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A COUPLED COINCIDENCE POINT THEOREM ON ORDERED PARTIAL B-METRIC-LIKE SPACES

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Abstract. In this paper, we prove a coupled coincidence point theorem in ordered partial b-metric-like spaces besides furnishing an illustrative example to demonstrate our main result.

1. Introduction

The concept of b-metric space was introduced by Czerwik [3] which runs as follows:
Definition 1.1([3]): A b-metric on a non empty set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ and $k \geq 1$, the following three conditions are satisfied:
(i) $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq k[d(x, z) + d(z, y)]$.
As usual, the pair (X, d) is called a b-metric space.
Example 1.2: Let $X = \mathcal{R}$ and $d(x, y) = (x - y)^2$ for all $x, y \in X$. Then d is a b-metric with $k = 2$ but not a metric as $d(1, -1) > d(1, 0) + d(0, -1)$.
Ali Alghamdi et al.[1] introduced the concept of b-metric-like spaces and proved some fixed point theorems involving a single map.
Definition 1.3([1]): A b-metric-like on a non empty set X is a function $d : X \times X \to [0, \infty)$ such that for all $x, y, z \in X$ and a constant $k \geq 1$, the following three conditions are satisfied:
(i) $d(x, y) = 0$ implies $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq k[d(x, z) + d(z, y)]$.
The pair (X, d) is called a b-metric-like space.
Example 1.4: Let $X = [0, \infty)$ and $d(x, y) = (x + y)^2$ for all $x, y \in X$. Then d is a b-metric-like space with $k = 2$ but not a b- metric .
Matthews [6] introduced the concept of a partial metric space which runs as follows:
Definition 1.5([6]): A mapping $p : X \times X \to [0, \infty)$ (where X is a nonempty set)

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is said to be a partial metric on X if (for any $x, y, z \in X$) the following conditions are satisfied:
(i) $x = y \Leftrightarrow p(x,x) = p(x,y) = p(y,y)$,
(ii) $p(x,x) \le p(x,y)$, $p(y,y) \le p(x,y)$,
(iii) $p(x,y) = p(y,x)$,
(iv) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.
The pair (X,p) is called a partial metric space.
In [2],Bhaskar and Lakshmikantham introduced the concept of coupled fixed points and obtained some coupled fixed point theorems. Later Lakshmikantham and Ciric [5] introduced the following definitions.
Definition 1.6([5]): An element $(x,y) \in X \times X$ is called
(i) a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $gx = F(x,y)$ and $gy = F(y,x)$.
(ii) a common coupled fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x,y)$ and $y = gy = F(y,x)$.
Definition 1.7([5]): Let $(X, \preceq)$ be a partially ordered set with $F : X \times X \to X$ and $g : X \to X$. Then F is said to have mixed g-monotone property if for any $x, y \in X$, we have

$$(i) x_1, x_2 \in X, gx_1 \preceq gx_2 \Rightarrow F(x_1,y) \preceq F(x_2,y)$$
$$(ii) y_1, y_2 \in X, gy_1 \preceq gy_2 \Rightarrow F(x,y_1) \succeq F(x,y_2).$$

In the sequel, we need the following lemma.
Lemma 1.8([4]): Let X be a non-empty set and $g : X \to X$ be a mapping. Then there exists a subset E of X such that $g(E) = g(X)$ and the mapping $g : E \to X$ is one-one.
Note that for $x, y \in [0,\infty)$ with $x \le y$, we have $\frac{x}{1+x} \le \frac{y}{1+y}$.

2. Main Result

Now, we give the following definition (by combining Definitons 1.3 and 1.5)
Definition 2.1: A partial b-metric-like on a non empty set X is a function $p : X \times X \to [0,\infty)$, wherein for all $x, y, z \in X$ and a constant $k \ge 1$, the following conditions are satisfied:
(p_1) $p(x,y) = 0$ implies $x = y$,
(p_2) $p(x,x) \le p(x,y)$,$p(y,y) \le p(x,y)$,
(p_3) $p(x,y) = p(y,x)$,
(p_4) $p(x,y) \le k[p(x,z) + p(z,y) - p(z,z)]$.
The pair (X,p,k) is called a partial b-metric-like space.
Definition 2.2: Let (X,p,k) be a partial b-metric-like space and $\{x_n\}$ a sequence in X with $x \in X$. Then the sequence $\{x_n\}$ is said to be convergent to x if $\lim_{n\to\infty} p(x_n,x) = p(x,x)$.
Definition 2.3: Let (X,p,k) be a partial b-metric-like space.
(i) A sequence $\{x_n\}$ in (X,p,k) is said to be Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite .
(ii) A partial b-metric-like space (X,p,k) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, to a point $x \in X$ so that $\lim_{n,m\to\infty} p(x_n,x_m) = p(x,x) = \lim_{n\to\infty} p(x_n,x)$.
One can easily verify the following remark.

