=U|T|^p$ is paranormal. From this fact, a p-hyponormal operator is a p-paranormal operator for $p>0$. Recently Fujii et al. [8] introduced a family $\{A(p,q); p,q>0\}$ of new classes of operators. For $p,q>0$, an operator T belongs to $A(p,q)$ if T satisfies an operator inequality

$$(|T^*|^q|T|^{2p}|T^*|^q)^{\frac{q}{q+p}} \geq |T^*|^{2q}.$$

Note that $A(k,1)$ is the classes $A(k)$ due to Furuta-Ito-Yamazaki. Namely the family $\{A(p,q): p,q>0\}$ is a generalization of $\{A(k): k>0\}$ exactly. Also, in [8] they discussed inclusion relations between $A(p,q)$ and p-paranormal operators. And we proved that every p-paranormality has monotone increasing property on $p>0$ and every p-paranormal operator is normaloid.

2. Relation between two operator inequality

Aluthge and Wang [2] introduced w-hyponormal operators defined via Aluthge transformation as follows. An operator T is said to be *w-hyponormal* if

$$|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|,$$

where the polar decomposition of T is $T=U|T|$ and $\widetilde{T}=|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ is the Aluthge transformation of T. As a generalization of w-hyponormality, Ito [13] introduced class $wA(p,q)$ as follows.

Definition 2.1. ([13]) An operator T belongs to *class* $wA(p,q)$ for $p>0$ and $q>0$ if

$$(|T^*|^q|T|^{2p}|T^*|^q)^{\frac{q}{p+q}} \geq |T^*|^{2q} \tag{2.1}$$

and

$$|T|^{2p} \geq (|T|^p|T^*|^{2q}|T|^p)^{\frac{p}{p+q}}. \tag{2.2}$$

He pointed out the following fact which states that $wA(p,q)$ can be expressed via generalized Aluthge transformation.

Proposition 2.2. *([13])An operator T belongs to class $wA(p,q)$for $p>0$ and $q>0$ if and only if*

$$|\widetilde{T}_{p,q}|^{\frac{2q}{p+q}} \geq |T|^{2q} \qquad \text{and} \qquad |T|^{2p} \geq |\widetilde{T}^*_{p,q}|^{\frac{2p}{p+q}},$$

where the polar decomposition of T is $T = U|T|$ and $\widetilde{T}_{p,q}$ is the generalized Aluthge transformation of T, i.e.,

$$\widetilde{T}_{p,q} = |T|^p U |T|^q.$$

Lemma 2.3. *([10]) Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^* \tag{2.3}$$

holds for any real number λ.

Theorem 2.4. (Heinz-Löwner Inequality) If $A \geq B \geq 0$, then $A^{\alpha} \geq B^{\alpha}$ for any $\alpha \in [0,1]$.

Theorem 2.5. *([20]) Let A and B be positive operators. Then for each $p \geq 0$ and $r \geq 0$, the following assertion holds:*

If $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{r+p}} \geq B^r$, then $A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{r+p}}$.

We remark that in case A and B are positive and invertible,

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{r}{r+p}} \geq B^r \Longleftrightarrow A^p \geq (A^{\frac{p}{2}}B^rA^{\frac{p}{2}})^{\frac{p}{r+p}}$$

for each $p \geq 0$ and $r \geq 0$ by Lemma 2.3. By Using Theorem 2.5 we have.

Corollary 2.6. *([20])*

(i) *Class $A(p,q)$ coincides with class $wA(p,q)$ for each $p > 0$ and $q > 0$.*
(ii) *Class A coincides with class $wA(1,1)$.*
(iii) *Class $A(\frac{1}{2},\frac{1}{2})$ coincides with the class of w-hyponormal operators.*

Lemma 2.7. [13] *Let $A \geq 0$ and $T = U|T|$ be the polar decomposition of T. Then for each $\alpha > 0$ and $\beta > 0$, the following assertion holds:*

$$(U|T|^{\beta}A|T|^{\beta}U^*)^{\alpha} = U(|T|^{\beta}A|T|^{\beta})^{\alpha}U^*.$$

Definition 2.8. Let $p > 0$ and $q > 0$. An operator $T \in \mathbf{B}(\mathcal{H})$ is said to be absolute-(p,q)-paranormal if

$$\||T|^p|T^*|^qx\|^q \geq \||T^*|^qx\|^{p+q}$$

for every unit vector $x \in \mathcal{H}$.

Lemma 2.9. *([8, 20]) Let $T \in \mathbf{B}(\mathcal{H})$.*

(i) *For each $p > 0$. If T is absolute-p-paranormal, then T is absolute-$(p,1)$-paranormal.*
(ii) *For each $p > 0$ and $q > 0$. If T belongs to class $A(p,q)$, then T is absolute-(p,q)-paranormal.*
(iii) *For each $0 < p_1 \leq p_2$ and $0 < q_1 \leq q_2$. If T is absolute-(p_1,q_1)-paranormal, then T is absolute-(p_2,q_2)-paranormal.*
(iv) *For each $0 < p_1 \leq p_2$ and $0 < q_1 \leq q_2$. If T belongs to class $A(p_1,q_1)$, then T belongs to class $A(p_2,q_2)$.*

Proposition 2.10. *For each $p > 0$. If T belongs to class $A(p)$, then T belongs to class $A(q,q)$, where $q = \max\{1,p\}$.*

Proof. The proof follows easily from part(iv) of Lemma 2.9. □

In elementary algebra if $a > 0, b$ and c are real numbers, then the real quadratic form $at^2 + bt + c \geq 0$ for every real t if and only if $b^2 - 4ac \geq 0$. In an analogous manner, we have proved some results for absolute-p-paranormal operators.

