A NOTE ON A SYSTEM OF TWO NONLINEAR DIFFERENCE EQUATIONS

MAI NAM PHONG

Abstract. The goal of this paper is to study the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form:

\[ x_{n+1} = a + bx_{n-1} + cx_{n-1}e^{-y_n}, \quad y_{n+1} = \alpha + \beta y_{n-1} + \gamma y_{n-1}e^{-x_n}, \]

where \( a, b, c, \alpha, \beta, \gamma \in (0, \infty) \), and the initial values \( x_1, x_0, y_1, y_0 \) are positive real values.

1. Introduction

In [3], the authors studied the boundedness, the asymptotic behavior, the periodic character of the solutions and the stability character of the positive equilibrium of the difference equation:

\[ x_{n+1} = a + bx_{n-1}e^{-x_n}, \]

where \( a, b \) are positive constants and the initial values \( x_{-1}, x_0 \) are positive numbers. Furthermore, in [3] the authors used \( a \) as the immigration rate and \( b \) as the growth rate in the population model. In fact, this was a model suggested by the people from the Harvard School of Public Health; studying the population dynamics of one species \( x_n \).

In [4], the authors investigated the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form:

\[ x_{n+1} = a + bx_{n-1}e^{-y_n}, \]
\[ y_{n+1} = c + dy_{n-1}e^{-x_n}. \]

Motivated by these above papers we will extend the difference equation in [3] and the system of difference equations in [4] to a system of difference equations of exponential form:

\[ x_{n+1} = a + bx_{n-1} + cx_{n-1}e^{-y_n}, \]
\[ y_{n+1} = \alpha + \beta y_{n-1} + \gamma y_{n-1}e^{-x_n}, \]

(1.1)

2010 Mathematics Subject Classification. 39A10.

Key words and phrases. Equilibrium, asymptotic, positive solution, system of difference equations.

where $a$, $b$, $c$, $\alpha$, $\beta$, $\gamma$ are positive constants and the initial values $x_{-1}$, $x_0$, $y_{-1}$, $y_0$ are positive real values. Our goal in this paper is to investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system (1.1).

It is easy to see that in the special cases, when $b = \beta = 0$ the system (1.1) becomes the system in the paper [4] and when $x_n = y_n$, $a = \alpha$, $b = \beta = 0$, $c = \gamma$ we have the difference equation in [3].

Difference equations and systems of difference equations of exponential form can be found concerning the theory and applications of difference equations, see [2, 6, 9] and the references cited therein. Moreover, as difference equations have many applications in applied sciences, there are many papers and books that can be found in the following papers: [1, 3-5, 7].

2. **Asymptotic behavior of the solutions of system (1.1)**

First it is very important to establish the boundedness and persistence of solutions; in the first theorem we will study the boundedness and persistence of the positive solutions of system (1.1) by comparing them with solutions of a solvable system of difference equations. Our method is a modification in Theorem 2 in [8]. For related and similar results see, [4, 6, 9].

**Theorem 2.1.** Consider system (1.1) such that:

\[ b + ce^{-a} < 1, \quad \beta + \gamma e^{-a} < 1. \]  

Then every positive solution of (1.1) is bounded and persists.

**Proof.** Let $(x_n, y_n)$ be an arbitrary solution of (1.1). Thus from (1.1) we see that

\[ x_n \geq a, \quad y_n \geq a, \quad n = 1, 2, \ldots \]  

In addition, it follows from (1.1) and (2.2) that

\[ x_{n+1} \leq a + bx_{n-1} + cx_n e^{-a}, \quad y_{n+1} \leq a + \beta y_{n-1} + \gamma y_n e^{-a}, \quad n = 0, 1, 2, \ldots \]  

We will now consider the non-homogeneous difference equations

\[ z_{n+1} = a + bz_{n-1} + cz_n e^{-a}, \quad v_{n+1} = a + \beta v_{n-1} + \gamma v_n e^{-a}, \quad n = 0, 1, 2, \ldots \]  