Remark 2.4: Let (X,p,k) be a partial b-metric-like space and $\{x_n\}$ a sequence in X such that $\lim_{n\to\infty} p(x_n,x)=0$. Then
(i) x is unique,
(ii)$\frac{1}{k}p(x,y)\leq \lim_{n\to\infty} p(x_n,y)\leq kp(x,y)$ for all $y\in X$
(iii) $p(x_n,x_0)\leq kp(x_0,x_1)+k^2p(x_1,x_2)+\cdots+k^{n-1}p(x_{n-2},x_{n-1})+k^{n-1}p(x_{n-1},x_n)$ whenever $\{x_k\}_{k=0}^n\in X$.
Ali Alghamdi et al.[1] introduced the following class of functions.
Let $\Psi_{\mathcal{L}}^k$ be the class of those functions $\mathcal{L}:(0,\infty)\to(0,\frac{1}{k^2})$ which satisfy the condition $\mathcal{L}(t_n)\to(\frac{1}{k^2})^+\Rightarrow t_n\to 0$, where $k>0$.
Using these functions, we now prove a coupled coincidence point theorem in ordered partial b-metric-like spaces.
Let (X,p,k) be a partial b-metric-like space and $F:X\times X\to X$ and $g:X\to X$. For $x,y,u,v\in X$, we denote

$$M(x,y,u,v)=\max\left\{\begin{array}{c} p(gx,gu),p(gy,gv),p(gx,F(x,y)),p(gy,F(y,x)),\\ p(gu,F(u,v)),p(gv,F(v,u)),\\ \frac{1}{2k}[p(gx,F(u,v))+p(gu,F(x,y))],\\ \frac{1}{2k}[p(gy,F(v,u))+p(gv,F(y,x))] \end{array}\right\}.$$

Notice that $M(x,y,u,v)=M(y,x,v,u)$ for all $x,y,u,v\in X$.
Now, we are equipped to prove our main result as follows.
Theorem 2.5: Let $(X,p,k,\preceq)$ be an ordered partial b-metric -like space and $F:X\times X\to X$, $g:X\to X$ be the mappings which satisfy the following conditions:
(2.5.1)$F(X\times X)\subseteq g(X)$, $g(X)$ is complete,
(2.5.2) F has the mixed g-monotone property,
(2.5.3) $p(F(x,y),F(u,v))\leq \mathcal{L}(M(x,y,u,v))M(x,y,u,v)$
for all $x,y,u,v\in X$ with $gx\preceq gu$, $gy\succeq gv$, where $\mathcal{L}\in\Psi_{\mathcal{L}}^k$
(2.5.4) there exist two elements $x_0,y_0\in X$ such that $gx_0\preceq F(x_0,y_0)$ and $gy_0\succeq F(y_0,x_0)$,
(2.5.5) (a) Suppose F and g are continuous

or

(b) $g(X)$ has the following properties:
(i) If a non-decreasing sequence $\{a_n\}\to a$, then $a_n\preceq a$, $\forall\ n$,
(ii) If a non-increasing sequence $\{a_n\}\to a$, then $a\preceq a_n$, $\forall\ n$.
Then F and g have a coupled coincidence point in $X\times X$.
Proof . By (2.5.4), there exist two elements $x_0,y_0\in X$ such that $gx_0\preceq F(x_0,y_0)$ and $gy_0\succeq F(y_0,x_0)$. Since $F(X\times X)\subseteq g(X)$, we can choose $x_1,y_1\in X$ such that $gx_1=F(x_0,y_0)$ and $gy_1=F(y_0,x_0)$. Again we can choose $x_2,y_2\in X$ such that $gx_2=F(x_1,y_1)$ and $gy_2=F(y_1,x_1)$. Continuing this process indefinitely, we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that $gx_{n+1}=F(x_n,y_n)$ and $gy_{n+1}=F(y_n,x_n)$ for all $n\geq 0$.
Now for $n\geq 0$, we shall prove that

$$gx_n\preceq gx_{n+1} \text{ and } gy_n\succeq gy_{n+1}. \tag{1}$$

From (2.5.4), (1) holds for $n=0$. Suppose (1) holds for $n=m>0$. Now, by (2.5.2), we have

$$gx_{m+1}=F(x_m,y_m)\preceq F(x_{m+1},y_m)\preceq F(x_{m+1},y_{m+1})=gx_{m+2} \text{ and}$$
$$gy_{m+1}=F(y_m,x_m)\succeq F(y_{m+1},x_m)\succeq F(y_{m+1},x_{m+1})=gy_{m+2}.$$

