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CERTAIN SUBCLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY SĂLĂGEAN OPERATOR

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Abstract. Making use of Sălăgean differential operator, in this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses. Also consequences of the results are pointed out.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$.

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\beta)$ of starlike functions of order β in $\mathbb{U}$ and the class $\mathcal{K}(\beta)$ of convex functions of order β in $\mathbb{U}$. By definition, we have

$$\mathcal{S}^*(\beta) := \left\{ f : f \in \mathcal{S} \,\text{and}\, \Re\left(\frac{zf'(z)}{f(z)}\right) > \beta;\ z \in \mathbb{U};\ \ 0 \leq \beta < 1 \right\} \tag{2}$$

and

$$\mathcal{K}(\beta) := \left\{ f : f \in \mathcal{S} \,\text{and}\, \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta;\ z \in \mathbb{U};\ \ 0 \leq \beta < 1 \right\}. \tag{3}$$

It readily follows from the definitions (2) and (3) that

$$f \in \mathcal{K}(\beta) \Longleftrightarrow zf' \in \mathcal{S}^*(\beta).$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z,\ z \in \mathbb{U}$$

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and

$$f(f^{-1}(w)) = w,\ |w| < r_0(f);\ r_0(f) \geq \frac{1}{4},$$

where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots. \tag{4}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). Examples of functions in the class Σ are

$$\frac{z}{1-z},\ -\log(1-z),\ \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \text{ and } \frac{z}{1-z^2}$$

are also not members of Σ (see [7, 21]).

In 1967, Lewin [8] investigated the bi-univalent function class Σ and showed that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Netanyahu [12], on the other hand, showed that $\max_{f\in\Sigma} |a_2| = \frac{4}{3}$. Brannan and Taha [4] (see also [23]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α $(0 \leq \alpha < 1)$, respectively (see [3]). Thus, following Brannan and Taha [4] (see also [23]), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma(\alpha)$ of strongly bi-starlike of order α $(0 < \alpha \leq 1)$, if

$$f \in \Sigma,\ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{U};\ 0 < \alpha \leq 1$$

and

$$\left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2},\ w \in \mathbb{U};\ 0 < \alpha \leq 1,$$

where the function g is given by

$$g(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots \tag{5}$$

the extension of f^{-1} to $\mathbb{U}$.

Similarly, a function $f \in \mathcal{A}$ is in the class $\mathcal{K}_\Sigma(\alpha)$ of strongly bi-convex functions of order $\alpha(0 < \alpha \leq 1)$ if

$$f \in \Sigma,\ \left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\alpha\pi}{2},\ z \in \mathbb{U};\ 0 < \alpha \leq 1$$

and

$$\left|\arg\left(1 + \frac{wg''(w)}{g'(w)}\right)\right| < \frac{\alpha\pi}{2},\ w \in \mathbb{U};\ 0 < \alpha \leq 1,$$

where the function g is extension of f^{-1} to $\mathbb{U}$.

The classes $\mathcal{S}^*_\Sigma(\beta)$ and $\mathcal{K}_\Sigma(\beta)$ of bi-starlike functions of order β and bi-convex functions of order β, corresponding (respectively) to the function classes $\mathcal{S}^*(\beta)$ and $\mathcal{K}(\beta)$ defined by (2) and (3), were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, Brannan and Taha [4] found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details see [4, 23]).

Recently, many authors investigated bounds for various subclasses of biunivalent functions ([1], [5] - [7], [9] - [11], [13], [14], [16] and [18]- [22]). But The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|$ for $n \in \mathbb{N} \setminus \{1,2\}$; $\mathbb{N} := \{1,2,3,\dots\}$ is presumably still an open problem.

In 1983, Sălăgean [17] introduced differential operator $\mathcal{D}^k : \mathcal{A} \to \mathcal{A}$ defined by

$$\begin{aligned}
\mathcal{D}^0 f(z) &= f(z),\\
\mathcal{D}^1 f(z) &= \mathcal{D}f(z) = zf'(z),\\
\mathcal{D}^k f(z) &= \mathcal{D}(\mathcal{D}^{k-1} f(z))\\
&= z(\mathcal{D}^{k-1} f(z))',\ k \in \mathbb{N} = \{1,2,3,\dots\}.
\end{aligned}$$

We note that

$$\mathcal{D}^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n,\ k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{6}$$

The object of the present paper is to introduce two new subclasses of the function class Σ associated with Sălăgean differential operator and find estimate on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ. In order to derive our main results, we have to recall here the following lemma:

Lemma 1[15] If $h \in \wp$ then

$$|c_k| \le 2 \quad for\ each \quad k,$$

where $\wp$ is the family of all functions h analytic in $\mathbb{U}$ for which

$$\Re\{h(z)\} > 0,$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \dots$ for $z \in \mathbb{U}$.