Therefore, from (2.4) an arbitrary solution $(z_n, v_n)$ of (2.4) is given by

\[ z_n = r_1 \left( b + ce^{-a} \right)^{n/2} + r_2 (-1)^n \left( b + ce^{-a} \right)^{n/2} + \frac{a}{1 - b - ce^{-a}}, \quad n = 1, 2, \ldots, \]

\[ v_n = s_1 \left( b + \gamma e^{-a} \right)^{n/2} + s_2 (-1)^n \left( b + \gamma e^{-a} \right)^{n/2} + \frac{a}{1 - \beta - \gamma e^{-a}}, \quad n = 1, 2, \ldots, \]  

where $r_1$, $r_2$, $s_1$, $s_2$ depend on the initial values $z_{-1}$, $z_0$, $v_{-1}$, $v_0$. Thus we see that relations (2.1) and (2.5) imply that $z_n$ and $v_n$ are bounded sequences. Now we will consider the solution $(z_n, v_n)$ of (2.4) such that

\[ z_{-1} = x_{-1}, \quad z_0 = x_0, \quad v_{-1} = y_{-1}, \quad v_0 = y_0. \]  

Thus from (2.3) and (2.6) we get

\[ x_n \leq z_n, \quad y_n \leq v_n, \quad n = 1, 2, \ldots \]  

Therefore, it follows that $x_n$ and $y_n$ are bounded sequences. Hence from (2.2) the proof of this theorem is now complete. \qed
In the next theorem we will study the existence of invariant intervals of system (1.1).

**Theorem 2.2.** Consider system (1.1) where relations (2.1) hold. Then the following statements are true:

(i) The set

\[
\left[a, a + \frac{a}{1 - b - ce^{-\alpha}}\right] \times \left[\alpha, \frac{\alpha}{1 - \beta - \gamma e^{-\alpha}}\right]
\]

is an invariant set for (1.1).

(ii) Let \(\epsilon\) be an arbitrary positive number and \((x_n, y_n)\) be an arbitrary solution of (1.1). We then consider the sets

\[
I_1 = \left[a, a + \epsilon \right] \times \left[\alpha, \frac{\alpha + \epsilon}{1 - \beta - \gamma e^{-\alpha}}\right], \quad I_2 = \left[\alpha, \alpha + \epsilon \right] \times \left[\frac{\alpha + \epsilon}{1 - b - ce^{-\alpha}}\right].
\]

Then there exists an \(n_0\) such that for all \(n \geq n_0\)

\[
x_n \in I_1, \quad y_n \in I_2.
\]

**Proof.** (i) Let \((x_n, y_n)\) be a solution of (1.1) with initial values \(x_{-1}, x_0, y_{-1}, y_0\) such that

\[
x_{-1}, x_0 \in \left[a, \frac{a}{1 - b - ce^{-\alpha}}\right]; \quad y_{-1}, y_0 \in \left[\alpha, \frac{\alpha}{1 - \beta - \gamma e^{-\alpha}}\right].
\]

Then from (1.1) and (2.10) we get

\[
a \leq x_1 = a + bx_{-1} + cx_{-1}e^{-y_0} \leq a \left[1 + \frac{1}{1 - b - ce^{-\alpha}}(b + ce^{-\alpha})\right] = \frac{a}{1 - b - ce^{-\alpha}},
\]

\[
\alpha \leq y_1 = \alpha + \beta y_{-1} + \gamma y_{-1}e^{-y_0} \leq \alpha \left[1 + \frac{1}{1 - \beta - \gamma e^{-\alpha}}(\beta + \gamma e^{-\alpha})\right] = \frac{\alpha}{1 - \beta - \gamma e^{-\alpha}}.
\]

Then it follows by induction that

\[
a \leq x_n \leq \frac{a}{1 - b - ce^{-\alpha}}, \quad \alpha \leq y_n \leq \frac{\alpha}{1 - \beta - \gamma e^{-\alpha}}, \quad n = 1, 2, \ldots.
\]

This completes the proof of statement (i).