Thus (1) holds for $n=m+1$. Hence by mathematical induction, (1) holds for all

$n \geq 0$.
In case, $gx_{n+1} = gx_n$ and $gy_{n+1} = gy_n$ for some n, then (x_n, y_n) is a coupled coincidence point of F and g. Otherwise, assume that $gx_n \neq gx_{n+1}$ or $gy_n \neq gy_{n+1}$ for all n. Consider

$$\begin{aligned} p(gx_n, gx_{n+1}) &= p(F(x_{n-1}, y_{n-1}), F(x_n, y_n)) \\ &\leq \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n))M(x_{n-1}, y_{n-1}, x_n, y_n) \end{aligned}$$

where

$$M(x_{n-1}, y_{n-1}, x_n, y_n) = \max \left\{ \begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), p(gx_{n-1}, gx_n), \\ p(gy_{n-1}, gy_n), p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}), \\ \frac{1}{2k}[p(gx_{n-1}, gx_{n+1}) + p(gx_n, gx_n)], \\ \frac{1}{2k}[p(gy_{n-1}, gy_{n+1}) + p(gy_n, gy_n)] \end{array} \right\},$$

$$\begin{aligned} \tfrac{1}{2k}[p(gx_{n-1}, gx_{n+1}) + p(gx_n, gx_n)] &\leq \tfrac{1}{2k} \left\{ \begin{array}{c} k[p(gx_{n-1}, gx_n) + p(gx_n, gx_{n+1}) - p(gx_n, gx_n)] \\ +kp(gx_n, gx_n) \end{array} \right\} \\ &\leq \max\left\{p(gx_{n-1}, gx_n), p(gx_n, gx_{n+1})\right\}, \end{aligned}$$

and

$$\begin{aligned} M(x_{n-1}, y_{n-1}, x_n, y_n) &\leq \max \left\{ \begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), \\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \end{array} \right\} \\ &\leq M(x_{n-1}, y_{n-1}, x_n, y_n). \end{aligned}$$

Thus

$$M(x_{n-1}, y_{n-1}, x_n, y_n) = \max \left\{ \begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), \\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \end{array} \right\}. \tag{2}$$

So,

$$p(gx_n, gx_{n+1}) \leq \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) \max \left\{ \begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), \\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \end{array} \right\}.$$

Similarly by using $M(y_{n-1}, x_{n-1}, y_n, x_n) = M(x_{n-1}, y_{n-1}, x_n, y_n)$, we can show that

$$p(gy_n, gy_{n+1}) \leq \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) \max \left\{ \begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), \\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \end{array} \right\}.$$

Thus

$$\max \left\{ \begin{array}{c} p(gx_n, gx_{n+1}), \\ p(gy_n, gy_{n+1}) \end{array} \right\} \leq \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) \max \left\{ \begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), \\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \end{array} \right\}. \tag{3}$$

If $\max \left\{ \begin{array}{c} p(gx_{n-1}, gx_n), p(gy_{n-1}, gy_n), \\ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \end{array} \right\} = \max \left\{ \ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \ \right\}$
then using $\mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) < \frac{1}{k^2}$, we get a contradiction from (3).
Hence

$$\max \left\{ \begin{array}{c} p(gx_n, gx_{n+1}), \\ p(gy_n, gy_{n+1}) \end{array} \right\} \leq \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) \max \left\{ \begin{array}{c} p(gx_{n-1}, gx_n), \\ p(gy_{n-1}, gy_n), \end{array} \right\}.$$

Put $p_n = \max\left\{ \ p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1}) \ \right\}$. Then

$$p_n \leq \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n))p_{n-1} < p_{n-1} \tag{4}$$