2. Coefficient bounds for the function class $\mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$

Definition 1 A function $f(z)$ given by (1) is said to be in the class $\mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$ if the following conditions are satisfied:

$$f \in \Sigma,\ \left|\arg\left(\frac{\mathcal{D}^{k+1} f(z)}{(1-\lambda)\mathcal{D}^k f(z) + \lambda \mathcal{D}^{k+1} f(z)}\right)\right| < \frac{\alpha\pi}{2},\ 0 < \alpha \le 1;\ \ 0 \le \lambda < 1,\ z \in \mathbb{U} \tag{7}$$

and

$$\left|\arg\left(\frac{\mathcal{D}^{k+1} g(w)}{(1-\lambda)\mathcal{D}^k g(w) + \lambda \mathcal{D}^{k+1} g(w)}\right)\right| < \frac{\alpha\pi}{2},\ 0 < \alpha \le 1;\ \ 0 \le \lambda < 1,\ w \in \mathbb{U}, \tag{8}$$

where the function g is given by (5).

Remark 1 Taking $\lambda = 0$ in the class $\mathcal{S}_{\Sigma}^{k,\lambda}(\alpha)$, we have $\mathcal{S}_{\Sigma}^{k,0}(\alpha) = \mathcal{S}_{\Sigma}^{k}(\alpha)$ and $f \in \mathcal{S}_{\Sigma}^{k}(\alpha)$ if the following conditions are satisfied:

$$f \in \Sigma,\ \left|\arg\left(\frac{\mathcal{D}^{k+1} f(z)}{\mathcal{D}^k f(z)}\right)\right| < \frac{\alpha\pi}{2},\ 0 < \alpha \le 1;\ z \in \mathbb{U} \tag{9}$$

and

$$\left|\arg\left(\frac{\mathcal{D}^{k+1} g(w)}{\mathcal{D}^k g(w)}\right)\right| < \frac{\alpha\pi}{2},\ 0 < \alpha \le 1;\ w \in \mathbb{U}, \tag{10}$$

where the function g is given by (5).

We note that for $k = 0$ and $\lambda = 0$ the class $\mathcal{S}_\Sigma^{0,0}(\alpha) = \mathcal{S}_\Sigma^*(\alpha)$ is class of strongly bi-starlike functions of order $\alpha(0 < \alpha \le 1)$. When $k = 1$ and $\lambda = 0$ the class $\mathcal{S}_\Sigma^{1,0}(\alpha) = \mathcal{K}_\Sigma(\alpha)$ is class of strongly bi-convex functions of order $\alpha(0 < \alpha \le 1)$. For $k = 0$ the class was introduced and studied in [11].

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{S}_\Sigma^{k,\lambda}(\alpha)$.

Theorem 1 Let $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^{k,\lambda}(\alpha)$, $0 < \alpha \le 1$ and $0 \le \lambda < 1$. Then

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2 - 1) - (\alpha - 1)(1-\lambda)^2]2^{2k}}} \tag{11}$$

and

$$|a_3| \le \frac{\alpha}{3^k(1-\lambda)} + \frac{4\alpha^2}{2^{2k}(1-\lambda)^2}. \tag{12}$$

Proof. It follows from (7) and (8) that

$$\frac{\mathcal{D}^{k+1}f(z)}{(1-\lambda)\mathcal{D}^k f(z) + \lambda \mathcal{D}^{k+1} f(z)} = [p(z)]^\alpha \tag{13}$$

and

$$\frac{\mathcal{D}^{k+1}g(w)}{(1-\lambda)\mathcal{D}^k g(w) + \lambda \mathcal{D}^{k+1} g(w)} = [q(w)]^\alpha \tag{14}$$

where $p(z)$ and $q(w)$ in $\wp$ and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots \tag{15}$$

and

$$q(z) = 1 + q_1 w + q_2 w^2 + \dots. \tag{16}$$

Now, equating the coefficients in (13) and (14), we get

$$2^k(1-\lambda)a_2 = \alpha p_1 \tag{17}$$

$$2^{2k}(\lambda^2 - 1)a_2^2 + 3^k(2 - 2\lambda)a_3 = \frac{1}{2}\left[\alpha(\alpha - 1)p_1^2 + 2\alpha p_2\right] \tag{18}$$

$$-2^k(1-\lambda)a_2 = \alpha q_1 \tag{19}$$

and

$$2(1-\lambda)(2a_2^2 - a_3)3^k + (\lambda^2 - 1)2^{2k}a_2^2 = \frac{1}{2}\left[\alpha(\alpha-1)q_1^2 + 2\alpha q_2\right]. \tag{20}$$