Lemma 2.11. *For each $p > 0$. Then $T \in \mathbf{B}(\mathcal{H})$ is absolute-p-paranormal if and only if*

$$|T^*|^p|T|^{2p}|T^*|^p - 2\lambda^p|T^*|^{2p} + \lambda^{2p}I \geq 0$$

for all $\lambda > 0$.

Proof. Let $q = \max\{1, p\}$. If T is absolute-p-paranormal, then T is absolute-$(p, 1)$-paranormal by (i) of Lemma 2.9 and hence T is absolute-(q, q)-paranormal. For each unit vector $x \in \mathcal{H}$, we have

$$\begin{aligned} &\left\langle (|T^*|^q|T|^{2q}|T^*|^q - 2\lambda^q|T^*|^{2q} + \lambda^{2q}I)x, x\right\rangle \geq 0 \\ \Leftrightarrow &\left\langle (|T^*|^q|T|^q)x, x\right\rangle - 2\lambda^q \left\langle (|T^*|^{2q}x, x)\right\rangle + \lambda^{2q}\left\langle x, x\right\rangle \geq 0 \\ \Leftrightarrow &\, \||T^*|^q|T|^q x\|^2 - 2\lambda^q \||T^*|^q x\|^2 + \lambda^{2q}\|x\|^2 \geq 0. \end{aligned}$$

By the above argument, this will happen only if

$$4\||T^*|^q x\|^4 - 4\||T|^q|T^*|^q x\| \leq 0.$$

This implies that

$$\||T|^q|T^*|^q x\|^{\frac{1}{2}} \geq \||T^*|^q x\|,$$

so T is absolute-q-paranormal. □

Lemma 2.12. *Let $T = U|T|$ be the polar decomposition of T which belong to class $A(p, p)$ which belong to class $A(p, p)$ for $p > 0$. Then $\widetilde{T}_{p,p} = |T|^pU|T|^p$ is semi-hyponormal.*

Proof.

$$\begin{aligned} \left(\widetilde{T^*}_{p,p}\widetilde{T}_{p,p}\right)^{\frac{1}{2}} &= \left(|T|^pU^*|T|^{2p}U|T|^p\right)^{\frac{1}{2}} \\ &= \left(U^*U|T|^pU^*|T|^{2p}U|T|^pU^*U\right)^{\frac{1}{2}} \\ &= U^*\left(U|T|^pU^*|T|^{2p}U|T|^p\right)^{\frac{1}{2}}U \qquad \text{(By Lemma 2.7)} \\ &= U^*\left(|T^*|^pU^*|T|^{2p}U|T^*|^p\right)^{\frac{1}{2}}U \\ &\geq U^*|T^*|^{2p}U \\ |\widetilde{T}_{p,p}| &\geq |T|^{2p} \qquad (A.1) \end{aligned}$$

and the last inequality holds by definition of class $A(p, p)$ and Löwner-Heinz theorem.

On the other hand

$$\begin{aligned} \left(\widetilde{T}_{p,p}\widetilde{T^*}_{p,p}\right)^{\frac{1}{2}} &= \left(|T|^pU|T|^{2p}U^*|T|^p\right)^{\frac{1}{2}} \\ &= \left(|T|^p|T^*|^{2p}|T|^p\right)^{\frac{1}{2}} \\ |\widetilde{T^*}_{p,p}| &\leq |T|^{2p} \qquad (A.2) \end{aligned}$$

and the last inequality holds by definition of class $A(p, p)$ and Löwner-Heinz theorem. Therefore (A.1) and (A.2) ensure

$$|\widetilde{T}_{p,p}| \geq |T|^{2p} \geq |\widetilde{T^*}_{p,p}|.$$

That is, $\widetilde{T}_{p,p}$ is semi-hyponormal. □

Lemma 2.13. *([20]) Let $A \geq 0$ and $B \geq 0$. If*

$$B^{\frac{1}{2}}AB^{\frac{1}{2}} \geq B^2 \qquad (A.3)$$

and

$$A^{\frac{1}{2}}BA^{\frac{1}{2}} \geq A^2, \qquad (A.4)$$

then $A = B$.