Thus $\{p_n\}$ is a non-increasing sequence of non-negative real numbers and hence also converges to some real number $s \geq 0$. Suppose $s > 0$.
From (4), we have $\quad s \leq \lim_{n\to\infty} \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n))s \quad$ so that

$1 \leq \lim_{n\to\infty} \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n))$.
Now we have
$\frac{1}{k^2} \leq \lim_{n\to\infty} \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) \leq \frac{1}{k^2}$ which in turn yields that
$\lim_{n\to\infty} \mathcal{L}(M(x_{n-1}, y_{n-1}, x_n, y_n)) = \frac{1}{k^2}$. Hence $\lim_{n\to\infty} M(x_{n-1}, y_{n-1}, x_n, y_n) = 0$.
Thus from(2), we have

$$\lim_{n\to\infty} \max \{p(gx_n, gx_{n+1}), p(gy_n, gy_{n+1})\} = 0. \tag{5}$$

Also, from (p_2) we have

$$\lim_{n\to\infty} \max \{p(gx_n, gx_n), p(gy_n, gy_n)\} = 0. \tag{6}$$

Now, we prove that

$$\lim_{n,m\to\infty} \max \{p(gx_n, gx_m), p(gy_n, gy_m)\} = 0. \tag{7}$$

Suppose (7) is not true. Then

$$\lim_{n,m\to\infty} \max \{p(gx_n, gx_m), p(gy_n, gy_m)\} > 0. \tag{8}$$

Let $m > n$. Then from (1), we have $gx_n \preceq gx_m$ and $gy_n \succeq gy_m$.
From (2.5.3), we have

$$\begin{aligned} p(gx_{n+1}, gx_{m+1}) &= p(F(x_n, y_n, F(x_m, y_m)) \\ &\leq \mathcal{L}(M(x_n, y_n, x_m, y_m))M(x_n, y_n, x_m, y_m) \end{aligned} \tag{9}$$

where

$$\lim_{n,m\to\infty} M(x_n, y_n, x_m, y_m) = \lim_{n,m\to\infty} \max \left\{ \begin{array}{c} p(gx_n, gx_m), p(gy_n, gy_m), p(gx_n, gx_{n+1}), \\ p(gy_n, gy_{n+1}), p(gx_m, gx_{m+1}), p(gy_m, gy_{m+1}), \\ \frac{1}{2k}[p(gx_n, gx_{m+1}) + p(gx_m, gx_{n+1})], \\ \frac{1}{2k}[p(gy_n, gy_{m+1}) + p(gy_m, gy_{n+1})] \end{array} \right\}.$$

But
$\frac{1}{2k}[p(gx_n, gx_{m+1}) + p(gx_m, gx_{n+1})]$
$\leq \frac{1}{2k} k \left[\begin{array}{c} p(gx_n, gx_m) + p(gx_m, gx_{m+1}) - p(gx_m, gx_m) + \\ p(gx_m, gx_n) + p(gx_n, gx_{n+1}) - p(gx_n, gx_n) \end{array} \right]$.
Hence

$$\begin{aligned} \lim_{n,m\to\infty} M(x_n, y_n, x_m, y_m) &\leq \lim_{n,m\to\infty} \max \{p(gx_n, gx_m), p(gy_n, gy_m)\} \ \mathit{from}\ (5),\ (6) \\ &\leq \lim_{n,m\to\infty} M(x_n, y_n, x_m, y_m). \end{aligned}$$

Hence

$$\lim_{n,m\to\infty} M(x_n, y_n, x_m, y_m) = \lim_{n,m\to\infty} \max \{p(gx_n, gx_m), p(gy_n, gy_m)\} \tag{10}$$

From (9), we have

$$\lim_{n,m\to\infty} p(gx_{n+1}, gx_{m+1}) \leq \lim_{n,m\to\infty} \mathcal{L}(M(x_n, y_n, x_m, y_m)) \lim_{n,m\to\infty} \max \left\{ \begin{array}{c} p(gx_n, gx_m), \\ p(gy_n, gy_m) \end{array} \right\}.$$

Similarly, we can show that

$$\lim_{n,m\to\infty} p(gy_{n+1}, gy_{m+1}) \leq \lim_{n,m\to\infty} \mathcal{L}(M(x_n, y_n, x_m, y_m)) \lim_{n,m\to\infty} \max \left\{ \begin{array}{c} p(gx_n, gx_m), \\ p(gy_n, gy_m) \end{array} \right\}.$$