From (17) and (19), we get

$$p_1 = -q_1 \tag{21}$$

and

$$2^{2k+1}(1-\lambda)^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{22}$$

From (18), (20) and (22), we obtain

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2 - 1) - (\alpha - 1)(1-\lambda)^2]2^{2k}}.$$

Applying Lemma 1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha(1-\lambda)3^k + [2\alpha(\lambda^2 - 1) - (\alpha - 1)(1-\lambda)^2]2^{2k}}}.$$

This gives the bound on $|a_2|$ as asserted in (11).

Next, in order to find the bound on $|a_3|$, by subtracting (20) from (18), we get

$$3^k(4-4\lambda)a_3 - 3^k(4-4\lambda)a_2^2 = \alpha(p_2-q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2). \tag{23}$$

It follows from (21), (22) and (23) that

$$a_3 = \frac{\alpha(p_2-q_2)}{3^k(4-4\lambda)} + \frac{\alpha^2(p_1^2+q_1^2)}{2^{2k+1}(1-\lambda)^2}. \tag{24}$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \le \frac{\alpha}{3^k(1-\lambda)} + \frac{4\alpha^2}{2^{2k}(1-\lambda)^2}.$$

This completes the proof of Theorem 1.
Taking $\lambda = 0$ in Theorem 1, we obtain the following corollary.
Corollary 1 Let $f(z)$ given by (1) be in the class $\mathcal{S}_\Sigma^k(\alpha)$ and $0 < \alpha \le 1$. Then

$$|a_2| \le \frac{2\alpha}{\sqrt{4\alpha 3^k + (1-3\alpha)2^{2k}}} \tag{25}$$

and

$$|a_3| \le \frac{4\alpha^2}{2^{2k}} + \frac{\alpha}{3^k}. \tag{26}$$

Putting $k = 0$ in Corollary 1, we obtain the coefficient estimates for well-known class $\mathcal{S}_\Sigma^{0,0}(\alpha) = \mathcal{S}_\Sigma^*(\alpha)$ of strongly bi-starlike functions of order α as in [4]. Considering $k = 1$ in Corollary 1, we obtain well-known class $\mathcal{S}_\Sigma^{1,0}(\alpha) = \mathcal{K}_\Sigma(\alpha)$ of strongly bi-convex functions of order α and coincide with results in [4].

3. Coefficient bounds for the function class $\mathcal{M}_\Sigma^{k,\lambda}(\beta)$

Definition 2 A function $f(z)$ given by (1) is said to be in the class $\mathcal{M}_\Sigma^{k,\lambda}(\beta)$ if the following conditions are satisfied:

$$f \in \Sigma,\ \Re\left(\frac{\mathcal{D}^{k+1}f(z)}{(1-\lambda)\mathcal{D}^k f(z) + \lambda\mathcal{D}^{k+1}f(z)}\right) > \beta,\ 0 \le \beta < 1;\ 0 \le \lambda < 1,\ z \in \mathbb{U} \tag{27}$$

and

$$\Re\left(\frac{\mathcal{D}^{k+1}g(w)}{(1-\lambda)\mathcal{D}^k g(w) + \lambda\mathcal{D}^{k+1}g(w)}\right) > \beta,\ 0 \le \beta < 1;\ 0 \le \lambda < 1,\ w \in \mathbb{U}, \tag{28}$$

where the function g is given by (5).
Remark 2 Taking $\lambda = 0$ in the class $\mathcal{M}_\Sigma^{k,\lambda}(\beta)$, we have $\mathcal{M}_\Sigma^{k,0}(\beta) = \mathcal{M}_\Sigma^k(\beta)$ and $f \in \mathcal{M}_\Sigma^k(\beta)$ if the following conditions are satisfied:

$$f \in \Sigma,\ \Re\left(\frac{\mathcal{D}^{k+1}f(z)}{\mathcal{D}^k f(z)}\right) > \beta,\ 0 \le \beta < 1;\ z \in \mathbb{U} \tag{29}$$

and

$$\Re\left(\frac{\mathcal{D}^{k+1}g(w)}{\mathcal{D}^k g(w)}\right) > \beta,\ 0 \le \beta < 1;\ w \in \mathbb{U}, \tag{30}$$

where the function g is given by (5).