Lemma 2.14. *Let $T \in \mathbf{B}(\mathcal{H})$. If T belongs to class A and T^* belongs to class A. Then T is normal.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Since T belongs to class A, then

$$\begin{aligned} |T^2| &= (T^{*2}T^2)^{\frac{1}{2}} = (T^*|T|^2T)^{\frac{1}{2}} \\ &= (U^*|T^*||T|^2|T^*|U)^{\frac{1}{2}} \\ &= U^*(|T^*||T|^2|T^*|)^{\frac{1}{2}}U \qquad \text{(by Lemma 2.7)} \\ &\geq |T|^2 \end{aligned}$$

Hence

$$(|T^*||T|^2|T^*|)^{\frac{1}{2}} \geq U|T|^2U^* = |T^*|^2,$$

so that by Theorem 2.5, we have

$$\left(|T^*||T|^{2p}|T^*|\right)^{\frac{1}{2}} \geq |T^*| \quad \text{and} \quad |T|^2 \geq \left(|T||T^*|^{2p}|T|\right)^{\frac{1}{2}}. \qquad (A.5)$$

On the other hand, if T^* belongs to class A, then

$$\left(|T||T^*|^2|T|\right)^{\frac{1}{2}} \geq |T|^2,$$

so that by Theorem 2.5, we have

$$\left(|T||T^*|^2|T|\right)^{\frac{1}{2}} \geq |T|^2 \quad \text{and} \quad |T^*|^2 \geq \left(|T^*||T|^2|T^*|\right)^{\frac{1}{2}}. \qquad (A.6)$$

Therefore

$$|T||T^*|^2|T| = |T|^4 \quad \text{and} \quad |T^*||T|^2|T^*| = |T^*|^4$$

hold by (A.5), (A.6), and then $|T| = |T^*|$ by Lemma 2.13. That is, T is normal. □

Lemma 2.15. *Let $T = U|T|$ be the polar decomposition of T. Then T belongs to class $A(p,p)$ if and only if $T_p = U|T|^p$ belongs to class A, for every $p > 0$.*

Proof. Let $p > 0$ and let $T = U|T|$ be the polar decomposition of T. Then if T_p belongs to class A, we have

$$\begin{aligned} |T_p^2| &= (U^*|T^*|^pU^*|T^*|^pU|T|^pU|T|^p)^{\frac{1}{2}} \\ &= U^*\left(|T^*|^p|T|^{2p}|T^*|^p\right)^{\frac{1}{2}}U \qquad \text{(by Lemma 2.7)} \\ &\geq |T_p|^2 \end{aligned}$$

Thus

$$\left(|T^*|^p|T|^{2p}|T^*|^p\right)^{\frac{1}{2}} \geq |T^*|^{2p},$$

and hence T belongs to class $A(p,p)$.

Conversely, if T belongs to class $A(p,p)$, then

$$\begin{aligned} \left(|T^*|^p|T|^{2p}|T^*|^p\right)^{\frac{1}{2}} &\geq |T^*|^{2p} \\ &\geq U|T|^{2p}U^* \\ U^*\left(|T^*|^p|T|^{2p}|T^*|^p\right)^{\frac{1}{2}}U &\geq |T|^{2p} \\ \left(U^*|T^*|^p|T|^{2p}|T^*|^pU\right)^{\frac{1}{2}} &\geq |T|^{2p} \end{aligned}$$

so that $|T_p^2| \geq |T_p|^2$. That is, T_p belongs to class A. □

Theorem 2.16. *Let $p > 0$. If T belongs to class $A(p)$ and T^* belongs to class $A(p)$. Then T is normal.*

Proof. Let $q = \max\{p, 1\}$. If T belongs to class $A(p)$, then T belongs to class $A(p,1)$ and hence by Lemma 2.9 T belongs to class $A(q,q)$, so that T_q belongs to class A. By a similar arguments T_q^* belongs to class A. Therefore, it follows from Lemma 2.14, T_q is normal, and so that T is normal. □

Lemma 2.17. *([10])Let S be invertible operator. Then*

$$(S^*S)^\lambda = S^*(SS^*)^{\lambda-1}S \quad \textit{holds for any real number } \lambda.$$

Theorem 2.18. *Let $p > 0$ and $T \in \mathbf{B}(\mathcal{H})$ be an invertible belongs to class $A(p,p)$, then T^{-1} belongs to class $A(p,p)$.*

Proof. First, we remark that T is invertible if and only if T_p is invertible. Now

$$|T_p|^2 = T_p^* T_P \le (T_p^{*2}T_p^2)^{\frac{1}{2}} = T_p^{*2}(T_p^2 T_p^{*2})^{-\frac{1}{2}}T_p^2 \qquad \text{(by Lemma 2.17)}$$
$$T_p^{*-1}T_p^{-1} \le (T_p^{*-2}T_p^2)^{\frac{1}{2}} = |T_p^{-2}|.$$

Hence T_p^{-1} belongs to class A, and so that T^{-1} belongs to class $A(p,p)$. □

Lemma 2.19. *([20]) Let A and B be positive operators. Then for each $p > 0, r \ge 0$ and $\lambda > 0$, the following assertion holds.*

$$\textit{If} \quad \frac{rB^{\frac{r}{2}}A^pB^{\frac{r}{2}} + p\lambda^{p+r}I}{(p+r)\lambda^p} \ge B^r, \quad \textit{then} \quad A^p \ge \frac{(p+r)\lambda^p A^{\frac{p}{2}}B^rA^{\frac{p}{2}}}{rA^{\frac{p}{2}}B^rA^{\frac{p}{2}} + p\lambda^{p+r}I}.$$

Lemma 2.20. *([3]) Let A and B be positive operators. If*

$$\frac{A^2+\lambda^2 I}{2\lambda} \ge B \qquad \textit{and} \qquad B \ge \frac{2\lambda A^2}{A^2+\lambda^2 I}$$

hold for all $\lambda > 0$, then $A = B$.