Thus

$$\lim_{n,m\to\infty}\max\left\{\begin{array}{c} p(gx_{n+1},gx_{m+1}),\\ p(gy_{n+1},gy_{m+1})\end{array}\right\}\leq \lim_{n,m\to\infty}\mathcal{L}(M(x_n,y_n,x_m,y_m))\lim_{n,m\to\infty}\max\left\{\begin{array}{c} p(gx_n,gx_m),\\ p(gy_n,gy_m)\end{array}\right\} \tag{11}$$

We have
$p(gx_n,gx_m)\leq kp(gx_n,gx_{n+1})+k^2p(gx_{n+1},gx_{m+1})+k^2p(gx_{m+1},gx_m)$
which implies that $\frac{1}{k^2}\lim\limits_{n,m\to\infty}p(gx_n,gx_m)\leq \lim\limits_{n,m\to\infty}p(gx_{n+1},gx_{m+1})\ from(5)$.
Similarly, $\frac{1}{k^2}\lim\limits_{n,m\to\infty}p(gy_n,gy_m)\leq \lim\limits_{n,m\to\infty}p(gy_{n+1},gy_{m+1})$.
Thus by using (11), we have

$$\begin{aligned}\frac{1}{k^2}\lim_{n,m\to\infty}\max\left\{\begin{array}{c} p(gx_n,gx_m),\\ p(gy_n,gy_m)\end{array}\right\} &\leq \lim_{n,m\to\infty}\max\left\{\begin{array}{c} p(gx_{n+1},gx_{m+1}),\\ p(gy_{n+1},gy_{m+1})\end{array}\right\}\\ &\leq \lim_{n,m\to\infty}\mathcal{L}(M(x_n,y_n,x_m,y_m))\lim_{n,m\to\infty}\max\left\{\begin{array}{c} p(gx_n,gx_m),\\ p(gy_n,gy_m)\end{array}\right\},\end{aligned}$$

which in turn implies from (8) that
$\frac{1}{k^2}\leq\lim\limits_{n,m\to\infty}\mathcal{L}(M(x_n,y_n,x_m,y_m))\leq\frac{1}{k^2}$ so that $\lim\limits_{n\to\infty}M(x_n,y_n,x_m,y_m)=0$.
It is a contradiction to (8) in view of (10).
Hence (7) holds. Thus $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, there exist $r_1,r_2,z_1,z_2\in X$ such that $gx_n\to r_1=gz_1$ and $gy_n\to r_2=gz_2$.
Suppose (2.5.5)(a) holds.
From Lemma 1.8, there exists a subset $E\subseteq X$ such that $g(E)=g(X)$ and the mapping $g:E\to X$ is one-one. Without loss of generality, we are able to choose $E\subseteq X$ such that $z_1,z_2\in E$. Now define $G:g(E)\times g(E)\to X$ by

$$G(ga,gb)=F(a,b)\text{ for all }ga,gb\in g(E)\text{ where }a,b\in E.$$

Since F and g are continuous, it follows that G is continuous. As $g:E\to X$ is one-one and $F(X\times X)\subseteq g(X)$, G is well defined. Again since F and g are continuous, it follows that G is continuous. Since $\{x_n\},\{y_n\}\subset X$ and $g(E)=g(X)$, there exists $\{a_n\},\{b_n\}\subset E$ such that $g(x_n)=g(a_n)$ and $g(y_n)=g(b_n)$ for all n. So we have

$$F(z_1,z_2)=G(gz_1,gz_2)=\lim_{n\to\infty}G(ga_n,gb_n)=\lim_{n\to\infty}F(a_n,b_n)=\lim_{n\to\infty}ga_{n+1}=gz_1,$$

$$F(z_2,z_1)=G(gz_2,gz_1)=\lim_{n\to\infty}G(gb_n,ga_n)=\lim_{n\to\infty}F(b_n,a_n)=\lim_{n\to\infty}gb_{n+1}=gz_2.$$

Thus (z_1,z_2) is a coupled coincidence point of F and g.
Suppose (2.5.5) (b) holds.
From (1)and (i) and (ii) of (2.5.5)(b), we have $gx_n\preceq gz_1$ and $gy_n\succeq gz_2$ for all n.
From definition of completeness of $g(X)$ and from (7), we have

$$\lim_{n\to\infty}p(gx_n,gz_1)=p(gz_1,gz_1)=\lim_{n,m\to\infty}p(gx_n,gx_m)=0 \tag{12}$$

$$\lim_{n\to\infty}p(gy_n,gz_2)=p(gz_2,gz_2)=\lim_{n,m\to\infty}p(gy_n,gy_m)=0 \tag{13}$$

