A COUPLED COINCIDENCE POINT THEOREM ON ORDERED
PARTIAL B-METRIC-LIKE SPACES

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Abstract. In this paper, we prove a coupled coincidence point theorem in
ordered partial \( b \)-metric-like spaces besides furnishing an illustrative example
to demonstrate our main result.

1. Introduction

The concept of \( b \)-metric space was introduced by Czerwik [3] which runs as
follows:

**Definition 1.1** ([3]): A \( b \)-metric on a non empty set \( X \) is a function \( d : X \times X \to [0, \infty) \) such that for all \( x, y, z \in X \) and \( k \geq 1 \), the following three conditions are
satisfied:

(i) \( d(x, y) = 0 \) if and only if \( x = y \),
(ii) \( d(x, y) = d(y, x) \),
(iii) \( d(x, y) \leq k[d(x, z) + d(z, y)] \).

As usual, the pair \((X, d)\) is called a \( b \)-metric space.

**Example 1.2**: Let \( X = \mathbb{R} \) and \( d(x, y) = (x - y)^2 \) for all \( x, y \in X \). Then \( d \) is a
\( b \)-metric with \( k = 2 \) but not a metric as \( d(1, -1) > d(1, 0) + d(0, -1) \).

Ali Alghamdi et al. [1] introduced the concept of \( b \)-metric-like spaces and proved
some fixed point theorems involving a single map.

**Definition 1.3** ([1]): A \( b \)-metric-like on a non empty set \( X \) is a function \( d : X \times X \to [0, \infty) \) such that for all \( x, y, z \in X \) and a constant \( k \geq 1 \), the following
three conditions are satisfied:

(i) \( d(x, y) = 0 \) implies \( x = y \),
(ii) \( d(x, y) = d(y, x) \),
(iii) \( d(x, y) \leq k[d(x, z) + d(z, y)] \).

The pair \((X, d)\) is called a \( b \)-metric-like space.

**Example 1.4**: Let \( X = [0, \infty) \) and \( d(x, y) = (x + y)^2 \) for all \( x, y \in X \). Then \( d \) is a
\( b \)-metric-like space with \( k = 2 \) but not a \( b \)-metric.

Matthews [6] introduced the concept of a partial metric space which runs as follows:

**Definition 1.5** ([6]): A mapping \( p : X \times X \to [0, \infty) \) (where \( X \) is a nonempty set)

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property.

One can easily verify the following remark.

Definition 2.1: Let \((X, p, k)\) be a partial b-metric-like space and \(\{x_n\}\) a sequence in \(X\). Then the sequence \(\{x_n\}\) is said to be convergent to \(x \in X\) if \(\lim_{n \to \infty} p(x_n, x) = p(x, x)\).

Definition 2.2: Let \((X, p, k)\) be a partial b-metric-like space and \(\{x_n\}\) a sequence in \(X\) with \(x \in X\). Then the sequence \(\{x_n\}\) is said to be convergent to \(x\) if \(\lim_{n \to \infty} p(x_n, x) = p(x, x)\).

Definition 2.3: Let \((X, p, k)\) be a partial b-metric-like space.

(i) A sequence \(\{x_n\}\) in \((X, p, k)\) is said to be Cauchy sequence if \(\lim_{n,m \to \infty} p(x_n, x_m)\) exists and is finite.

(ii) A partial b-metric-like space \((X, p, k)\) is said to be complete if every Cauchy sequence \(\{x_n\}\) in \(X\) converges to a point \(x \in X\) so that \(\lim_{n,m \to \infty} p(x_n, x_m) = p(x, x) = \lim_{n \to \infty} p(x_n, x)\).

One can easily verify the following remark.
Remark 2.4: Let \((X, p, k)\) be a partial b-metric-like space and \(\{x_n\}\) a sequence in \(X\) such that \(\lim_{n \to \infty} p(x_n, x) = 0\). Then

(i) \(x\) is unique,
(ii) \(k p(x, y) \leq \lim_{n \to \infty} p(x_n, y) \leq kp(x, y)\) for all \(y \in X\)
(iii) \(p(x_n, x_0) \leq kp(x_0, x_1)+k^2p(x_1, x_2)+\cdots+k^{n-1}p(x_{n-2}, x_{n-1})+k^{n-1}p(x_{n-1}, x_n)\) whenever \(\{x_k\}_{k=0}^\infty \in X\).