Theorem 2.21. *Let $q > 0$. If T is absolute-q-paranormal and if T^* is absolute-q-paranormal, then T is normal.*

Proof. Let $p = \max\{q, 1\}$. If T is an absolute-q-paranormal then T is an absolute-(p,p)-paranormal by (iii) of Lemma 2.9. By Lemma 2.11, we have

$$p|T^*|^p|T|^{2p}|T^*|^p - 2p\lambda^p|T^*|^{2p} + p\lambda^{2p}I \ge 0 \quad \text{for all} \quad \lambda > 0.$$

This is equivalent to

$$\frac{|T^*|^p|T|^{2p}|T^*|^p + \lambda^{2p}I}{2\lambda^p} \ge |T^*|^{2p},$$

so that by Lemma 2.19, we have

$$\frac{|T^*|^p|T|^{2p}|T^*|^p + \lambda^{2p}I}{2\lambda^p} \ge |T^*|^{2p} \quad \text{and} \quad |T|^{2p} \ge \frac{2\lambda^p|T|^p|T^*|^{2p}|T|^p}{|T|^p|T^*|^{2p}|T|^p + \lambda^{2p}I}. \tag{A.7}$$

On the other hand, If T^* is absolute-q-paranormal then T^* is absolute-(p,p)-paranormal by (iii) of Lemma 2.9. By Lemma 2.11, we have

$$p|T|^p|T^*|^{2p}|T|^p - 2p\lambda^p|T|^{2p} + p\lambda^{2p}I \ge 0 \quad \text{for all } \lambda > 0 .$$

This is equivalent to

$$\frac{|T|^p|T^*|^{2p}|T|^p + \lambda^{2p}I}{2\lambda^p} \ge |T|^{2p},$$

so that by Lemma 2.19, we have

$$\frac{|T|^p|T^*|^{2p}|T|^p + \lambda^{2p}I}{2\lambda^p} \ge |T|^{2p} \quad \text{and} \quad |T^*|^{2p} \ge \frac{2\lambda^p|T^*|^p|T|^{2p}|T^*|^p}{|T^*|^p|T|^{2p}|T^*|^p + \lambda^{2p}I}. \tag{A.8}$$

Hence $\left(|T^*|^p|T|^{2p}|T^*|^p\right)^{\frac{1}{2}} = |T^*|^{2p}$ and $\left(|T|^p|T^*|^{2p}|T|^p\right)^{\frac{1}{2}} = |T|^{2p}$ hold by (A.7) and (A.8) and Lemma 2.20 and then $|T| = |T^*|$ by Lemma 2.13. Therefore T is normal. □

Corollary 2.22. *If T and T^* are paranormal, then T is normal.*

3. GENERALIZED p-PARANORMAL OPERATORS

Rai [18] has defined a bounded operator T on a Hilbert space $\mathcal{H}$ as *generalized paranormal* if for every unit vector $x \in \mathcal{H}$ and $M > 0$, T satisfies $\left\|T^2x\right\| \geq \frac{1}{M}\left\|Tx\right\|^2$. He also proved a result for every unit vector x,

$$\left\|T^{k+1}x\right\|^2 \geq \frac{1}{M^{2k-1}}\left\|T^kx\right\|^2\left\|T^2x\right\|,$$

where T is a bounded linear operator on $\mathcal{H}$, $M > 0$ and $k \geq 1$.

On the basis of the above result, we define the generalized n-paranormal operator as follows:

Definition 3.1. A bounded linear operator T on $\mathcal{H}$ is called *generalized n-paranormal* operator if for every unit vector $x \in \mathcal{H}$, $M > 0$, and a positive integer n such that $n \geq 2$, T satisfies

$$\left\|T^nx\right\| \geq \frac{1}{M^{\frac{n}{2}}}\left\|Tx\right\|^n.$$

Definition 3.2. Let $T \in \mathbf{B}(\mathcal{H})$. An operator T belongs to *generalized class A* operator if for $M > 0$, T satisfies

$$|T^2| \geq \frac{1}{M}|T|^2.$$

Theorem 3.3. *If T satisfies $|T^n|^{\frac{2}{n}} \geq \frac{1}{M}|T|^2$ for some positive integer n such that $n \geq 2$, and $M > 0$, then T is a generalized n-paranormal operator.*

In case $n = 2$, Theorem 3.3 means every generalized class A is a generalized paranormal operator. We need the following lemma in order to give a proof of Theorem 3.3.

Lemma 3.4. ***(Hölder-McCarthy Inequality)*** *Let T be a positive operator. Then the following inequalities hold for all $x \in \mathcal{H}$:*
(i) $\langle T^rx, x\rangle \leq \langle Tx, x\rangle^r\left\|x\right\|^{2(1-r)}$ *for* $0 < r \leq 1$.
(ii) $\langle T^rx, x\rangle \geq \langle Tx, x\rangle^r\left\|x\right\|^{2(1-r)}$ *for* $r \geq 1$.