Now

$$\begin{aligned}p(gx_{n+1},F(z_1,z_2)) &= p(F(x_n,y_n),F(z_1,z_2))\\ &\leq \mathcal{L}(M(x_n,y_n,z_1,z_2))M(x_n,y_n,z_1,z_2)\end{aligned} \tag{14}$$

$$\lim_{n\to\infty} M(x_n, y_n, z_1, z_2)$$
$$= \lim_{n\to\infty} \max \left\{ \begin{array}{c} p(gx_n, gz_1), p(gy_n, gz_2), p(gx_n, gx_{n+1}), \\ p(gy_n, gy_{n+1}), p(gz_1, F(z_1, z_2)), p(gz_2, F(z_2, z_1)), \\ \frac{1}{2k}[p(gx_n, F(z_1, z_2)) + p(gz_1, gx_{n+1})], \\ \frac{1}{2k}[p(gy_n, F(z_2, z_1)) + p(gz_2, gy_{n+1})] \end{array} \right\}.$$

$$\leq \max \left\{ \begin{array}{c} 0, 0, 0, 0, p(gz_1, F(z_1, z_2)), p(gz_2, F(z_2, z_1)), \\ \frac{1}{2k}[kp(gz_1, F(z_1, z_2)) + 0], \\ \frac{1}{2k}[kp(gz_2, F(z_2, z_1)) + 0] \end{array} \right\}$$

$$from\ (12), (13), (5)\ and\ Remark\ 2.4$$

$$= \max \{p(gz_1, F(z_1, z_2)), p(gz_2, F(z_2, z_1))\}$$
$$\leq \lim_{n\to\infty} M(x_n, y_n, z_1, z_2).$$

Hence

$$\lim_{n\to\infty} M(x_n, y_n, z_1, z_2) = \max \{p(gz_1, F(z_1, z_2)), p(gz_2, F(z_2, z_1))\} \tag{15}$$

Now

$$\begin{aligned} \frac{1}{k^2} p(gz_1, F(z_1, z_2)) &\leq \frac{1}{k} p(gz_1, F(z_1, z_2)) \\ &\leq \lim_{n\to\infty} p(gx_{n+1}, F(z_1, z_2))\ from\ Remark\ 2.4 \\ &\leq \lim_{n\to\infty} \mathcal{L}(M(x_n, y_n, z_1, z_2)) \max \left\{ \begin{array}{c} p(gz_1, F(z_1, z_2)), \\ p(gz_2, F(z_2, z_1)) \end{array} \right\} from\ (14), (15) \end{aligned}$$

Similarly we can show that

$$\frac{1}{k^2} p(gz_2, F(z_2, z_1)) \leq \lim_{n\to\infty} \mathcal{L}(M(x_n, y_n, z_1, z_2)) \max \left\{ \begin{array}{c} p(gz_1, F(z_1, z_2)), \\ p(gz_2, F(z_2, z_1)) \end{array} \right\}.$$

Thus

$$\frac{1}{k^2} \max \left\{ \begin{array}{c} p(gz_1, F(z_1, z_2)), \\ p(gz_2, F(z_2, z_1)) \end{array} \right\} \leq \lim_{n\to\infty} \mathcal{L}(M(x_n, y_n, z_1, z_2)) \max \left\{ \begin{array}{c} p(gz_1, F(z_1, z_2)), \\ p(gz_2, F(z_2, z_1)) \end{array} \right\}.$$

If $\max \left\{ \begin{array}{c} p(gz_1, F(z_1, z_2)), \\ p(gz_2, F(z_2, z_1)) \end{array} \right\} > 0$, then from property of $\mathcal{L}$, we have
$\lim_{n\to\infty} M(x_n, y_n, z_1, z_2) = 0$ which is a contradiction in view of (15).
Hence $gz_1 = F(z_1, z_2)$ and $gz_2 = F(z_2, z_1)$.
Thus (z_1, z_2) is a coupled coincidence point of F and g. This completes the proof.
Now, we furnish an example to illustrate Theorem 2.5.