Ali Alghamdi et al.\cite{1} introduced the following class of functions.

Let \(\Psi_k^L\) be the class of those functions \(\mathcal{L} : (0, \infty) \to (0, \frac{1}{k^2})\) which satisfy the condition \(\mathcal{L}(t_n) \to \frac{1}{k^2} \Rightarrow t_n \to 0\), where \(k > 0\).

Using these functions, we now prove a coupled coincidence point theorem in ordered partial b-metric-like spaces.

Let \((X, p, k)\) be a partial b-metric-like space and \(F : X \times X \to X\) and \(g : X \to X\). For \(x, y, u, v \in X\), we denote

\[
M(x, y, u, v) = \max \left\{ \frac{p(gx, gy), p(gy, gu), p(gx, F(x, y)), p(gy, F(y, x))}{p(gu, F(u, v)), p(gv, F(v, u))}, \frac{1}{2k}[p(gx, F(x, v)) + p(gy, F(y, x))], \frac{1}{2k}[p(gy, F(y, u)) + p(gv, F(y, x))] \right\}.
\]

Notice that \(M(x, y, u, v) = M(y, x, v, u)\) for all \(x, y, u, v \in X\).

Now, we are equipped to prove our main result as follows.

**Theorem 2.5:** Let \((X, p, k, \preceq)\) be an ordered partial b-metric-like space and \(F : X \times X \to X\), \(g : X \to X\) be the mappings which satisfy the following conditions:

(2.5.1) \(F(X \times X) \subseteq g(X)\), \(g(X)\) is complete,
(2.5.2) \(F\) has the mixed \(-g\)-monotone property,
(2.5.3) \(p(F(x, y), F(u, v)) \leq \mathcal{L}(M(x, y, u, v))M(x, y, u, v)\)
for all \(x, y, u, v \in X\) with \(gx \preceq gu\), \(gy \geq gv\), where \(\mathcal{L} \in \Psi_k^L\).
(2.5.4) There exist two elements \(x_0, y_0 \in X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(gy_0 \geq F(y_0, x_0)\).
(2.5.5) (a) Suppose \(F\) and \(g\) are continuous

or

(b) \(g(X)\) has the following properties:

(i) If a non-decreasing sequence \(\{a_n\} \to a\), then \(a_n \preceq a\), \(\forall n\).
(ii) If a non-increasing sequence \(\{a_n\} \to a\), then \(a \preceq a_n\), \(\forall n\).

Then \(F\) and \(g\) have a coupled coincidence point in \(X \times X\).

**Proof.** By (2.5.4), there exist two elements \(x_0, y_0 \in X\) such that \(gx_0 \preceq F(x_0, y_0)\) and \(gy_0 \geq F(y_0, x_0)\). Since \(F(X \times X) \subseteq g(X)\), we can choose \(x_1, y_1 \in X\) such that \(gx_1 = F(x_0, y_0)\) and \(gy_1 = F(y_0, x_0)\). Again we can choose \(x_2, y_2 \in X\) such that \(gx_2 = F(x_1, y_1)\) and \(gy_2 = F(y_1, x_1)\). Continuing this process indefinitely, we construct sequences \(\{x_n\}\) and \(\{y_n\}\) in \(X\) such that \(gx_{n+1} = F(x_n, y_n)\) and \(gy_{n+1} = F(y_n, x_n)\) for all \(n \geq 0\).