Proof of Theorem 3.3. Suppose T satisfies

$$|T^n|^{\frac{2}{n}} \geq \frac{1}{M}|T|^2, \qquad \text{(B.1)}$$

for some positive integer n such that $n \geq 2$ and $M > 0$. Then for every unit vector $x \in \mathcal{H}$,

$$\begin{aligned}\left\|T^n\right\|^2 &= \left\langle |T^n|^2x, x\right\rangle \\ &\geq \left\langle |T^n|^{\frac{2}{n}}x, x\right\rangle^n && \text{(by (ii) of Theorem 3.4)} \\ &\geq \left\langle \frac{1}{M}|T|^2x, x\right\rangle^n && \text{(by (B.1))} \\ &= \frac{1}{M^n}\left\|Tx\right\|^{2n}.\end{aligned}$$

Hence we have

$$\left\|T^n\right\| \geq \frac{1}{M^{\frac{n}{2}}}\left\|Tx\right\|^n \quad \text{for every unit vector } x \in \mathcal{H},$$

so that T is generalized n-paranormal for positive integer n such that $n \geq 2$ and $M > 0$. □

Definition 3.5. An operator T on a Hilbert space $\mathcal{H}$ is called a generalized n-perinormal operator if for positive integer n such that $n \geq 2$ and $M > 0$, T satisfies

$$M^nT^{*n}T^n - (T^*T)^n \geq 0.$$

Remark 3.6. We note the following:

(a) The class of generalized 2-perinormal coincides with the class of generalized quasi-hyponormal one, i.e., $M^2T^{*2}T^2 - (T^*T)^2 \geq 0$, [18].

(b) we easily obtain the following result by Löwner-Heinz theorem : For each positive integer n such that $n \geq 2$ and $M > 0$, every generalized n-perinormal operator satisfies

$$|T^n|^{\frac{2}{n}} \geq \frac{1}{M}|T|^2.$$

Definition 3.7. An operator T on a Hilbert space $\mathcal{H}$ is called a generalized k-quasi-hyponormal operator if for positive integer k such that $k \geq 2$ and $M > 0$, T satisfies

$$M^{k+1}T^{*k}(T^*T)T^k - T^{*k}TT^*T^k \geq 0.$$

Theorem 3.8. *Let $T \in \mathbf{B}(\mathcal{H})$. If T is generalized k-quasi-hyponormal, then T is generalized $(k+1)$-paranormal.*

Proof. If T is a generalized k-quasi-hyponormal then the following relation holds for every unit vector $x \in \mathcal{H}$

$$\left\|T^{k+1}x\right\| \geq \frac{1}{M^{(\frac{k+1}{2})}}\left\|T^*T^kx\right\|. \tag{B.2}$$

To prove T is generalized $(k+1)$-paranormal it suffices to prove that

$$\left\|T^{k+1}x\right\| \geq \frac{1}{M^{(\frac{k+1}{2})}}\left\|Tx\right\|^{k+1}.$$

We know that for any bounded linear operator T on a Hilbert space $\mathcal{H}$

$$\left\|Tx\right\|^{k+1} \leq \left\|T^*T^kx\right\|. \tag{B.3}$$

Therefore, from (B.2) and (B.3), we get

$$\left\|T^{k+1}x\right\| \geq \frac{1}{M^{(\frac{k+1}{2})}}\left\|Tx\right\|^{k+1}.$$

Hence T is a generalized $(k+1)$-paranormal operator. □

Remark 3.9. Note that if $k = 1$ in Definition 3.7, then T is a generalized quasi-hyponormal operator.

Corollary 3.10. *Every generalized quasi-hyponormal is a generalized paranormal.*

Theorem 3.11. *An operator T is a generalized n-paranormal if and only if*

$$T^{*n}T^n + \frac{2\lambda}{M^{\frac{n}{2}}}(T^*T)^{\frac{n}{2}} + \lambda^2 I \geq 0,$$

for all real $\lambda, M > 0$ and positive integer n such that $n \geq 2$.

Proof. Let x be any unit vector in $\mathcal{H}$, then we have

$$\left\langle \left(T^{*n}T^n + \frac{2\lambda}{M^{\frac{n}{2}}}(T^*T)^{\frac{n}{2}} + \lambda^2 I\right)x, x\right\rangle \geq 0$$

or

$$\langle T^{*n}T^n x, x\rangle + \frac{2\lambda}{M^{\frac{n}{2}}}\left\langle (T^*T)^{\frac{n}{2}}x, x\right\rangle + \lambda^2\langle x, x\rangle \geq 0. \tag{B.3.a}$$

But

$$\begin{aligned}\left\langle (T^*T)^{\frac{n}{2}}x, x\right\rangle &\geq \langle T^*Tx, x\rangle^{\frac{n}{2}} && \text{(by (ii) of Lemma 3.4)}\\ &\geq \|Tx\|^n\end{aligned}$$

Hence

$$\left\langle (T^*T)^{\frac{n}{2}}x, x\right\rangle \geq \|Tx\|^n. \tag{B.3.b}$$

Then (B.3.a) and (B.3.b) ensure

$$\|T^nx\|^2 + \frac{2\lambda}{M^{\frac{n}{2}}}\|Tx\|^n + \lambda^2\|x\|^2 \geq 0.$$

