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EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR TWO KINDS OF NONLINEAR NEUTRAL DIFFERENCE EQUATIONS WITH VARIABLE DELAY

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Abstract. In this paper, we consider two kinds of nonlinear neutral difference equations with variable delay. By choosing available operators and applying Krasnoselskii's fixed point theorem, we obtain sufficient conditions for the existence of positive periodic solutions to such equations.

1. INTRODUCTION

In recent years, there have been a few papers written on the existence of periodic solutions, nontrivial periodic solutions and positive periodic solutions for several classes of functional differential and difference equations with delays, which arise from a number of mathematical ecological models, economical and control models, physiological and population models and other models, see the references in this article and references therein.

In this paper, we are interested in the analysis of qualitative theory of positive periodic solutions of delay difference equations. Motivated by the papers [1]-[6],[8],[9] and the references therein, we concentrate on the existence of positive periodic solutions for the following two kinds of nonlinear neutral difference equations with variable delay

$$x(n+1) = a(n)\, x(n) + \triangle g(n, x(n-\tau(n))) + f(n, x(n-\tau(n))), \tag{1}$$

and

$$\begin{aligned} x(n+1) = a(n)\, x(n) &+ \triangle \sum_{r=-\infty}^{-1} Q(r)\, g(n, x(n+r)) \\ &+ b(n) \sum_{r=-\infty}^{-1} Q(r)\, f(n, x(n+r)), \end{aligned} \tag{2}$$

where

$$g, f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R},$$

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with $\mathbb{Z}$ is the set of integers and $\mathbb{R}$ is the set of real numbers. Throughout this paper $\triangle$ denotes the forward difference operator $\triangle x(n) = x(n+1) - x(n)$ for any sequence $\{x(n),\ n \in \mathbb{Z}\}$. Also, we define the operator E by $Ex(n) = x(n+1)$. For more on the calculus of difference equations, we refer the reader to [7].

The purpose of this paper is to use Krasnoselskii's fixed point theorem to show the existence of positive periodic solutions for equations (1) and (2). To apply Krasnoselaskii's fixed point theorem we need to construct two mappings, one is a contraction and the other is completely continuous.

The organization of this paper is as follows. In Section 2, we present the inversions of difference equations (1) and (2), and we give the Green's functions of (1) and (2), which play an important role in this paper. Also, we present the Krasnoselskii's fixed point theorem. For details on Krasnoselskii's theorem we refer the reader to [10]. In Section 3 and Section 4, we present our main results on existence of positive periodic solutions of (1) and (2), respectively.

2. PRELIMINARIES

Let T be an integer such that $T \geq 1$. Define $P_T = \{\varphi \in C(\mathbb{Z}, \mathbb{R}) : \varphi(n+T) = \varphi(n)\}$ where $C(\mathbb{Z}, \mathbb{R})$ is the space of all real valued functions. Then $(P_T, \|.\|)$ is a Banach space with the maximum norm

$$\|x\| = \sup_{n \in [0, T-1] \cap \mathbb{Z}} |x(n)|.$$

Since we are searching for the existence of periodic solutions for equations (1) and (2), it is natural to assume that

$$a(n+T) = a(n),\ b(n+T) = b(n),\ \tau(n+T) = \tau(n), \tag{3}$$

with τ being scalar sequence and $\tau(n) \geq \tau^* > 0$. Also, we assume

$$0 < a(n) < 1,\ Q(n) > 0,\ \sum_{r=-\infty}^{-1} Q(r) = 1. \tag{4}$$

We also assume that the functions $g(n, x)$ and $f(n, x)$ are continuous in x and periodic in n with period T, that is,

$$g(n+T, x) = g(n, x),\ f(n+T, x) = f(n, x). \tag{5}$$

The following lemmas are fundamental to our results.

