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A NOTE ON A SYSTEM OF TWO NONLINEAR DIFFERENCE EQUATIONS

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Abstract. The goal of this paper is to study the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form:

$$x_{n+1} = a + bx_{n-1} + cx_{n-1}e^{-y_n},\ y_{n+1} = \alpha + \beta y_{n-1} + \gamma y_{n-1}e^{-x_n},$$

where $a,\ b,\ c,\ \alpha,\ \beta,\ \gamma \in (0, \infty)$, and the initial values $x_{-1},\ x_0,\ y_{-1},\ y_0$ are positive real values.

1. Introduction

In [3], the authors studied the boundedness, the asymptotic behavior, the periodic character of the solutions and the stability character of the positive equilibrium of the difference equation:

$$x_{n+1} = a + bx_{n-1}e^{-x_n},$$

where $a,\ b$ are positive constants and the initial values $x_{-1},\ x_0$ are positive numbers. Furthermore, in [3] the authors used a as the immigration rate and b as the growth rate in the population model. In fact, this was a model suggested by the people from the Harvard School of Public Health; studying the population dynamics of one species x_n.

In [4], the authors investigated the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system of two difference equations of exponential form:

$$x_{n+1} = a + bx_{n-1}e^{-y_n},$$
$$y_{n+1} = c + dy_{n-1}e^{-x_n}.$$

Motivated by these above papers we will extend the difference equation in [3] and the system of difference equations in [4] to a system of difference equations of exponential form:

$$\begin{aligned} x_{n+1} &= a + bx_{n-1} + cx_{n-1}e^{-y_n}, \\ y_{n+1} &= \alpha + \beta y_{n-1} + \gamma y_{n-1}e^{-x_n}, \end{aligned} \tag{1.1}$$

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where $a,\ b,\ c,\ \alpha,\ \beta,\ \gamma$ are positive constants and the initial values $x_{-1},\ x_0,\ y_{-1},\ y_0$ are positive real values. Our goal in this paper is to investigate the boundedness, the persistence and the asymptotic behavior of the positive solutions of the system (1.1).

It is easy to see that in the special cases, when $b = \beta = 0$ the system (1.1) becomes the system in the paper [4] and when $x_n = y_n,\ a = \alpha,\ b = \beta = 0,\ c = \gamma$ we have the difference equation in [3].

Difference equations and systems of difference equations of exponential form can be found in the following papers: [1, 3-5, 7]. Moreover, as difference equations have many applications in applied sciences, there are many papers and books that can be found concerning the theory and applications of difference equations, see [2, 6, 9] and the references cited therein.

2. Asymptotic behavior of the solutions of system (1.1)

First it is very important to establish the boundedness and persistence of solutions; in the first theorem we will study the boundedness and persistence of the positive solutions of system (1.1) by comparing them with solutions of a solvable system of difference equations. Our method is a modification in Theorem 2 in [8]. For related and similar results see, [4, 6, 9].

Theorem 2.1. *Consider system* (1.1) *such that:*

$$b + ce^{-\alpha} < 1,\ \beta + \gamma e^{-a} < 1. \tag{2.1}$$

Then every positive solution of (1.1) *is bounded and persists.*

Proof. Let (x_n, y_n) be an arbitrary solution of (1.1). Thus from (1.1) we see that

$$x_n \geq a,\ y_n \geq \alpha,\ n = 1, 2, \ldots. \tag{2.2}$$

In addition, it follows from (1.1) and (2.2) that

$$x_{n+1} \leq a + bx_{n-1} + cx_{n-1}e^{-\alpha},\ y_{n+1} \leq \alpha + \beta y_{n-1} + \gamma y_{n-1}e^{-a},\ n = 0, 1, 2, \ldots. \tag{2.3}$$

We will now consider the non-homogeneous difference equations

$$z_{n+1} = a + bz_{n-1} + cz_{n-1}e^{-\alpha},\ v_{n+1} = \alpha + \beta v_{n-1} + \gamma v_{n-1}e^{-a},\ n = 0, 1, 2, \ldots. \tag{2.4}$$