By the above argument, this will happen only if

$$\frac{4}{M^n}\|Tx\|^{2n} - 4\|T^nx\|^2 \leq 0$$

or

$$\|T^n x\| \geq \frac{1}{M^{\frac{n}{2}}} \|Tx\|^n .$$

Hence the proof of the theorem is achieved. □

Definition 3.12. An operator T on a Hilbert space $\mathcal{H}$ is called a generalized (p,k)-quasi-hyponormal operator if for positive integer k such that $k \geq 1$, $0 < p \leq 1$ and $M > 0$, T satisfies

$$M^{(k+1)p} T^{*k} \left((T^*T)^p - (TT^*)^p)\right) T^k \geq 0.$$

Lemma 3.13. *Let $T \in \mathbf{B}(\mathcal{H})$ be a generalized (p,k)-quasihyponormal operator for $0 < p \leq 11$ and a positive integer k and $M > 0$. Then the following assertions hold.*

(i) $\frac{1}{M^{(\frac{n+1}{2})}} \|T^n\|^2 \leq \left\|T^{n-1}x\right\| \left\|T^{n+1}x\right\|$ *for all unit vectors $x \in \mathcal{H}$ and all positive integers $n \geq k$.*

(ii) *If $T^n = 0$ for some positive integer $n \geq k$, then $T^k = 0$.*

Proof. (i) Since it is obvious that generalized (p,k)-quasihyponormal operators are generalized $(p,k+1)$-quasihyponormal, we may assume that $k = n$. Since

$$\begin{aligned}\langle T^{*n}(TT^*)^p T^n x, x\rangle &= \langle (T^*T)^{p+1} T^{n-1}x, T^{n-1}x\rangle \\ &\geq \|T^{n-1}x\|^{-2p} \langle T^*TT^{n-1}x, T^{n-1}x\rangle^{p+1} \quad \text{(by Hölder-McCarthy Inequality)} \\ &= \|T^{n-1}x\|^{-2p} \|T^n x\|^{2p+2}\end{aligned}$$

and

$$\begin{aligned}\langle T^{*n}(T^*T)^p T^n x, x\rangle &= \langle (T^*T)^p T^n x, T^n x\rangle \\ &\leq \|T^n x\|^{2-2p} \langle T^*TT^n x, T^n x\rangle \qquad \text{(by Hölder-McCarthy Inequality)} \\ &= \|T^n x\|^{2-2p} \|T^{n+1}x\|^{2p}.\end{aligned}$$

But T is (p,n)-quasihyponormal operator. Then

$$\langle M^{(n+1)p} T^{*n}((T^*T)^p - (TT^*)^p) T^n x, x\rangle \geq 0.$$

Hence

$$\frac{1}{M^{(\frac{n+1}{2})}} \|T^n x\|^2 \leq \|T^{n-1}x\| \|T^{n+1}x\|.$$

(ii) If $T^{k+1} = 0$, then $T^k = 0$ by (i). The rest of the proof is similar.

□

Definition 3.14. Let $p > 0$. A bounded linear operator T on $\mathcal{H}$ is called *generalized p-paranormal* operator if for every unit vector $x \in \mathcal{H}$ and $M > 0$, T satisfies

$$\||T|^p U |T|^p x\| \geq \frac{1}{M^p} \||T|^p x\|^2 .$$

Theorem 3.15. *An operator T is generalized p-paranormal if and only if*

$$|T|^p U^* |T|^{2p} U |T|^p + \frac{2\lambda}{M^p} |T|^{2p} + \lambda^2 I \geq 0$$

for all real λ, $M > 0$.

Proof.

$$|T|^p U^* |T|^{2p} U |T|^p + \frac{2\lambda}{M^p} |T|^{2p} + \lambda^2 I \geq 0$$

implies that

$$\left\langle \left(|T|^p U^* |T|^{2p} U |T|^p + \frac{2\lambda}{M^p} |T|^{2p} + \lambda^2 I \right) x, x \right\rangle \geq 0$$

or

$$\left\langle (|T|^p U^* |T|^{2p} U |T|^p) x, x \right\rangle + \frac{2\lambda}{M^p} \left\langle (|T|^{2p}) x, x \right\rangle + \lambda^2 \langle x, x\rangle \geq 0$$

or

$$\||T|^pU|T|^px\|^2 + \frac{2\lambda}{M^p}\||T|^px\|^2 + \lambda^2\|x\|^2 \geq 0.$$

By the above argument, this will happen only if

$$\frac{4}{M^{2p}}\||T|^p\|^4 - 4\||T|^pU|T|^px\|^2 \leq 0$$

or

$$\||T|^pU|T|^px\| \geq \frac{1}{M^p}\||T|^p\|^2$$

Hence the theorem. □

Corollary 3.16. *Every generalized 1-paranormal operator is a generalized paranormal operator.*

Lemma 3.17. *([7]) Let $T = U|T|$ be the polar decomposition of a p-paranormal operator, then $U|T|^p$ is a paranormal operator.*

Lemma 3.18. *Let $T = U|T|$ be the polar decomposition of T. Then T is a generalized p-paranormal operator if and only if $T_p = U|T|^p$ is generalized paranormal operator for any $p > 0$.*