(ii) Let \((x_n, y_n)\) be an arbitrary solution of (1.1). Therefore, from Theorem 2.1 we get

\[
0 < l_1 = \lim_{n \to \infty} \inf x_n, \quad 0 < l_2 = \lim_{n \to \infty} \inf y_n
\]

\[
0 < L_1 = \lim_{n \to \infty} \sup x_n < \infty, \quad 0 < L_2 = \lim_{n \to \infty} \sup y_n < \infty.
\]

It follows from (1.1) and (2.11) that

\[
L_1 \leq a + bL_1 + cL_1e^{-l_2}, \quad l_1 \geq a + bl_1 + cl_1e^{-L_2},
\]

\[
L_2 \leq \alpha + \beta L_2 + \gamma L_2e^{-l_1}, \quad l_2 \geq \alpha + \beta l_2 + l_2\gamma e^{-L_1}.
\]
which imply that
\[ a \leq L_1 \leq \frac{a}{1 - b - e^{-a}}, \quad \alpha \leq L_2 \leq \frac{\alpha}{1 - \beta - \gamma e^{-\alpha}}. \quad (2.12) \]

Thus from (1.1), we see that there exists an \( n_0 \) such that (2.9) holds true. This completes the proof of the theorem. \( \square \)

In the next two theorems we will study the asymptotic behavior of the positive solutions of (1.1). The next lemma is a slight modification of Theorem 1.16 of [2] and for readers convenience we state it without its proof.

**Lemma 2.1.** Let \( f : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+, \ g : \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) be a continuous function, \( \mathbb{R}^+ = (0, \infty) \) and \( a_1, b_1, a_2, b_2 \) be positive numbers such that \( a_1 < b_1, \ a_2 < b_2 \). Suppose that
\[ f : [a_1, b_1] \times [a_2, b_2] \rightarrow [a_1, b_1], \ g : [a_1, b_1] \times [a_2, b_2] \rightarrow [a_2, b_2]. \quad (2.13) \]

In addition, assume that \( f(x, y) \) (res. \( g(x, y) \)) is decreasing with respect to \( y \) (res. \( x \)) for every \( x \) (res. \( y \)) and increasing with respect to \( x \) (res. \( y \)) for every \( y \) (res. \( x \)). Finally suppose that if \( m, M, r, R \) are real numbers such that
\[ M = f(M, m), \ m = f(m, R), \ R = g(R, m), \ r = g(r, M), \quad (2.14) \]

then \( m = M \) and \( r = R \). Then the following system of difference equations
\[ x_{n+1} = f(x_{n-1}, y_n), \quad y_{n+1} = g(x_n, y_{n-1}) \quad (2.15) \]

has a unique positive equilibrium \((\bar{x}, \bar{y})\) and every positive solution \((x_n, y_n)\) of the system (2.15) which satisfies
\[ x_{n_0} \in [a_1, b_1], \ x_{n_0+1} \in [a_1, b_1], \ y_{n_0} \in [a_2, b_2], \ y_{n_0+1} \in [a_2, b_2], \ n_0 \in \mathbb{N} \quad (2.16) \]

converges to the unique positive equilibrium of (2.15).

**Theorem 2.3.** Consider system (1.1) such that the following relations hold: If \( \alpha(1 - b) \geq a(1 - \beta) \) then
\[ c < e^{\frac{-a(1 - 2\beta) + \sqrt{a^2(1 - 2\beta)^2 + 4(1 - b)^2}}{2}}, \quad \gamma < e^{\frac{\alpha(a(1 - b) - \sqrt{a^2(1 - b)^2 - a^2(1 - \beta)^2})}{2} - a(1 - 2b) + \sqrt{a^2(1 - 2b)^2 + 4(1 - \beta)^2}} \quad (2.17) \]

and if \( c(1 - b) \leq a(1 - \beta) \) then
\[ \gamma < e^{\frac{-a(1 - 2\beta) + \sqrt{a^2(1 - 2\beta)^2 + 4(1 - \beta)^2}}{2}}, \quad c < e^{\frac{a(a(1 - \beta) - \sqrt{a^2(1 - \beta)^2 - a^2(1 - b)^2})}{2} - a(1 - 2\beta) + \sqrt{a^2(1 - 2\beta)^2 + 4(1 - b)^2}} \quad (2.18) \]

Then system (1.1) has a unique positive equilibrium \((\bar{x}, \bar{y})\) and every positive solution of (1.1) tends to the unique positive equilibrium \((\bar{x}, \bar{y})\) as \( n \rightarrow \infty \).