Lemma 2.1. *Suppose (3)–(5) hold. If $x \in P_T$, then x is a solution of equation (1) if and only if*

$$\begin{aligned} x(t) &= g(n, x(n-\tau(n))) \\ &\quad + \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, x(u-\tau(u))) - (1-a(u))\, g(u, x(u-\tau(u)))\right], \end{aligned} \tag{6}$$

where

$$G(n,u) = \frac{\prod_{s=u+1}^{n+T-1} a(s)}{1 - \prod_{s=n}^{n+T-1} a(s)}. \tag{7}$$

Proof. We consider two cases, $n \geq 1$ and $n \leq 0$. Let $x \in P_T$ be a solution of (1). For $n \geq 1$ equation (1) is equivalent to

$$\triangle\left[x(n)\prod_{s=0}^{n-1}a^{-1}(s)\right] = \left[\triangle g(n, x(n-\tau(n))) + f(n, x(n-\tau(n)))\right]\prod_{s=0}^{n}a^{-1}(s). \tag{8}$$

By summing (8) from n to $n+T-1$, we obtain

$$\begin{aligned}&\sum_{u=n}^{n+T-1}\triangle\left[x(u)\prod_{s=0}^{u-1}a^{-1}(s)\right]\\ &= \sum_{u=n}^{n+T-1}\left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u)))\right]\prod_{s=0}^{u}a^{-1}(s).\end{aligned}$$

As a consequence, we arrive at

$$\begin{aligned}&x(n+T)\prod_{s=0}^{n+T-1}a^{-1}(s) - x(n)\prod_{s=0}^{n-1}a^{-1}(s)\\ &= \sum_{u=n}^{n+T-1}\left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u)))\right]\prod_{s=0}^{u}a^{-1}(s).\end{aligned}$$

Since $x(n+T) = x(n)$, we obtain

$$\begin{aligned}&x(n)\left[\prod_{s=0}^{n+T-1}a^{-1}(s) - \prod_{s=0}^{n-1}a^{-1}(s)\right]\\ &= \sum_{u=n}^{n+T-1}\left[\triangle g(u, x(u-\tau(u))) + f(u, x(u-\tau(u)))\right]\prod_{s=0}^{u}a^{-1}(s).\end{aligned} \tag{9}$$

Rewrite

$$\begin{aligned}&\sum_{u=n}^{n+T-1}\triangle g(u, x(u-\tau(u)))\prod_{s=0}^{u}a^{-1}(s)\\ &= \sum_{u=n}^{n+T-1}E\left[\prod_{s=0}^{u-1}a^{-1}(s)\right]\triangle g(u, x(u-\tau(u))).\end{aligned}$$

Performing a summation by parts on the on the above equation, we get

$$\begin{aligned}
&\sum_{u=n}^{n+T-1} \triangle g\left(u, x\left(u-\tau\left(u\right)\right)\right) \prod_{s=0}^{u} a^{-1}\left(s\right) \\
&= g\left(n, x\left(n-\tau\left(n\right)\right)\right) \left[\prod_{s=0}^{n+T-1} a^{-1}\left(s\right) - \prod_{s=0}^{n-1} a^{-1}\left(s\right)\right] \\
&- \sum_{u=n}^{n+T-1} g\left(u, x\left(u-\tau\left(u\right)\right)\right) \triangle \left[\prod_{s=0}^{u-1} a^{-1}\left(s\right)\right] \\
&= g\left(n, x\left(n-\tau\left(n\right)\right)\right) \left[\prod_{s=0}^{n+T-1} a^{-1}\left(s\right) - \prod_{s=0}^{n-1} a^{-1}\left(s\right)\right] \\
&- \sum_{u=n}^{n+T-1} g\left(u, x\left(u-\tau\left(u\right)\right)\right) \left[1-a\left(u\right)\right] \prod_{s=0}^{u} a^{-1}\left(s\right). \qquad (10)
\end{aligned}$$