Proof. Let $p > 0$. Let $x \in \mathcal{H}$ be a unit vector. Then

$$\begin{aligned}\|T_p^2x\|^2 &= \langle T_p^2x, T_p^2x\rangle \\ &= \langle U|T|^pU|T|^px, U|T|^pU|T|^px\rangle \\ &= \langle(|T|^pU^*|T|^p)(|T|^pU|T|^p)x, x\rangle \\ &= \||T|^pU|T|^p\|^2 \qquad \text{(B.4)}\end{aligned}$$

and

$$\|T_px\|^2 = \langle T_px, T_px\rangle = \langle U|T|^px, U|T|^px\rangle = \||T|^px\|^2 \qquad \text{(B.5).}$$

Therefore, (B.4) and (B.5) ensure

$$\|T_p^2x\| \geq \frac{1}{M^p}\|T_px\|^2.$$

Hence the theorem. □

Young et al. [21] proved that the inverse of an invertible p-paranormal operator is also p-paranormal. We have a generalization for generalized p-paranormal operators.
It is easy that if $T = U|T|$ is invertible, then U is unitary and $T^{-1} = U^*|T^*|^{-1}$ is the polar decomposition.

Theorem 3.19. *Let $T = U|T|$ be invertible generalized p-paranormal for $p > 0$. Then T^{-1} is also generalized p-paranormal.*

Proof. Suppose that $T = U|T|$ is invertible generalized p-paranormal operator. Then

$$U|T|^{-k} = |T^*|^{-k}U \quad \text{and} \quad |T^*|^{-k} = U|T|^{-k}U^*$$

for all $k > 0$. Since T is a generalized p-paranormal, we have

$$\begin{aligned}&\lambda^2|T^{-1}|^pU^*|T^{-1}|^{2p}U|T^{-1}|^p + \frac{2\lambda}{M^{\frac{p}{2}}}|T^{-1}|^{2p} + I \\ &= I + \frac{2\lambda}{M^{\frac{p}{2}}}U|T|^{-2p}U^* + \lambda^2U|T|^{-p}U|T|^{-2p}U^*|T|^{-p}U^* \\ &= U|T|^{-p}U|T|^{-p}\left(|T|^pU^*|T|^{2p}U|T|^p + \frac{2\lambda}{M^{\frac{p}{2}}}|T|^{2p} + \lambda^2I\right)|T|^{-p}U^*|T|^{-p}U^*\end{aligned}$$

is positive for all real $\lambda, M > 0$. By Theorem 3.15, T^{-1} is generalized p-paranormal. □

It is known that if T is p-paranormal, $T \otimes I$ is also p-paranormal. However the tensor product of two doubly commuting p-paranormal operators is not necessarily p-paranormal [21].

Theorem 3.20. *If T is p-paranormal, $T \otimes I$ is also p-paranormal for $p > 0$.*

Proof. Let $T = U|T|$ be the polar decomposition of T. Then

$$T \otimes I = (U \otimes I)(|T| \otimes I)$$

is the polar decomposition of $T \otimes I$. Since

$$\begin{aligned} &|T \otimes I|^p (U \otimes I)^* |T \otimes I|^{2p} (U \otimes I)|T \otimes I|^p + \frac{2\lambda}{M^p}|T \otimes I|^{2p} + \lambda^2 \\ &= (|T|^p U^* |T|^{2p} U|T|^p + \frac{2\lambda}{M^p}|T|^{2p} + \lambda^2) \otimes I \end{aligned}$$

is positive for all real λ, $M > 0$. Hence $T \otimes I$ is p-paranormal. □

Theorem 3.21. *Let* $0 < p < 1$. *Every generalized p-paranormal operator is generalized paranormal.*

Proof. First of al, we note that the Hölder inequality by McCarthy (ii) of Lemma 3.4 has the folowing form;

$$\|S^p y\| \le \|Sy\|^p \|y\|^{1-p}$$

for all $y \in \mathcal{H}$. Putting $S = |T|$ and $y = U|T|^p$ in part (ii) of Lemma 3.4, we have

$$\||T|^p U|T|^p x\| \le \||T|U|T|^p x\|^p \||T|^p x\|^{1-p}$$

Since the left hand side of the above inequality is greater than $|||T|^p x||^2/M^p||x||$ by the generalized absolute-p- paranormality, it follows that

$$\||T|^p x\|^{1+p} \le \||T|U|T|^p x\|^p \|x\| . \qquad \text{(B.6)}$$

Hence, if we replace x by $|T|^{1-p}x$ in (B.6), then

$$\frac{1}{M^p} \|Tx\|^{p+1} \le \left\||T|^{1-p}x\right\| \left\|T^2 x\right\|^p .$$

Applying part (ii) of Lemma 3.4 again, it follows that

$$\left\||T|^{1-p}x\right\| \le \|Tx\|^{1-p} \|x\|^p .$$

Therefore it implies that

$$\frac{1}{M^p} \|Tx\|^{p+1} \le \left\||T|^{1-p}x\right\| \left\|T^2 x\right\|^p \le \|Tx\|^{1-p} \|x\|^p \left\|T^2 x\right\|^p ,$$

so that

$$\frac{1}{M} \|Tx\|^2 \le \left\|T^2 x\right\| \|x\| .$$

This completes the proof. □

Theorem 3.22. *Let* T *be a generalized p-paranormal operator, then*

$$\left\|T^3 x\right\| \ge \frac{1}{M^2} \left\|T^2 x\right\| \|Tx\| , \quad \text{for every unit vector } x \in \mathcal{H}.$$