Now for \(n \geq 0\), we shall prove that

\[
gx_n \preceq gx_{n+1} \text{ and } gy_n \succeq gy_{n+1}.
\]

From (2.5.4), (1) holds for \(n = 0\). Suppose (1) holds for \(n = m > 0\). Now, by (2.5.2), we have

\[
gx_{m+1} = F(x_m, y_m) \preceq F(x_{m+1}, y_{m+1}) \preceq F(x_{m+1}, y_m) = gx_{m+2} \text{ and } gy_{m+1} = F(y_m, x_m) \preceq F(y_{m+1}, x_{m+1}) \preceq F(y_{m+1}, x_m) = gy_{m+2}.
\]

Thus (1) holds for \(n = m + 1\). Hence by mathematical induction, (1) holds for all
\[ p(g_{x_n}, g_{x_{n+1}}) = p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \leq \mathcal{L}(M(x_n, y_n, x, y)) \]

where

\[ M(x_n, y_n, x, y) = \max \left\{ \begin{array}{l}
\frac{1}{\mathcal{L}} p(g_{x_n-1}, g_{x_n}), p(g_{y_n-1}, g_{y_n}), p(g_{x_n-1}, g_{y_n}), \\
\frac{1}{\mathcal{L}} p(g_{y_n-1}, g_{x_n}), p(g_{x_n}, g_{x_{n+1}}), p(g_{y_n}, g_{y_{n+1}}), \\
\frac{1}{\mathcal{L}} p(g_{x_n-1}, g_{x_{n+1}}) + p(g_{x_n}, g_{x_n}), \\
\frac{1}{\mathcal{L}} p(g_{y_n-1}, g_{y_{n+1}}) + p(g_{y_n}, g_{y_n}) \end{array} \right\} \]

and

\[ \frac{1}{\mathcal{L}} p(g_{x_n-1}, g_{x_{n+1}}) + p(g_{x_n}, g_{x_n}) \leq \frac{1}{\mathcal{L}} \left\{ \begin{array}{l}
k[p(g_{x_n-1}, g_{x_n}) + p(g_{x_n}, g_{x_{n+1}}) - p(g_{x_n}, g_{x_n})] \\
+k[p(g_{x_n-1}, g_{x_n})] \end{array} \right\} \]

\[ \leq \max \{p(g_{x_n-1}, g_{x_n}), p(g_{x_n}, g_{x_{n+1}})\} \]

Thus

\[ M(x_n, y_n, x, y) = \max \left\{ \begin{array}{l}
p(g_{x_n-1}, g_{x_n}), p(g_{y_n-1}, g_{y_n}), \\
p(g_{x_n}, g_{x_{n+1}}), p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} \]

So,

\[ p(g_{x_n}, g_{x_{n+1}}) \leq \mathcal{L}(M(x_n, y_n, x, y)) \max \left\{ \begin{array}{l}
p(g_{x_n-1}, g_{x_n}), p(g_{y_n-1}, g_{y_n}), \\
p(g_{x_n}, g_{x_{n+1}}), p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} \]

Similarly by using \( M(y_n, x_n, y_n, x_n) = M(x_n, y_n, x, y) \), we can show that

\[ p(g_{y_n}, g_{y_{n+1}}) \leq \mathcal{L}(M(x_n, y_n, x, y)) \max \left\{ \begin{array}{l}
p(g_{x_n-1}, g_{x_n}), p(g_{y_n-1}, g_{y_n}), \\
p(g_{x_n}, g_{x_{n+1}}), p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} \]

Thus

\[ \max \left\{ \begin{array}{l}
p(g_{x_n}, g_{x_{n+1}}), \\
p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} \leq \mathcal{L}(M(x_n, y_n, x, y)) \max \left\{ \begin{array}{l}
p(g_{x_n-1}, g_{x_n}), p(g_{y_n-1}, g_{y_n}), \\
p(g_{x_n}, g_{x_{n+1}}), p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} \]

If \( \max \left\{ \begin{array}{l}
p(g_{x_n-1}, g_{x_n}), p(g_{y_n-1}, g_{y_n}), \\
p(g_{x_n}, g_{x_{n+1}}), p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} = \max \left\{ \begin{array}{l}
p(g_{x_n}, g_{x_{n+1}}), p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} \)

then using \( \mathcal{L}(M(x_n, y_n, x, y)) < \frac{1}{\mathcal{L}} \), we get a contradiction from (3).