Therefore, from (2.4) an arbitrary solution (z_n, v_n) of (2.4) is given by

$$\begin{aligned} z_n &= r_1(b + ce^{-\alpha})^{n/2} + r_2(-1)^n(b + ce^{-\alpha})^{n/2} + \frac{a}{1 - b - ce^{-\alpha}},\ n = 1, 2, \ldots, \\ v_n &= s_1(\beta + \gamma e^{-a})^{n/2} + s_2(-1)^n(\beta + \gamma e^{-a})^{n/2} + \frac{\alpha}{1 - \beta - \gamma e^{-a}},\ n = 1, 2, \ldots, \end{aligned} \tag{2.5}$$

where $r_1,\ r_2,\ s_1,\ s_2$ depend on the initial values $z_{-1},\ z_0,\ v_{-1},\ v_0$. Thus we see that relations (2.1) and (2.5) imply that z_n and v_n are bounded sequences. Now we will consider the solution (z_n, v_n) of (2.4) such that

$$z_{-1} = x_{-1},\ z_0 = x_0,\ v_{-1} = y_{-1},\ v_0 = y_0. \tag{2.6}$$

Thus from (2.3) and (2.6) we get

$$x_n \leq z_n,\ y_n \leq v_n,\ n = 1, 2, \ldots. \tag{2.7}$$

Therefore, it follows that x_n and y_n are bounded sequences. Hence from (2.2) the proof of this theorem is now complete. □

In the next theorem we will study the existence of invariant intervals of system (1.1).

Theorem 2.2. *Consider system* (1.1) *where relations* (2.1) *hold. Then the following statements are true:*
(i) The set

$$\left[a,\ \frac{a}{1-b-ce^{-\alpha}}\right] \times \left[\alpha,\ \frac{\alpha}{1-\beta-\gamma e^{-a}}\right]$$

is an invariant set for (1.1).
(ii) Let ϵ *be an arbitrary positive number and* (x_n, y_n) *be an arbitrary solution of* (1.1). *We then consider the sets*

$$I_1 = \left[a,\ \frac{a+\epsilon}{1-b-ce^{-\alpha}}\right],\ I_2 = \left[\alpha,\ \frac{\alpha+\epsilon}{1-\beta-\gamma e^{-a}}\right]. \tag{2.8}$$

Then there exists an n_0 *such that for all* $n \geq n_0$

$$x_n \in I_1, \qquad y_n \in I_2. \tag{2.9}$$

Proof. (i) Let (x_n, y_n) be a solution of (1.1) with initial values $x_{-1},\ x_0,\ y_{-1},\ y_0$ such that

$$x_{-1},\ x_0 \in \left[a,\ \frac{a}{1-b-ce^{-\alpha}}\right]; \quad y_{-1},\ y_0 \in \left[\alpha,\ \frac{\alpha}{1-\beta-\gamma e^{-a}}\right]. \tag{2.10}$$

Then from (1.1) and (2.10) we get

$$\begin{aligned} a \leq x_1 = a + bx_{-1} + cx_{-1}e^{-y_0} &\leq a\left[1 + \frac{1}{1-b-ce^{-\alpha}}(b+ce^{-\alpha})\right] \\ &= \frac{a}{1-b-ce^{-\alpha}}, \\ \alpha \leq y_1 = \alpha + \beta y_{-1} + \gamma y_{-1}e^{-x_0} &\leq \alpha\left[1 + \frac{1}{1-\beta-\gamma e^{-a}}(\beta+\gamma e^{-a})\right] \\ &= \frac{\alpha}{1-\beta-\gamma e^{-a}}. \end{aligned}$$

Then it follows by induction that

$$a \leq x_n \leq \frac{a}{1-b-ce^{-\alpha}}, \quad \alpha \leq y_n \leq \frac{\alpha}{1-\beta-\gamma e^{-a}},\ n = 1, 2, \ldots.$$