Proof. For a unit vector x in $\mathcal{H}$, we may assume that $\|Tx\| \ne 0$, we have

$$\begin{aligned} \left\|T^3 x\right\| &= \|Tx\| \left\|T^2 \frac{Tx}{\|Tx\|}\right\| \ge \frac{1}{M} \|Tx\| \left\|T \frac{Tx}{\|Tx\|}\right\|^2 \qquad \text{(by Theorem 3.21)} \\ &\ge \frac{1}{M} \|Tx\| \left\|\frac{T^2 x}{\|Tx\|}\right\|^2 \ge \frac{1}{M} \frac{\|Tx\| \left\|T^2 x\right\|^2}{\|Tx\|^2} \\ &\ge \frac{1}{M^2} \frac{\|Tx\| \left\|T^2 x\right\| \|Tx\|^2}{\|Tx\|^2} \\ \left\|T^3 x\right\| &\ge \frac{1}{M^2} \left\|T^2 x\right\| \|Tx\| . \end{aligned}$$

Hence the theorem. □

Theorem 3.23. *Let T be a generalized p-paranormal operator, then*

$$\left\|T^{k+1}x\right\|^2 \geq \frac{1}{M^{2k-1}}\left\|T^k x\right\|^2\left\|T^2 x\right\|$$

for a positive integer $k \geq 1$ and every unit vector $x \in \mathcal{H}$.

Proof. We will use induction to establish the inequality

$$\left\|T^{k+1}x\right\|^2 \geq \frac{1}{M^{2k-1}}\left\|T^k x\right\|^2\left\|T^2 x\right\| \quad \text{for a positive integer } k \geq 1 \qquad (B.7)$$

In case $k = 1$,

$$\left\|T^2 x\right\|^2 = \left\|T^2 x\right\|\left\|T^2 x\right\| \geq \frac{1}{M^2}\left\|Tx\right\|^4 \qquad \text{(B.8)}$$

hold by Theorem 3.21. Now suppose that (B.7) holds for some $k \geq 1$ and assume that $\|Tx\| \neq 0$, then

$$\begin{aligned}
\left\|T^{k+2}x\right\|^2 &= \|Tx\|^2\left\|T^{k+1}\frac{Tx}{\|Tx\|}\right\|^2 \\
&\geq \|Tx\|^2\frac{1}{M^{2k-1}}\left\|T^k\frac{Tx}{\|Tx\|}\right\|^2\left\|T^2\frac{Tx}{\|Tx\|}\right\| && \text{(by (B.8))} \\
&\geq \|Tx\|^2\frac{1}{M^{2k-1}}\left\|\frac{T^{k+1}x}{\|Tx\|}\right\|^2\frac{\left\|T^3 x\right\|}{\|Tx\|} \\
&\geq \frac{1}{M^{2k-1}}\left\|T^{k+1}x\right\|^2\frac{1}{M^2}\frac{\left\|T^2 x\right\|\|Tx\|}{\|Tx\|} && \text{(by Theorem 3.21)} \\
&\geq \frac{1}{M^{2k+1}}\left\|T^{k+1}x\right\|^2\left\|T^2 x\right\|.
\end{aligned}$$

That is,

$$\left\|T^{k+2}x\right\|^2 \geq \frac{1}{M^{2k+1}}\left\|T^{k+1}x\right\|^2\left\|T^2 x\right\|.$$

The proof is complete. □

Lemma 3.24. *Let $T \in \mathbf{B}(\mathcal{H})$. If T is a generalized p-paranormal operator T, then T satisfies*

$$\left\|T^{n+1}x\right\| \geq \frac{1}{M^{n(n+1)/2}}\left\|Tx\right\|^{n+1},$$

for any unit vector $x \in \mathcal{H}, M > 0$, and positive integer n such that $n \geq 1$.

Proof. For $n = 1$ the statement is trivial. If the statement is true for $n - 1$, then we have

$$\begin{aligned}
\left\|T^{n+1}x\right\| &= \|Tx\|\left\|T^n\frac{Tx}{\|Tx\|}\right\| \\
&\geq \frac{1}{M^{n(n-1)/2}}\|Tx\|\left\|\frac{T^2 x}{\|Tx\|}\right\|^n \\
&\geq \frac{1}{M^{n(n-1)/2}}\|Tx\|\frac{1}{M^n}\frac{\|Tx\|^{2n}}{\|Tx\|^n} \\
&\geq \frac{1}{M^{n(n+1)/2}}\|Tx\|^{n+1}.
\end{aligned}$$

Hence the lemma. □

Theorem 3.25. *If a generalized p-paranormal operator T has compact power T^k, then T is compact.*

Proof. We shall prove that $T^n \in \mathfrak{C} \Rightarrow T \in \mathfrak{C}$, where $\mathfrak{C}$ is an algebra of all compact operators. Let us suppose that

$$x_n \to 0 \quad \text{(weakly)}, \qquad \|x_n\| = 1.$$

By Lemma 3.24, T satisfies

$$\|T^n x_n\| \geq \frac{1}{M^{n(n-1)/2}} \|Tx_n\|^n ,$$

which tell us that Tx_n converges strongly to 0, since

$$\|T^n x_n\| \to 0 \quad \text{by compactness of } T^n.$$

Therefore, T is compact. □

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