Hence

\[ \max \left\{ \begin{array}{l}
p(g_{x_n}, g_{x_{n+1}}), \\
p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} \leq \mathcal{L}(M(x_n, y_n, x, y)) \max \left\{ \begin{array}{l}
p(g_{x_n-1}, g_{x_n}), \\
p(g_{y_n-1}, g_{y_n}) \end{array} \right\} \]

Put \( p_n = \max \left\{ \begin{array}{l}
p(g_{x_n}, g_{x_{n+1}}), p(g_{y_n}, g_{y_{n+1}}) \end{array} \right\} \). Then

\[ p_n \leq \mathcal{L}(M(x_n, y_n, x, y)) p_{n-1} < p_{n-1} \]

Thus \( \{p_n\} \) is a non-increasing sequence of non-negative real numbers and hence also converges to some real number \( s \geq 0 \). Suppose \( s > 0 \).

From (4), we have \( s \leq \lim_{n \to \infty} \mathcal{L}(M(x_n, y_n, x, y)) s \) so that
Now we have
\[ \frac{1}{k} \leq \lim_{n \to \infty} \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) \leq \frac{1}{k^2} \] which in turn yields that
\[ \lim_{n \to \infty} \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) = \frac{1}{k}. \] Hence
\[ \lim_{n \to \infty} M(x_{n-1}, y_{n-1}, x_n, y_n) = 0. \]
Thus from (2), we have
\[ \lim_{n \to \infty} \max \{ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \} = 0. \] (5)
Also, from (p2) we have
\[ \lim_{n \to \infty} \max \{ p(gx_n, gx_n), p(gy_n, gy_n) \} = 0. \] (6)
Now, we prove that
\[ \lim_{n, m \to \infty} \max \{ p(gx_n, gx_m), p(gy_n, gy_m) \} = 0. \] (7)
Suppose (7) is not true. Then
\[ \lim_{n, m \to \infty} \max \{ p(gx_n, gx_m), p(gy_n, gy_m) \} > 0. \] (8)
Let \( m > n \). Then from (1), we have \( gx_n \leq gx_m \) and \( gy_n \leq gy_m \).
From (2.5.3), we have
\[ p(gx_{n+1}, gx_{m+1}) = p(F(x_n, y_n, F(x_m, y_m)) \leq \mathcal{L}(M(x_n, y_n, x_m, y_m)) M(x_n, y_n, x_m, y_m) \] (9)
where
\[ \lim_{n, m \to \infty} M(x_n, y_n, x_m, y_m) = \lim_{n, m \to \infty} \max \left\{ \frac{p(gx_n, gx_m), p(gy_n, gy_m), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), p(gx_m, gx_{m+1}), p(gy_m, gy_{m+1}),}{\frac{1}{k} [p(gx_n, gx_{m+1}) + p(gx_m, gx_{n+1})],} \right\}. \]
But
\[ \frac{1}{k} [p(gx_n, gx_{m+1}) + p(gx_m, gx_{n+1})] \]
\[ \leq \frac{1}{k} \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) \leq \frac{1}{k} \max \left\{ p(gx_n, gx_m), p(gy_n, gy_m) \right\} \] from (5), (6)
\[ \leq \lim_{n, m \to \infty} M(x_n, y_n, x_m, y_m) \] from (5), (6)
\[ \leq \lim_{n, m \to \infty} M(x_n, y_n, x_m, y_m) \] from (5), (6)
\[ \leq \lim_{n, m \to \infty} M(x_n, y_n, x_m, y_m). \]
Hence
\[ \lim_{n, m \to \infty} M(x_n, y_n, x_m, y_m) = \lim_{n, m \to \infty} \max \{ p(gx_n, gx_m), p(gy_n, gy_m) \} \] (10)
From (9), we have
\[ \lim_{n, m \to \infty} p(gx_{n+1}, gx_{m+1}) \leq \lim_{n, m \to \infty} \mathcal{L}(M(x_n, y_n, x_m, y_m)) \lim_{n, m \to \infty} \max \left\{ p(gx_n, gx_m), p(gy_n, gy_m) \right\}. \]
Thus
\[
\lim_{n,m \to \infty} \max \left\{ \frac{p(gx_{n+1}, gx_{m+1})}{p(gy_{n+1}, gy_{m+1})} \right\} \leq \lim_{n,m \to \infty} \mathcal{L}(M(x_n, y_n, x_m, y_m)) \lim_{n,m \to \infty} \max \left\{ \frac{p(gx_n, gx_m)}{p(gy_n, gy_m)} \right\}
\]