**Proof.** We consider the functions
\[ f(x, y) = a + bx + cxe^{-y}, \ g(x, y) = \alpha + \beta y + \gamma ye^{-x} \quad (2.19) \]

where
\[ x \in I_1, \ y \in I_2 \quad (2.20) \]
$I_1$, $I_2$ are defined in (2.8). Then from (2.17), (2.18)-(2.20), we see that for $x \in I_1$, $y \in I_2$
\[ a \leq f(x, y) \leq a + b \frac{a + \epsilon}{1 - b - ce^{-\alpha}} + c \frac{a + \epsilon}{1 - b - ce^{-\alpha}} e^{-\alpha} \]
\[ = a + \epsilon \frac{b + ce^{-\alpha}}{1 - b - ce^{-\alpha}} < a + \epsilon \frac{a + \epsilon}{1 - b - ce^{-\alpha}}, \]
\[ \alpha \leq g(x, y) \leq \alpha + \beta \frac{\alpha + \epsilon}{1 - \beta - \gamma e^{-a}} + \frac{\gamma}{1 - \beta - \gamma e^{-a}} e^{-a} \]
\[ = \frac{\alpha + \epsilon (\beta + \gamma e^{-a})}{1 - \beta - \gamma e^{-a}} < \frac{\alpha + \epsilon}{1 - \beta - \gamma e^{-a}}, \]
and so $f : I_1 \times I_2 \rightarrow I_1$, $g : I_1 \times I_2 \rightarrow I_2$. Let $(x_n, y_n)$ be an arbitrary solution of (1.1). Therefore, as relations (2.17), (2.18) imply conditions (2.1), from Theorem 2.2 there exists an $n_0$ such that relations (2.9) hold.

Let $m$, $M$, $r$, $R$ be positive real numbers such that
\[ M = a + bM + cMe^{-r}, \quad m = a + bm + cm e^{-R}, \]
\[ R = \alpha + \beta R + \gamma Re^{-m}, \quad r = \alpha + \beta r + \gamma re^{-M}. \]

From (2.21) it follows that
\[ r = \ln \left[ \frac{cM}{M(1-b) - a} \right], \quad R = \ln \left[ \frac{cm}{m(1-b) - a} \right], \]
\[ m = \ln \left[ \frac{\gamma R}{R(1-\beta) - \alpha} \right], \quad M = \ln \left[ \frac{\gamma r}{r(1-\beta) - \alpha} \right]. \]

Thus we see that relations (2.21) and (2.22) imply
\[ (1 - b - ce^{-r}) \ln \left[ \frac{\gamma r}{r(1-\beta) - \alpha} \right] = a, \quad (1 - b - ce^{-R}) \ln \left[ \frac{\gamma R}{R(1-\beta) - \alpha} \right] = a, \]
\[ (1 - \beta - \gamma e^{-m}) \ln \left[ \frac{cm}{m(1-b) - a} \right] = \alpha, \quad (1 - \beta - \gamma e^{-M}) \ln \left[ \frac{cM}{M(1-b) - a} \right] = \alpha. \]

We then consider the function
\[ F(x) = (1 - \beta - \gamma e^{-x}) \ln \left[ \frac{cx}{x(1-b) - a} \right] - \alpha \]
Let $z$ be a solution of $F(x) = 0$. We claim that
\[ F'(z) < 0. \]

From (2.24) we see that
\[ F'(z) = -a \frac{(1 - \beta - \gamma e^{-z})}{z[z(1-b) - a]} + \gamma e^{-z} \ln \left[ \frac{cz}{z(1-b) - a} \right]. \]