Substituting (10) into (9), we obtain

$$\begin{aligned}
&x\left(n\right) \left[\prod_{s=0}^{n+T-1} a^{-1}\left(s\right) - \prod_{s=0}^{n-1} a^{-1}\left(s\right)\right] \\
&= g\left(n, x\left(n-\tau\left(n\right)\right)\right) \left[\prod_{s=0}^{n+T-1} a^{-1}\left(s\right) - \prod_{s=0}^{n-1} a^{-1}\left(s\right)\right] \\
&- \sum_{u=n}^{n+T-1} g\left(u, x\left(u-\tau\left(u\right)\right)\right) \left[1-a\left(u\right)\right] \prod_{s=0}^{u} a^{-1}\left(s\right) \\
&+ \sum_{u=n}^{n+T-1} f\left(u, x\left(u-\tau\left(u\right)\right)\right) \prod_{s=0}^{u} a^{-1}\left(s\right).
\end{aligned}$$

Dividing both sides of the above equation by $\prod_{s=0}^{n+T-1} a^{-1}\left(s\right) - \prod_{s=0}^{n-1} a^{-1}\left(s\right)$, we obtain (6).

Now for $n \leq 0$, equation (1) is equivalent to

$$\triangle \left[x\left(n\right) \prod_{s=n}^{0} a^{-1}\left(s\right)\right] = \left[\triangle g\left(n, x\left(n-\tau\left(n\right)\right)\right) + f\left(n, x\left(n-\tau\left(n\right)\right)\right)\right] \prod_{s=n+1}^{0} a^{-1}\left(s\right).$$

Summing the above expression from n to $n+T-1$, we obtain (6) by a similar argument. This completes the proof. □

Lemma 2.2. *Suppose (3)–(5) hold. If $x \in P_T$, then x is a solution of equation (2) if and only if*

$$\begin{aligned} x(t) &= \sum_{r=-\infty}^{-1} Q(r)\, g(n, x(n+r)) \\ &\quad + \sum_{u=n}^{n+T-1} G(n,u) \left[b(u) \sum_{r=-\infty}^{-1} Q(r)\, f(u, x(u+r)) \right. \\ &\quad \left. - (1-a(u)) \sum_{r=-\infty}^{-1} Q(r)\, g(u, x(u+r)) \right], \end{aligned} \tag{11}$$

where G is given by (7).

The proof is similar to that of Lemma 2.1, and hence we omit it.
It is easy to see that for all $n, u \in \mathbb{Z}$,

$$G(n+T, u+T) = G(n,u), \tag{12}$$

and

$$\sum_{u=n}^{n+T-1} G(n,u)(1-a(u)) = 1. \tag{13}$$

Lastly in this section, we state Krasnoselskii's fixed point theorem which enables us to prove the existence of positive periodic solutions to (1) and (2). For its proof we refer the reader to [10].

Theorem 2.3 (Krasnoselskii)**.** *Let $\mathbb{D}$ be a closed convex nonempty subset of a Banach space $(\mathbb{B}, \|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{D}$ into $\mathbb{B}$ such that*
(i) $x, y \in \mathbb{D}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathbb{D}$,
(ii) $\mathcal{A}$ is completely continuous,
(iii) $\mathcal{B}$ is a contraction mapping.
Then there exists $z \in \mathbb{D}$ with $z = \mathcal{A}z + \mathcal{B}z$.

3. Existence of positive periodic solutions for (1.1)

To apply Theorem 2.3, we need to define a Banach space $\mathbb{B}$, a closed convex subset $\mathbb{D}$ of $\mathbb{B}$ and construct two mappings, one is a contraction and the other is completely continuous. So, we let $(\mathbb{B}, \|.\|) = (P_T, \|.\|)$ and $\mathbb{D} = \{\varphi \in \mathbb{B} : L \le \varphi \le K\}$, where L is non-negative constant and K is positive constant. We express equation (6) as

$$\varphi(n) = (\mathcal{B}_1\varphi)(n) + (\mathcal{A}_1\varphi)(n) := (H_1\varphi)(n),$$

where $\mathcal{A}_1, \mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ are defined by

$$(\mathcal{A}_1\varphi)(n) = \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right], \tag{14}$$

and

$$(\mathcal{B}_1\varphi)(n) = g(n, \varphi(n-\tau(n))). \tag{15}$$

In this section we obtain the existence of a positive periodic solution of (1) by considering the two cases; $g(n,x) \ge 0$ and $g(n,x) \le 0$ for all $n \in \mathbb{Z}$, $x \in \mathbb{D}$. We assume that function $g(n,x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant k such that

$$|g(n,x)-g(n,y)|\leq k\|x-y\|\,, \text{ for all } n\in[0,T-1]\cap\mathbb{Z},\ x,y\in\mathbb{D}. \tag{16}$$