We have
\[
p(gx_n, gx_m) \leq kp(gx_n, gx_{n+1}) + k^2 p(gx_{n+1}, gx_m) + k^2 p(gx_{m+1}, gx_m)
\]
which implies that
\[
\frac{1}{k^2} \lim_{n,m \to \infty} p(gx_n, gx_m) \leq \lim_{n,m \to \infty} p(gx_{n+1}, gx_{m+1}) \text{ from (5)}.
\]
Similarly, \(\frac{1}{k^2} \lim_{n,m \to \infty} p(gy_n, gy_m) \leq \lim_{n,m \to \infty} p(gy_{n+1}, gy_{m+1})\).

Thus by using (11), we have
\[
\frac{1}{k^2} \lim_{n,m \to \infty} \max \left\{ \frac{p(gx_n, gx_m)}{p(gy_n, gy_m)} \right\} \leq \lim_{n,m \to \infty} \max \left\{ \frac{p(gx_{n+1}, gx_{m+1})}{p(gy_{n+1}, gy_{m+1})} \right\}
\]
\[
\leq \lim_{n,m \to \infty} \mathcal{L}(M(x_n, y_n, x_m, y_m)) \lim_{n,m \to \infty} \max \left\{ \frac{p(gx_n, gx_m)}{p(gy_n, gy_m)} \right\},
\]

which in turn implies from (8) that
\[
\frac{1}{k^2} \leq \lim_{n,m \to \infty} \mathcal{L}(M(x_n, y_n, x_m, y_m)) \leq \frac{1}{k^2}
\]
since \(\lim_{n \to \infty} M(x_n, y_n, x_m, y_m) = 0\).

It is a contradiction to (8) in view of (10).

Hence (7) holds. Thus \(\{gx_n\}\) and \(\{gy_n\}\) are Cauchy sequences in \(g(X)\). Since \(g(X)\) is complete, there exist \(r_1, r_2, z_1, z_2 \in X\) such that \(gx_n \to r_1 = g z_1\) and \(gy_n \to r_2 = g z_2\).

Suppose (2.5.5)(a) holds.

From Lemma 1.8, there exists a subset \(E \subseteq X\) such that \(g(E) = g(X)\) and the mapping \(g : E \to X\) is one-one. Without loss of generality, we are able to choose \(E \subseteq X\) such that \(z_1, z_2 \in E\). Now define \(G : g(E) \times g(E) \to X\) by
\[
G(ga, gb) = F(a, b) \text{ for all } ga, gb \in g(E) \text{ where } a, b \in E.
\]

Since \(F\) and \(g\) are continuous, it follows that \(G\) is continuous. As \(g : E \to X\) is one-one and \(F(X \times X) \subseteq g(X)\), \(G\) is well defined. Again since \(F\) and \(g\) are continuous, it follows that \(G\) is continuous. Since \(\{x_n\}, \{y_n\} \subseteq X\) and \(g(E) = g(X)\), there exists \(\{a_n\}, \{b_n\} \subseteq E\) such that \(gx_n = g(a_n)\) and \(gy_n = g(b_n)\) for all \(n\). So we have
\[
F(z_1, z_2) = G(gz_1, gz_2) = \lim_{n \to \infty} G(ga_n, gb_n) = \lim_{n \to \infty} F(a_n, b_n) = \lim_{n \to \infty} ga_{n+1} = gz_1,
\]
\[
F(z_2, z_1) = G(gz_2, gz_1) = \lim_{n \to \infty} G(gb_n, ga_n) = \lim_{n \to \infty} F(b_n, a_n) = \lim_{n \to \infty} gb_{n+1} = gz_2.
\]

Thus \((z_1, z_2)\) is a coupled coincidence point of \(F\) and \(g\).