Since $z$ satisfies equation $F(x) = 0$, then it follows that
\[ \ln \left[ \frac{cz}{z(1-b) - a} \right] = \frac{\alpha}{(1 - \beta - \gamma e^{-z})} \]
Therefore, relations (2.26) and (2.27) imply that
\[ F'(z) = -a \frac{(1 - \beta - \gamma e^{-z})}{z[z(1-b) - a]} + \frac{\alpha \gamma e^{-z}}{1 - \beta - \gamma e^{-z}}. \]
Using (2.28), to prove our claim (2.25), it suffices to prove that
\[ H(z) - G(z) < 0, \quad H(z) = \alpha \gamma z [z(1 - b) - a], \quad G(z) = a e^z (1 - \beta - \gamma e^{-z})^2. \tag{2.29} \]
From (2.29) we get
\[ H'(z) = \alpha [2z(1 - b) - a], \quad G'(z) = (1 - \beta)^2 a e^z - a\gamma^2 e^{-z}, \quad H''(z) = 2\alpha \gamma (1 - b), \]
\[ G''(z) = (1 - \beta)^2 a e^z + a\gamma^2 e^{-z}, \quad H'''(z) = 0, \quad G'''(z) = (1 - \beta)^2 a e^z - a\gamma^2 e^{-z}. \tag{2.30} \]
Moreover if \( \alpha(1 - b) \geq \alpha(1 - \beta) \), then from (2.17) it follows that
\[ 0 < \gamma < e^a \frac{\alpha(1 - b) - \sqrt{\alpha^2 (1 - b)^2 - a^2 (1 - \beta)^2}}{a} \]
and we get
\[ a\gamma^2 - 2\alpha \gamma (1 - b)e^a + (1 - \beta)^2 ae^{2a} > 0. \tag{2.34} \]
If \( \alpha(1 - b) \leq \alpha(1 - \beta) \) we can easily prove that (2.34) holds true. Then from (2.33) and (2.34) we get \( H''(a) - G''(a) < 0 \) and so from (2.32) it follows that
\[ H''(z) - G''(z) < 0. \tag{2.35} \]
Therefore from (2.35) and since \( z > a \) it follows
\[ H'(z) - G'(z) < H'(a) - G'(a). \tag{2.36} \]
Hence using (2.30) we get
\[ H'(a) - G'(a) = a\alpha \gamma (1 - 2b) - (1 - \beta)^2 a e^a + a\gamma^2 e^{-a} \]
\[ = ae^{-a} [\gamma^2 + \alpha(1 - 2b)e^a - (1 - \beta)^2 e^{2a}]. \tag{2.37} \]
Now observe that from (2.17), (2.18) we have
\[ 0 < \gamma < e^a \frac{-\alpha(1 - 2b) + \sqrt{\alpha^2 (1 - 2b)^2 + 4(1 - \beta)^2}}{2} \]
and so
\[ \gamma^2 + \alpha(1 - 2b)e^a - (1 - \beta)^2 e^{2a} < 0 \tag{2.38} \]
Therefore relations (2.37) and (2.38) imply that \( H'(a) - G'(a) < 0 \) and so from (2.36) it follows that
\[ H'(z) - G'(z) < 0. \tag{2.39} \]
Hence from (2.39) and as \( z > a \), we get

\[
H(z) - G(z) < H(a) - G(a) = -a^2 b x < 0. 
\quad (2.40)
\]

Thus from (2.40), we get \( H(z) - G(z) < 0 \) which implies that (2.25) is true. Since (2.25) holds, it is known that there exists an \( \epsilon \) such that for \( x \in (z - \epsilon, z + \epsilon) \)

\[
F'(x) < 0. 
\quad (2.41)
\]

Therefore from (2.41) the function \( F \) is decreasing in the interval \( (z - \epsilon, z + \epsilon) \). Suppose that \( F \) has roots greater than the root \( z \). Let \( z_1 \) be the smallest root of \( F \) such that \( z_1 > z \). From the argument above, we can show that there exists an \( \epsilon_1 \) such that \( F \) is decreasing in the interval \( (z - \epsilon_1, z + \epsilon_1) \). Since \( F(z + \epsilon) < 0 \), \( F(z_1 - \epsilon) > 0 \) and \( F \) is continuous, we see that \( F \) must have a root in the interval \( (z + \epsilon, z_1 - \epsilon) \). This is clearly a contradiction since \( z_1 \) is the smallest root of \( F \) such that \( z_1 > z \). Similarly we can prove that \( F \) has no solutions in \((0, z)\). Therefore equation \( F(x) = 0 \) must have a unique solution. Hence from (2.23) and (2.24) \( m, M \) are the solutions of the equation \( F(x) = 0 \). Thus we see that \( m = M \).