We note that for $k = 0$, $\lambda = 0$ the class $\mathcal{M}_\Sigma^{0,0}(\beta) = \mathcal{S}_\Sigma^*(\beta)$ is class of bi-starlike functions of order $\beta (0 \le \beta < 1)$. When $k - 1 = \lambda = 0$ the class $\mathcal{M}_\Sigma^{1,0}(\beta) = \mathcal{K}_\Sigma(\beta)$

is class of bi-convex functions of order $\beta(0 \leq \beta < 1)$. For $k = 0$ the class was introduced in [11].

Next, we find the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{M}_{\Sigma}^{k,\lambda}(\beta)$.

Theorem 2 Let $f(z)$ given by (1) be in the class $\mathcal{M}_{\Sigma}^{k,\lambda}(\beta)$, $0 \leq \beta < 1$ and $0 \leq \lambda < 1$. Then

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2^{2k}(\lambda^2-1)+2(1-\lambda)3^k}} \tag{31}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{2^{2k}(1-\lambda)^2} + \frac{(1-\beta)}{3^k(1-\lambda)}. \tag{32}$$

Proof. It follows from (27) and (28) that there exists $p, q \in \wp$ such that

$$\frac{\mathcal{D}^{k+1}f(z)}{(1-\lambda)\mathcal{D}^k f(z) + \lambda \mathcal{D}^{k+1}f(z)} = \beta + (1-\beta)p(z) \tag{33}$$

and

$$\frac{\mathcal{D}^{k+1}g(w)}{(1-\lambda)\mathcal{D}^k g(w) + \lambda \mathcal{D}^{k+1}g(w)} = \beta + (1-\beta)q(w), \tag{34}$$

where $p(z)$ and $q(w)$ have the forms (15) and (16), respectively. Equating coefficients in (33) and (34), we get

$$2^k(1-\lambda)a_2 = (1-\beta)p_1 \tag{35}$$

$$2^{2k}(\lambda^2-1)a_2^2 + 3^k(2-2\lambda)a_3 = (1-\beta)p_2 \tag{36}$$

$$-2^k(1-\lambda)a_2 = (1-\beta)q_1 \tag{37}$$

and

$$2(1-\lambda)(2a_2^2-a_3)3^k + (\lambda^2-1)2^{2k}a_2^2 = (1-\beta)q_2. \tag{38}$$

From (35) and (37), we get

$$p_1 = -q_1 \tag{39}$$

and

$$2^{2k+1}(1-\lambda)^2 a_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{40}$$

Also, from (36), (38) and (40), we obtain

$$a_2^2 = \frac{(1-\beta)(p_2+q_2)}{2^{2k+1}(\lambda^2-1)+4(1-\lambda)3^k}.$$

Applying Lemma 1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2^{2k}(\lambda^2-1)+2(1-\lambda)3^k}}.$$

This gives the bound on $|a_2|$ as asserted in (31).

Next, in order to find the bound on $|a_3|$, by subtracting (38) from (36), we get

$$3^k(4-4\lambda)a_3 - 3^k(4-4\lambda)a_2^2 = (1-\beta)(p_2-q_2). \tag{41}$$

It follows from (39), (40) and (41) that

$$a_3 = \frac{(1-\beta)^2(p_1^2+q_1^2)}{2^{2k+1}(1-\lambda)^2} + \frac{(1-\beta)(p_2-q_2)}{3^k(4-4\lambda)}. \tag{42}$$

Applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leq \frac{4(1-\beta)^2}{2^{2k}(1-\lambda)^2} + \frac{(1-\beta)}{3^k(1-\lambda)}.$$

This completes the proof of Theorem 2.

When $\lambda = 0$ in the Theorem 2, we get the following corollary.

Corollary 2 Let $f(z)$ given by (1) be in the class $\mathcal{M}_{\Sigma}^{k}(\beta)$ and $0 \leq \beta < 1$. Then

$$|a_2| \leq \sqrt{\frac{1-\beta}{3^k - 2^{2k-1}}} \tag{43}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{2^{2k}} + \frac{1-\beta}{3^k}. \tag{44}$$

Putting $k = 0$ in Corollary 2, we have the coefficients estimates for the well-known class $\mathcal{M}_{\Sigma}^{0,0}(\beta) = \mathcal{S}_{\Sigma}^{*}(\beta)$ of bi-starlike functions of order β as in [4]. Further, taking $k = 1$ in Corollary 2, we obtain the estimates for the well-known class $\mathcal{M}_{\Sigma}^{1,0}(\beta) = \mathcal{K}_{\Sigma}(\beta)$ of bi-convex functions of order β and our results reduces to [4].

Remark 3 For $k = 0$ the results obtained in this paper are coincide with the results discussed in [11]. Further, for the different choice of k the results discussed in this paper would lead to many known and new results.

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