Suppose (2.5.5) (b) holds.

From (1) and (i) and (ii) of (2.5.5)(b), we have \(gx_n \leq gz_1\) and \(gy_n \geq gz_2\) for all \(n\).

From definition of completeness of \(g(X)\) and from (7), we have
\[
\lim_{n \to \infty} p(gx_n, gz_1) = p(gz_1, gz_1) = \lim_{n,m \to \infty} p(gx_n, gx_m) = 0 \quad \text{(12)}
\]
\[
\lim_{n \to \infty} p(gy_n, gz_2) = p(gz_2, gz_2) = \lim_{n,m \to \infty} p(gy_n, gy_m) = 0 \quad \text{(13)}
\]

Now
\[
p(gx_{n+1}, F(z_1, z_2)) = p(F(x_n, y_n), F(z_1, z_2)) \leq \mathcal{L}(M(x_n, y_n, z_1, z_2)) M(x_n, y_n, z_1, z_2) \quad \text{(14)}
\]
\[
\lim_{n \to \infty} M(x_n, y_n, z_1, z_2) = \lim_{n \to \infty} \max \left\{ \begin{array}{l}
p(gx_n, g(z_1)), p(gy_n, g(z_2)), p(gx_n, g(z_2)), \\
p(gy_n, g(x_n+1)), p(g(z_1, z_2)), p(g(z_2, z_1)), \\
\frac{1}{k} [kp(gz_1, F(z_1, z_2)) + 0], \\
\frac{1}{k} [kp(gz_2, F(z_2, z_1)) + 0]
\end{array} \right\}
\]

from (12), (13), (5) and Remark 2.4

\[
= \max \{p(gz_1, F(z_1, z_2)), p(gz_2, F(z_2, z_1))\} \leq \lim_{n \to \infty} M(x_n, y_n, z_1, z_2).
\]

Hence

\[
\lim_{n \to \infty} M(x_n, y_n, z_1, z_2) = \max \{p(gz_1, F(z_1, z_2)), p(gz_2, F(z_2, z_1))\} \quad (15)
\]

Now
\[
\frac{1}{k^2} p(gz_1, F(z_1, z_2)) \leq \frac{1}{k^2} p(gz_1, F(z_1, z_2)) \leq \lim_{n \to \infty} p(gx_n+1, F(z_1, z_2)) \quad \text{from Remark 2.4}
\]

\[
\leq \lim_{n \to \infty} \mathcal{L}(M(x_n, y_n, z_1, z_2)) \max \left\{ \begin{array}{l}
p(gz_1, F(z_1, z_2)), \\
p(gz_2, F(z_2, z_1))
\end{array} \right\} \quad \text{from (14), (15)}
\]

Similarly we can show that
\[
\frac{1}{k^2} p(gz_2, F(z_2, z_1)) \leq \lim_{n \to \infty} \mathcal{L}(M(x_n, y_n, z_1, z_2)) \max \left\{ \begin{array}{l}
p(gz_1, F(z_1, z_2)), \\
p(gz_2, F(z_2, z_1))
\end{array} \right\}
\]

Thus
\[
\frac{1}{k^2} \max \left\{ \begin{array}{l}
p(gz_1, F(z_1, z_2)), \\
p(gz_2, F(z_2, z_1))
\end{array} \right\} \leq \lim_{n \to \infty} \mathcal{L}(M(x_n, y_n, z_1, z_2)) \max \left\{ \begin{array}{l}
p(gz_1, F(z_1, z_2)), \\
p(gz_2, F(z_2, z_1))
\end{array} \right\}
\]

If \( \max \left\{ \begin{array}{l}
p(gz_1, F(z_1, z_2)), \\
p(gz_2, F(z_2, z_1))
\end{array} \right\} > 0 \), then from property of \( \mathcal{L} \), we have
\[
\lim_{n \to \infty} M(x_n, y_n, z_1, z_2) = 0 \] which is a contradiction in view of (15).