Example 2.6: Let $X = [0, 1]$ and $p(x, y) = \max\{x^2, y^2\}$. Then p is a partial b-metric-like with $k = 2$. Define $x \preceq y$ as $x \leq y$. Consider the functions $F : X \times X \to X$ and $g : X \to X$ which are defined as $gx = x$ and

$$F(x, y) = \left\{ \begin{array}{c} \frac{x}{2\sqrt{1+y^2}}\ ,\ if\ x \leq y \\ 0\ ,\ if\ x > y \end{array} \right.$$

Let $\mathcal{L} : (0, \infty) \to (0, \frac{1}{4})$ be defined by $\mathcal{L}(t) = \frac{1}{4(1+t)}$.
Let $gx \preceq gu$ and $gy \succeq gv$.That is let $x \leq u$ and $y \geq v$.
Case(i): Assume $x \leq y$ and $u \leq v$.
Then $p(x, u) = \max\{x^2, u^2\} = u^2 \leq M(x, y, u, v)$.

$p(F(x,y),F(u,v)) = \max\left\{\frac{x^2}{4(1+y^2)}, \frac{u^2}{4(1+v^2)}\right\}$
$= \frac{u^2}{4(1+v^2)} \leq \frac{u^2}{4(1+u^2)} \leq \frac{M(x,y,u,v)}{4(1+M(x,y,u,v))} = L(M(x,y,u,v))M(x,y,u,v)$
Case(ii): Assume $x \leq y$ and $u > v$.
Then $p(gx, F(x,y)) = \max\{x^2, \frac{x^2}{4(1+y^2)}\} = x^2 \leq M(x,y,u,v)$.
$p(F(x,y),F(u,v)) = \frac{x^2}{4(1+y^2)}$
$\leq \frac{x^2}{4(1+x^2)} \leq \frac{M(x,y,u,v)}{4(1+M(x,y,u,v))} = L(M(x,y,u,v))M(x,y,u,v)$
Case(iii): Assume $x > y$ and $u > v$.
Then $p(F(x,y),F(u,v)) = 0 \leq L(M(x,y,u,v))M(x,y,u,v)$.
The case $x > y$ and $u \leq v$ does n't arise as $x \leq u$ and $y \geq v$. Thus the condition (2.5.3) is satisfied.One can easily verify the remaining conditions.Clearly $(0,0)$ is a coupled coincidence point of F and g.

Corollary 2.7: Let $(X,p,k,\preceq)$ be an ordered complete partial b-metric-like space and $F : X \times X \to X$ be a mapping satisfying

(2.7.1) F has the mixed monotone property,
(2.7.2) $p(F(x,y),F(u,v)) \leq \mathcal{L}(M(x,y,u,v))M(x,y,u,v)$
for all $x,y,u,v \in X$ with $x \preceq u$, $y \succeq v$, where $\mathcal{L} \in \Psi^k_{\mathcal{L}}$ and

$$M(x,y,u,v) = \max\left\{\begin{array}{c} p(x,u), p(y,v), p(x,F(x,y)), p(y,F(y,x)), \\ p(u,F(u,v)), p(v,F(v,u)), \\ \frac{1}{2k}[p(x,F(u,v)) + p(u,F(x,y))], \\ \frac{1}{2k}[p(y,F(v,u)) + p(v,F(y,x))] \end{array}\right\}$$

(2.7.3) there exist two elements $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $y_0 \succeq F(y_0,x_0)$,
(2.7.4) (a) Suppose F is continuous
or
(b) X has the following properties:
(i) If a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, $\forall\ n$,
(ii) If a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$, $\forall\ n$.

Then F has a coupled fixed point in $X \times X$.

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References

[1] M.Ali Alghamdi,N.Hussain and P.Salimi,Fixed point and coupled fixed point theorems on b-metric-like spaces,Journal of inequalities and Applications,(2013),2013:402,25 pages.
[2] T.G. Bhaskar and V. Lakshmikantham, Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Analysis, 65, (7) (2006), 1379 - 1393.
[3] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inf. Univ. Ostraviensis, 1(1993), 5 - 11.
[4] R. H. Haghi, Sh. Rezapour and N. Shahzad, Some fixed point generalizations are not real generalizations, Nonlinear Anal., 74 (2011), 1799 - 1803.
[5] V. Lakshmikantham and Lj. Ciric, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis: Theory, Methods and Applications, vol. 70, no. 12(2009), 4341-4349.
[6] S.G. Matthews, Partial metric topology, Proc. 8th Summer conference on General Topology and Applications, Ann. New York Acad. Sci., vol. 728, 1994, 183 - 197.

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