Denote

$$F(n,x)=\frac{f(n,x)}{1-a(n)}-g(n,x)\,.$$

In the case $g(n,x)\geq 0$, we assume that there exist a non-negative constant k_1 and positive constant k_2 such that

$$k_1x\leq g(n,x)\leq k_2x, \text{ for all } n\in[0,T-1]\cap\mathbb{Z},\ x\in\mathbb{D}, \tag{17}$$

$$k_2<1, \tag{18}$$

and for all $n\in[0,T-1]\cap\mathbb{Z}$, $x\in\mathbb{D}$

$$(1-k_1)\,L\leq F(n,x)\leq(1-k_2)\,K. \tag{19}$$

Lemma 3.1. *Suppose that the conditions (3)–(5) and (17)–(19) hold. Then* $\mathcal{A}_1:\mathbb{D}\to\mathbb{B}$ *is completely continuous.*

Proof. We first show that $(\mathcal{A}_1\varphi)(n+T)=(\mathcal{A}_1\varphi)(n)$.
Let $\varphi\in\mathbb{D}$. Then using (14) we arrive at

$$\begin{aligned}&(\mathcal{A}_1\varphi)(n+T)\\&=\sum_{u=n+T}^{n+2T-1}G(n+T,u)\left[f(u,\varphi(u-\tau(u)))-(1-a(u))\,g(u,\varphi(u-\tau(u)))\right].\end{aligned}$$

Let $j=u-T$, then

$$\begin{aligned}&(\mathcal{A}_1\varphi)(n+T)\\&=\sum_{j=n}^{n+T-1}G(n+T,j+T)\left[f(j+T,\varphi(j+T-\tau(j+T)))\right.\\&\left.-(1-a(j+T))\,g(j+T,\varphi(j+T-\tau(j+T)))\right]\\&=\sum_{j=n}^{n+T-1}G(n,j)\left[f(j,\varphi(j-\tau(j)))-(1-a(j))\,g(j,\varphi(j-\tau(j)))\right]\\&=(\mathcal{A}_1\varphi)(n)\,,\end{aligned}$$

by (3), (5) and (12).

To see that $\mathcal{A}_1(\mathbb{D})$ is uniformly bounded, we let $n\in[0,T-1]\cap\mathbb{Z}$ and for $\varphi\in\mathbb{D}$, we have by (19) that

$$\begin{aligned}&|(\mathcal{A}_1\varphi)(n)|\\&=\left|\sum_{u=n}^{n+T-1}G(n,u)\left[f(u,\varphi(u-\tau(u)))-(1-a(u))\,g(u,\varphi(u-\tau(u)))\right]\right|\\&\leq(1-k_2)\,K.\end{aligned}$$

From the estimation of $|(\mathcal{A}_1\varphi)(n)|$ it follows that

$$\|\mathcal{A}_1\varphi\|\leq(1-k_2)\,K.$$

This shows that $\mathcal{A}_1(\mathbb{D})$ is uniformly bounded.

Next, we show that $\mathcal{A}_1$ maps bounded subsets into compact sets. As $\mathcal{A}_1(\mathbb{D})$ is uniformly bounded in $\mathbb{R}^T$, then $\mathcal{A}_1(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. Therefore $\mathcal{A}_1$ is completely continuous. This completes the proof. $\square$

Lemma 3.2. *Suppose that (3)–(5) and (16) hold. If $\mathcal{B}_1$ is given by (15) with*

$$k < 1, \tag{20}$$

then $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction.