Similarly if we set

\[
G(x) = (1 - b - ce^{-x}) \ln \left[ \frac{\gamma x}{x(1 - \beta) - \alpha} \right] - a
\]

and using (2.17), (2.18) we can show that equation \( G(x) = 0 \) has a unique solution. Also as \( r, R \) are the solutions of equation \( G(x) = 0 \), it follows that \( r = R \). Therefore from Lemma 2.1 the proof of the theorem is complete.

**Theorem 2.4.** Consider system (1.1) and suppose that the constants \( a, b, c, \alpha, \beta, \gamma \) satisfy the following relations:

\[
\gamma < e^a \min \left\{ \frac{(1 - \beta)(1 - b - ce^{-\alpha})}{1 + a - b - ce^{-\alpha}}, \frac{(1 - b)(1 - \beta) - c(1 + \alpha - \beta)e^{-\alpha}}{1 - b - ce^{-\alpha}} \right\};
\quad (2.42)
\]

\[
c < e^a \frac{(1 - b)(1 - \beta)}{1 + \alpha - \beta}.
\]

Then system (1.1) has unique positive equilibrium \((\bar{x}, \bar{y})\) such that

\[
\bar{x} \in \left( a, \frac{a}{1 - b - ce^{-\alpha}} \right), \quad \bar{y} \in \left( \alpha, \frac{\alpha}{1 - \beta - \gamma e^{-\alpha}} \right).
\quad (2.43)
\]

Moreover every positive solution of (1.1) tends to the unique positive equilibrium \((\bar{x}, \bar{y})\) as \( n \to \infty \).

**Proof.** First we prove that (1.1) has a positive equilibrium such that relations (2.43) hold. First we consider the following system of algebraic equations

\[
x = a + b \bar{x} + c x e^{-\bar{y}}, \quad y = \alpha + \beta y + \gamma y e^{-x}
\quad (2.44)
\]

Observe that system (2.44) is equivalent to the following system

\[
x = \frac{a}{1 - b - ce^{-\bar{y}}}, \quad y = \frac{\alpha}{1 - \beta - \gamma e^{-x}}.
\quad (2.45)
\]

So we set

\[
F(x) = \frac{a}{1 - b - ce^{-f(x)}}, \quad f(x) = \frac{\alpha}{1 - \beta - \gamma e^{-x}}, \quad x \in \left( a, \frac{a}{1 - b - ce^{-\alpha}} \right).
\quad (2.46)
\]
Then from (2.46) we get
\[ F(a) = \frac{a[b + ce^{-f(a)}]}{1 - b - ce^{-f(a)}} > 0 \]
\[ F\left(\frac{a}{1 - b - ce^{-\alpha}}\right) = \frac{\alpha c e^{-f(a)} - e^{-\alpha}}{1 - b - ce^{-\alpha}} < 0 \] (2.47)

Therefore from (2.47) equation \( F(x) = 0 \) has a solution \( \bar{x} \in \left( a, \frac{a}{1 - b - ce^{-\alpha}} \right) \).