Hence \( gz_1 = F(z_1, z_2) \) and \( gz_2 = F(z_2, z_1) \).

Thus \( (z_1, z_2) \) is a coupled coincidence point of \( F \) and \( g \). This completes the proof.

Now, we furnish an example to illustrate Theorem 2.5.

Example 2.6: Let \( X = [0, 1] \) and \( p(x, y) = \max\{x^2, y^2\} \). Then \( p \) is a partial b-metric-like with \( k = 2 \). Define \( x \leq y \) as \( x \leq y \). Consider the functions \( F : X \times X \to X \) and \( g : X \to X \) which are defined as \( gx = x \) and
\[
F(x, y) = \begin{cases} 
\frac{x}{2\sqrt{1+y^2}}, & \text{if } x \leq y \\
0, & \text{if } x > y
\end{cases}
\]

Let \( \mathcal{L} : (0, \infty) \to (0, \frac{1}{2}) \) be defined by \( \mathcal{L}(t) = \frac{1}{2(t+1)} \).
Let \( gx \leq gu \) and \( gy \geq gv \). That is let \( x \leq u \) and \( y \geq v \).
Case(i): Assume \( x \leq y \) and \( u \leq v \).
Then \( p(x, u) = \max\{x^2, u^2\} = u^2 \leq M(x, y, u, v) \).
p(F(x, y), F(u, v)) = \max \left\{ \frac{x^2}{4(1+y^2)}, \frac{u^2}{4(1+v^2)} \right\}
= \frac{u^2}{4(1+v^2)} \leq \frac{x^2}{4(1+y^2)} \leq \frac{M(x,y,u,v)}{4(1+M(x,y,u,v))} = L(M(x, y, u, v))M(x, y, u, v)
Case (ii): Assume x \leq y and u \geq v.
Then p(F(x, y), F(u, v)) = \max \left\{ \frac{x^2}{4(1+y^2)} \right\} = x^2 \leq M(x, y, u, v).
P(F(x, y), F(u, v)) = \frac{x^2}{4(1+y^2)} \leq \frac{M(x,y,u,v)}{4(1+M(x,y,u,v))} = L(M(x, y, u, v))M(x, y, u, v)

Case (iii): Assume x > y and u > v.
Then p(F(x, y), F(u, v)) = 0 \leq L(M(x, y, u, v))M(x, y, u, v).
The case x > y and u \geq v does n’t arise as x \leq u and y \geq v. Thus the condition (2.5.3) is satisfied. One can easily verify the remaining conditions. Clearly (0, 0) is a coupled coincidence point of F and g.

Corollary 2.7: Let (X, p, k, \preceq) be an ordered complete partial b-metric-like space and F : X \times X \rightarrow X be a mapping satisfying

(2.7.1) F has the mixed monotone property,
(2.7.2) p(F(x, y), F(u, v)) \leq L(M(x, y, u, v))M(x, y, u, v)
for all x, y, u, v \in X with x \preceq u, y \succeq v, where L \in \Psi_1^k and

M(x, y, u, v) = \max \left\{ \frac{p(x, u), p(y, v), p(x, F(x, y)), p(y, F(y, x))}{p(u, F(u, v)), p(v, F(v, u))}, \right\}
\frac{1}{2} \left[ p(x, F(u, v)) + p(u, F(x, y)) \right], \frac{1}{2} \left[ p(y, F(v, u)) + p(v, F(y, x)) \right]

(2.7.3) there exist two elements x_0, y_0 \in X such that x_0 \preceq F(x_0, y_0) and y_0 \succeq F(y_0, x_0),
(2.7.4) (a) Suppose F is continuous
or
(b) X has the following properties:
(i) If a non-decreasing sequence \{x_n\} \rightarrow x, then x_n \preceq x, \forall n,
(ii) If a non-increasing sequence \{y_n\} \rightarrow y, then y \succeq y_n, \forall n.

Then F has a coupled fixed point in X \times X.

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