Proof. Let $\mathcal{B}_1$ be defined by (15). Obviously, $(\mathcal{B}_1\varphi)(n+T) = (\mathcal{B}_1\varphi)(n)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned}
|(\mathcal{B}_1\varphi)(n) - (\mathcal{B}_1\psi)(n)| &\leq |g(n, \varphi(n-\tau(n))) - g(n, \psi(n-\tau(n)))| \\
&\leq k\|\varphi - \psi\|.
\end{aligned}$$

Then $\|\mathcal{B}_1\varphi - \mathcal{B}_1\psi\| \leq k\|\varphi - \psi\|$. Thus $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction by (20). $\square$

Theorem 3.3. *Suppose (3)–(5) and (16)–(20) hold and there exists a $n_0 \in [0, T-1] \cap \mathbb{Z}$ such that $F(n_0, x) > (1-k_1)L$ for any $x \in \mathbb{D}$. Then equation (1) has a positive T-periodic solution x in the subset $\mathbb{D}_1 = \{\varphi \in \mathbb{B} : L < \varphi \leq K\}$.*

Proof. By Lemma 3.1, the operator $\mathcal{A}_1 : \mathbb{D} \to \mathbb{B}$ is completely continuous. Also, from Lemma 3.2, the operator $\mathcal{B}_1 : \mathbb{D} \to \mathbb{B}$ is a contraction. Moreover, if $\varphi, \psi \in \mathbb{D}$, we see that

$$\begin{aligned}
&(\mathcal{B}_1\psi)(n) + (\mathcal{A}_1\varphi)(n) \\
&= g(n, \psi(n-\tau(n))) \\
&+ \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\
&\leq k_2 K + (1-k_2) K \sum_{u=n}^{n+T-1} G(n,u)(1-a(u)) \\
&= k_2 K + (1-k_2) K = K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\mathcal{B}_1\psi)(n) + (\mathcal{A}_1\varphi)(n) \\
&= g(n, \psi(n-\tau(n))) \\
&+ \sum_{u=n}^{n+T-1} G(n,u)\left[f(u, \varphi(u-\tau(u))) - (1-a(u))\, g(u, \varphi(u-\tau(u)))\right] \\
&\geq k_1 L + (1-k_1) L \sum_{u=n}^{n+T-1} G(n,u)(1-a(u)) \\
&= k_1 L + (1-k_1) L = L.
\end{aligned}$$

This shows that $\mathcal{B}_1\psi + \mathcal{A}_1\varphi \in \mathbb{D}$. Clearly, all the hypotheses of the Krasnoselskii theorem are satisfied. Thus there exists a fixed point $x \in \mathbb{D}$ such that $x = \mathcal{A}_1 x + \mathcal{B}_1 x$. By Lemma 2.1 this fixed point is a solution of (1).

Next, we prove that $x \in \mathbb{D}_1$. We just need to prove that for all $n \in [0, T-1] \cap \mathbb{Z}$, $x(n) > L$. Otherwise, there exists $n^* \in [0, T-1] \cap \mathbb{Z}$ satisfying $x(n^*) = L$. From (6), we have

$$\begin{aligned}
L &= g\left(n^*, \psi\left(n^* - \tau\left(n^*\right)\right)\right) \\
&\quad + \sum_{u=n^*}^{n^*+T-1} G\left(n^*, u\right)\left[f\left(u, \varphi\left(u - \tau\left(u\right)\right)\right) - \left(1 - a\left(u\right)\right) g\left(u, \varphi\left(u - \tau\left(u\right)\right)\right)\right] \\
&\geq k_1 L + \sum_{u=n^*}^{n^*+T-1} G\left(n^*, u\right)\left(1 - a\left(u\right)\right)\left\{\frac{f\left(u, \varphi\left(u - \tau\left(u\right)\right)\right)}{\left(1 - a\left(u\right)\right)} - g\left(u, \varphi\left(u - \tau\left(u\right)\right)\right)\right\}.
\end{aligned}$$