Now we will prove that \( \bar{x} \) is the unique solution of \( F(x) = 0 \). From (2.42) and (2.46) it follows that
\[ F'(x) = \frac{aco\gamma e^{-f(x)-a}}{1 - b - ce^{-f(x)}[1 - \beta - \gamma e^{-x}]^2 - 1} < \frac{aco\gamma e^{-a-a}}{1 - b - ce^{-a}[1 - \beta - \gamma e^{-a}]^2 - 1} \] (2.48)

Moreover from (2.42) we get,
\[ \frac{a\gamma e^{-a}}{(1 - b - ce^{-\alpha})(1 - \beta - \gamma e^{-\alpha})} < 1, \quad \frac{ca\gamma e^{-a}}{(1 - b - ce^{-\alpha})(1 - \beta - \gamma e^{-\alpha})} < 1. \] (2.49)

Hence relations (2.48) and (2.49) imply that \( F'(x) < 0 \) which implies that equation \( F(x) = 0 \) has a unique solution \( \bar{x} \in \left( a, \frac{a}{1 - b - ce^{-\alpha}} \right) \). Then from (2.45) and (2.46) system (2.44) has a unique solution \( (\bar{x}, \bar{y}) \) such that (2.43) holds.

Let \( (x_n, y_n) \) be an arbitrary solution of (1.1). Using relations (2.42) and Theorem 2.1, we see that (2.11) hold which also imply that
\[ L_1 \leq \frac{a}{1 - b - ce^{-l_2}}, \quad l_1 \geq \frac{a}{1 - b - ce^{-L_2}}, \]
\[ L_2 \leq \frac{a}{1 - \beta - \gamma e^{-l_1}}, \quad l_2 \geq \frac{a}{1 - \beta - \gamma e^{-L_1}}. \] (2.50)

From (2.50) we get
\[ L_1 l_2 \leq \frac{aL_2}{1 - b - ce^{-l_2}}, \quad l_1 L_2 \geq \frac{aL_2}{1 - b - ce^{-L_2}}, \]
\[ L_2 l_1 \leq \frac{aL_1}{1 - \beta - \gamma e^{-l_1}}, \quad l_2 L_1 \geq \frac{aL_1}{1 - \beta - \gamma e^{-L_1}}, \]

and so we see that
\[ \frac{aL_1}{1 - \beta - \gamma e^{-L_1}} \leq \frac{al_2}{1 - b - ce^{-l_2}}, \quad \frac{aL_2}{1 - b - ce^{-L_2}} \leq \frac{al_1}{1 - \beta - \gamma e^{-l_1}}. \] (2.51)

Now we consider the functions
\[ f(x) = \frac{ax}{1 - \beta - \gamma e^{-x}}, \quad g(y) = \frac{ay}{1 - b - ce^{-y}}, \]
\[ x \in \left( a, \frac{a}{1 - b - ce^{-\alpha}} \right), \quad y \in \left( a, \frac{a}{1 - \beta - \gamma e^{-\alpha}} \right). \] (2.52)
Then from (2.52) it follows that
\[ f'(x) = \frac{\alpha [1 - \beta - \gamma(1 + x)e^{-x}]}{(1 - \beta - \gamma e^{-x})^2}, \quad g'(y) = \frac{\alpha [1 - b - c(1 + y)e^{-y}]}{(1 - b - ce^{-y})^2} \] (2.53)

From (2.42) and (2.49), consider \(x \in \left( a, \frac{\alpha}{1 - b - ce^{-a}} \right), \quad y \in \left( \alpha, \frac{\alpha}{1 - \beta - \gamma e^{-a}} \right)\)
we have:
\[
1 - \beta - \gamma(1 + x)e^{-x} > 1 - \beta - \gamma(1 + \frac{\alpha}{1 - b - ce^{-a}})e^{-a} > 0 \]
\[
1 - b - c(1 + y)e^{-y} > 1 - b - c(1 + \frac{\alpha}{1 - \beta - \gamma e^{-a}})e^{-a} > 0 \] (2.54)

Therefore from (2.53) and (2.54) we see that
\[ f'(x) > 0, \quad g'(y) > 0, \quad x \in \left( a, \frac{\alpha}{1 - b - ce^{-a}} \right), \quad y \in \left( \alpha, \frac{\alpha}{1 - \beta - \gamma e^{-a}} \right). \]

Hence, \(f, g\) are increasing function and this, together with (2.51) implies that \(l_1 = L_1\). Then, from (2.51) again, we see that \(l_2 = L_2\). Therefore, this completes the proof of this theorem. \(\square\)

In the last theorem of this section, we will study the global asymptotic stability of the positive equilibrium of (1.1).