This completes the proof of statement (i).
(ii) Let $(x_n,\ y_n)$ be an arbitrary solution of (1.1). Therefore, from Theorem 2.1 we get

$$\begin{aligned} &0 < l_1 = \lim_{n\to\infty} \inf x_n, \quad 0 < l_2 = \lim_{n\to\infty} \inf y_n \\ &0 < L_1 = \lim_{n\to\infty} \sup x_n < \infty, \quad 0 < L_2 = \lim_{n\to\infty} \sup y_n < \infty. \end{aligned} \tag{2.11}$$

It follows from (1.1) and (2.11) that

$$\begin{aligned} &L_1 \leq a + bL_1 + cL_1e^{-l_2}, \quad l_1 \geq a + bl_1 + cl_1e^{-L_2}, \\ &L_2 \leq \alpha + \beta L_2 + \gamma L_2 e^{-l_1}, \quad l_2 \geq \alpha + \beta l_2 + l_2\gamma e^{-L_1}. \end{aligned}$$

which imply that

$$a \leq L_1 \leq \frac{a}{1-b-ce^{-\alpha}}, \quad \alpha \leq L_2 \leq \frac{\alpha}{1-\beta-\gamma e^{-a}}. \tag{2.12}$$

Thus from (1.1), we see that there exists an n_0 such that (2.9) holds true. This completes the proof of the theorem. □

In the next two theorems we will study the asymptotic behavior of the positive solutions of (1.1). The next lemma is a slight modification of Theorem 1.16 of [2] and for readers convenience we state it without its proof.

Lemma 2.1. *Let $f : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $g : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a continuous function, $\mathbb{R}^+ = (0,\ \infty)$ and $a_1,\ b_1,\ a_2,\ b_2$ be positive numbers such that $a_1 < b_1,\ a_2 < b_2$. Suppose that*

$$f : [a_1,\ b_1] \times [a_2,\ b_2] \longrightarrow [a_1,\ b_1],\ g : [a_1,\ b_1] \times [a_2,\ b_2] \longrightarrow [a_2,\ b_2]. \tag{2.13}$$

In addition, assume that $f(x,\ y)$ (res. $g(x,\ y)$) is decreasing with respect to y (res. x) for every x (res. y) and increasing with respect to x (res. y) for every y (res. x). Finally suppose that if $m,\ M,\ r,\ R$ are real numbers such that

$$M = f(M,\ m), \quad m = f(m,\ R), \quad R = g(R,\ m), \quad r = g(r,\ M), \tag{2.14}$$

then $m = M$ and $r = R$. Then the following system of difference equations

$$x_{n+1} = f(x_{n-1}, y_n),\ y_{n+1} = g(x_n, y_{n-1}) \tag{2.15}$$

has a unique positive equilibrium $(\bar{x},\ \bar{y})$ and every positive solution $(x_n,\ y_n)$ of the system (2.15) which satisfies

$$x_{n_0} \in [a_1,\ b_1],\ x_{n_0+1} \in [a_1,\ b_1],\ y_{n_0} \in [a_2,\ b_2],\ y_{n_0+1} \in [a_2,\ b_2],\ n_0 \in \mathbb{N} \tag{2.16}$$

converges to the unique positive equilibrium of (2.15).

Theorem 2.3. *Consider system (1.1) such that the following relations hold:*
If $\alpha(1-b) \geq a(1-\beta)$ then

$$c < e^{\alpha}\frac{-a(1-2\beta)+\sqrt{a^2(1-2\beta)^2+4(1-b)^2}}{2},$$
$$\gamma < e^{a}\min\left\{\frac{\alpha(1-b)-\sqrt{\alpha^2(1-b)^2-a^2(1-\beta)^2}}{a},\ \frac{-\alpha(1-2b)+\sqrt{\alpha^2(1-2b)^2+4(1-\beta)^2}}{2}\right\}. \tag{2.17}$$

and if $c(1-b) \leq a(1-\beta)$ then

$$\gamma < e^{a}\frac{-\alpha(1-2b)+\sqrt{\alpha^2(1-2b)^2+4(1-\beta)^2}}{2},$$
$$c < e^{\alpha}\min\left\{\frac{a(1-\beta)-\sqrt{a^2(1-\beta)^2-\alpha^2(1-b)^2}}{\alpha},\ \frac{-a(1-2\beta)+\sqrt{a^2(1-2\beta)^2+4(1-b)^2}}{2}\right\}. \tag{2.18}$$