From $\sum_{u=n^*}^{n^*+T-1} G\left(n^*, u\right)\left(1 - a\left(u\right)\right) = 1$, it follows that

$$\sum_{u=n^*}^{n^*+T-1} G\left(n^*, u\right)\left(1 - a\left(u\right)\right)\left[F\left(u, x\right) - \left(1 - k_1\right) L\right] \leq 0.$$

Noting that $F(u, x) \geq (1 - k_1) L$ and $F(n_0, x) > (1 - k_1) L$, $n_0 \in [0, T-1] \cap \mathbb{Z}$, we obtain

$$\sum_{u=n^*}^{n^*+T-1} G\left(n^*, u\right)\left(1 - a\left(u\right)\right)\left[F\left(u, x\right) - \left(1 - k_1\right) L\right] > 0.$$

This is a contraction. So, $x \in \mathbb{D}_1$. The proof is complete. $\square$

In the case $g(n, x) \leq 0$, we substitute conditions (17)–(19) with the following conditions respectively. We assume that there exist a negative constant k_3 and a non-positive constant k_4 such that

$$k_3 x \leq g\left(n, x\right) \leq k_4 x, \text{ for all } n \in [0, T-1] \cap \mathbb{Z},\ x \in \mathbb{D}, \tag{21}$$

$$-k_3 < 1, \tag{22}$$

and for all $n \in [0, T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$

$$L - k_3 K \leq F\left(n, x\right) \leq K - k_4 L. \tag{23}$$

Theorem 3.4. *Suppose (3)–(5), (16) and (20)-(23) hold and there exists a $n_0 \in [0, T-1] \cap \mathbb{Z}$ such that $F(n_0, x) > L - k_3 K$ for any $x \in \mathbb{D}$. Then equation (1) has a positive T-periodic solution x in the subset $\mathbb{D}_1$.*

The proof follows along the lines of Theorem 3.3, and hence we omit it.

4. EXISTENCE OF POSITIVE PERIODIC SOLUTIONS FOR (1.2)

We express equation (11) as

$$\varphi\left(n\right) = \left(\mathcal{B}_2 \varphi\right)\left(n\right) + \left(\mathcal{A}_2 \varphi\right)\left(n\right) := \left(H_2 \varphi\right)\left(n\right),$$

where $\mathcal{A}_2, \mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ are defined by

$$\begin{aligned}
\left(\mathcal{A}_2 \varphi\right)\left(n\right) = \sum_{u=n}^{n+T-1} G\left(n, u\right)&\left[b\left(u\right) \sum_{r=-\infty}^{-1} Q\left(r\right) f\left(u, \varphi\left(u + r\right)\right)\right. \\
&\left. - \left(1 - a\left(u\right)\right) \sum_{r=-\infty}^{-1} Q\left(r\right) g\left(u, \varphi\left(u + r\right)\right)\right],
\end{aligned} \tag{24}$$

and

$$\left(\mathcal{B}_2\varphi\right)(n) = \sum_{r=-\infty}^{-1} Q(r)\, g(n, \varphi(n+r)). \tag{25}$$

Denote

$$H(n,x) = \frac{b(n)}{1-a(n)} f(n,x) - g(n,x).$$

We substitute conditions (19) and (23) with the following conditions respectively. we assume that for all $n \in [0, T-1] \cap \mathbb{Z}$, $x \in \mathbb{D}$,

$$(1-k_1)\, L \leq H(n,x) \leq (1-k_2)\, K, \tag{26}$$

and

$$L - k_3 K \leq H(n,x) \leq K - k_4 L. \tag{27}$$

Lemma 4.1. *Suppose that the conditions (3)–(5) and (17)–(18) and (26) hold. Then $\mathcal{A}_2 : \mathbb{D} \to \mathbb{B}$ is completely continuous.*