**Theorem 2.5.** Consider system (1.1) such that either (2.17) and (2.18) hold or (2.42) holds. Also suppose that the following relation holds true:
\[
b + ce^{-a} + \beta + \gamma e^{-a} + (b + ce^{-a})(\beta + \gamma e^{-a}) + \frac{\alpha c \alpha \gamma e^{-a - a}}{(1 - b - ce^{-a})(1 - \beta - \gamma e^{-a})} < 1. \] (2.55)

Then the unique positive equilibrium \((\bar{x}, \bar{y})\) of (1.1) is globally asymptotically stable.

**Proof.** First we will prove that \((\bar{x}, \bar{y})\) is locally asymptotically stable. The linearized system of (1.1) about \((\bar{x}, \bar{y})\) is
\[
x_{n+1} = (b + ce^{-\bar{y}})x_{n-1} - cxe^{-\bar{y}}y_n \\
y_{n+1} = (\beta + \gamma e^{-\bar{x}})y_{n-1} - \gamma xe^{-\bar{x}}x_n \] (2.56)

We clearly see that (2.56) is equivalent to the system
\[
w_{n+1} = Aw_n, \quad A = \begin{pmatrix}
0 & b_1 & a_1 & 0 \\
d_1 & 0 & 0 & c_1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}, \quad w_n = \begin{pmatrix}
x_n \\
y_n \\
x_{n-1} \\
y_{n-1}
\end{pmatrix}, \quad a_1 = b + ce^{-\bar{y}}, \quad b_1 = -cx e^{-\bar{y}}, \quad c_1 = \beta + \gamma e^{-\bar{x}}, \quad d_1 = -\gamma xe^{-\bar{x}}.
\]

Then the characteristic equation of \(A\) is
\[
\lambda^4 - (a_1 + c_1 + b_1 d_1)\lambda + a_1 c_1 = 0 \] (2.57)
Since $\bar{x}$, $\bar{y}$ satisfy (2.44) it is obvious that $\bar{x} > a$, $\bar{y} > \alpha$. Hence, from (2.55) and as $\bar{x}$, $\bar{y}$ satisfy (2.45) we get

$$|a_1| + |c_1| + |b_1d_1| + |a_1c_1| = b + ce^{-\bar{y}} + \beta + \gamma e^{-\bar{x}} + c\bar{e}e^{-\bar{y}}\bar{y}e^{-\bar{x}} + (b + ce^{-\bar{y}})(\beta + \gamma e^{-\bar{x}}) = b + ce^{-\bar{y}} + \beta + \gamma e^{-\bar{x}} + (b + ce^{-\bar{y}})(\beta + \gamma e^{-\bar{x}}) + \frac{ac\alpha\gamma e^{-\bar{x}} - \bar{y}}{(1 - b - ce^{-\bar{y}})(1 - \beta - \gamma e^{-\bar{x}})}$$

$$< b + ce^{-\alpha} + \beta + \gamma e^{-a} + (b + ce^{-\alpha})(\beta + \gamma e^{-a}) + \frac{ac\alpha\gamma e^{-\alpha}}{(1 - b - ce^{-\bar{y}})(1 - \beta - \gamma e^{-\bar{x}})} < 1.$$ (2.58)

Therefore, from (2.58) and Remark 1.3.1 of [9] all the roots (2.57) have the modulus are less than 1 which implies that ($\bar{x}$, $\bar{y}$) is locally asymptotically stable. Using Theorem 2.3 ($\bar{x}$, $\bar{y}$) is globally asymptotically stable. This completes the proof of the theorem.

ACKNOWLEDGEMENT

The author would like to thank the referees for their helpful suggestions for improvement of the manuscript.

REFERENCES

[8] S. Stević, On the recursive sequence $x_{n+1} = \frac{\alpha + bx_n - 1}{1 + g(x_n)}$, Indian J. Pure Appl. Math. 33(12)(2002), 1767-1774.

Mai Nam Phong
Department of Mathematical Analysis, University of Transport and Communications, Hanol, Vietnam
E-mail address: mnphong@gmail.com