Proof. Obviously, $\left(\mathcal{A}_2\varphi\right)(n+T) = \left(\mathcal{A}_2\varphi\right)(n)$. To see that $\mathcal{A}_2(\mathbb{D})$ is uniformly bounded, we let $n \in [0, T-1] \cap \mathbb{Z}$ and for $\varphi \in \mathbb{D}$, we have by (26) that

$$\begin{aligned}
&\left|\left(\mathcal{A}_2\varphi\right)(n)\right| \\
&\leq \left| \sum_{u=n}^{n+T-1} G(n,u) \left[b(u) \sum_{r=-\infty}^{-1} Q(r)\, f(u, \varphi(u+r)) \right.\right. \\
&\left.\left. - (1-a(u)) \sum_{r=-\infty}^{-1} Q(r)\, g(u, \varphi(u+r)) \right] \right| \\
&\leq \sum_{u=n}^{n+T-1} G(n,u)(1-a(u)) \sum_{r=-\infty}^{-1} Q(r) \left| \frac{b(u)}{1-a(u)} f(u, \varphi(u+r)) - g(u, \varphi(u+r)) \right| \\
&\leq (1-k_2)\, K \sum_{u=n}^{n+T-1} G(n,u)(1-a(u)) \sum_{r=-\infty}^{-1} Q(r) \\
&= (1-k_2)\, K.
\end{aligned}$$

From the estimation of $\left|\left(\mathcal{A}_2\varphi\right)(n)\right|$ it follows that

$$\left\|\mathcal{A}_2\varphi\right\| \leq (1-k_2)\, K.$$

This shows that $\mathcal{A}_2(\mathbb{D})$ is uniformly bounded.

Next, we show that $\mathcal{A}_2$ maps bounded subsets into compact sets. As $\mathcal{A}_2(\mathbb{D})$ is uniformly bounded in $\mathbb{R}^T$, then $\mathcal{A}_2(\mathbb{D})$ is contained in a compact subset of $\mathbb{B}$. Therefore $\mathcal{A}_2$ is completely continuous. This completes the proof. □

Lemma 4.2. *Suppose that (3)–(5), (16) and (20) hold. If $\mathcal{B}_2$ is given by (25), then $\mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ is a contraction.*

Proof. Let $\mathcal{B}_2$ be defined by (25). Obviously, $(\mathcal{B}_2\varphi)(n+T) = (\mathcal{B}_2\varphi)(n)$. So, for any $\varphi, \psi \in \mathbb{D}$, we have

$$\begin{aligned}
&\left|(\mathcal{B}_2\varphi)(n) - (\mathcal{B}_2\psi)(n)\right| \\
&\leq \left|\sum_{r=-\infty}^{-1} Q(r)\, g(n, \varphi(n+r)) - \sum_{r=-\infty}^{-1} Q(r)\, g(n, \psi(n+r))\right| \\
&\leq \sum_{r=-\infty}^{-1} Q(r) \left|g(n, \varphi(n+r)) - g(n, \psi(n+r))\right| \\
&\leq k\,\|\varphi - \psi\| \sum_{r=-\infty}^{-1} Q(r) \\
&= k\,\|\varphi - \psi\|.
\end{aligned}$$

Then $\|\mathcal{B}_2\varphi - \mathcal{B}_2\psi\| \leq k\,\|\varphi - \psi\|$. Thus $\mathcal{B}_2 : \mathbb{D} \to \mathbb{B}$ is a contraction by (20). □

Similar to the results in Section 3, we have

Theorem 4.3. *Suppose (3)–(5), (16)–(18), (20) and (26) hold and there exists a* $n_0 \in [0, T-1] \cap \mathbb{Z}$ *such that* $H(n_0, x) > (1 - k_1)\, L$ *for any* $x \in \mathbb{D}$. *Then equation (2) has a positive* T*-periodic solution* x *in the subset* $\mathbb{D}_1$.

Theorem 4.4. *Suppose (3)–(5), (16), (20)–(22) and (27) hold and there exists a* $n_0 \in [0, T-1] \cap \mathbb{Z}$ *such that* $H(n_0, x) > L - k_3 K$ *for any* $x \in \mathbb{D}$. *Then equation (2) has a positive* T*-periodic solution* x *in the subset* $\mathbb{D}_1$.

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