Then system (1.1) has a unique positive equilibrium $(\bar{x},\ \bar{y})$ and every positive solution of (1.1) tends to the unique positive equilibrium $(\bar{x},\ \bar{y})$ as $n \to \infty$.

Proof. We consider the functions

$$f(x,\ y) = a + bx + cxe^{-y}, \quad g(x,\ y) = \alpha + \beta y + \gamma y e^{-x} \tag{2.19}$$

where

$$x \in I_1, \quad y \in I_2, \tag{2.20}$$

$I_1,\ I_2$ are defined in (2.8). Then from (2.17), (2.18)-(2.20), we see that for $x \in I_1,\ y \in I_2$

$$\begin{aligned} a \leq f(x,\ y) &\leq a + b\frac{a+\epsilon}{1-b-ce^{-\alpha}} + c\frac{a+\epsilon}{1-b-ce^{-\alpha}}e^{-\alpha} \\ &= \frac{a+\epsilon(b+ce^{-\alpha})}{1-b-ce^{-\alpha}} < \frac{a+\epsilon}{1-b-ce^{-\alpha}}, \\ \alpha \leq g(x,\ y) &\leq \alpha + \beta\frac{\alpha+\epsilon}{1-\beta-\gamma e^{-a}} + \gamma\frac{\alpha+\epsilon}{1-\beta-\gamma e^{-a}}e^{-a} \\ &= \frac{\alpha+\epsilon(\beta+\gamma e^{-a})}{1-\beta-\gamma e^{-a}} < \frac{\alpha+\epsilon}{1-\beta-\gamma e^{-a}}, \end{aligned}$$

and so $f : I_1 \times I_2 \longrightarrow I_1,\ g : I_1 \times I_2 \longrightarrow I_2$. Let $(x_n,\ y_n)$ be an arbitrary solution of (1.1). Therefore, as relations (2.17), (2.18) imply conditions (2.1), from Theorem2.2 there exists an n_0 such that relations (2.9) hold.

Let $m,\ M,\ r,\ R$ be positive real numbers such that

$$\begin{aligned} M = a + bM + cMe^{-r},\ m = a + bm + cme^{-R}, \\ R = \alpha + \beta R + \gamma Re^{-m},\ r = \alpha + \beta r + \gamma re^{-M}. \end{aligned} \tag{2.21}$$

From (2.21) it follows that

$$\begin{aligned} r = \ln\left[\frac{cM}{M(1-b)-a}\right],\ R = \ln\left[\frac{cm}{m(1-b)-a}\right], \\ m = \ln\left[\frac{\gamma R}{R(1-\beta)-\alpha}\right],\ M = \ln\left[\frac{\gamma r}{r(1-\beta)-\alpha}\right]. \end{aligned} \tag{2.22}$$

Thus we see that relations (2.21) and (2.22) imply

$$\begin{aligned} (1-b-ce^{-r})\ln\left[\frac{\gamma r}{r(1-\beta)-\alpha}\right] = a,\ (1-b-ce^{-R})\ln\left[\frac{\gamma R}{R(1-\beta)-\alpha}\right] = a, \\ (1-\beta-\gamma e^{-m})\ln\left[\frac{cm}{m(1-b)-a}\right] = \alpha,\ (1-\beta-\gamma e^{-M})\ln\left[\frac{cM}{M(1-b)-a}\right] = \alpha \end{aligned} \tag{2.23}$$

We then consider the function

$$F(x) = (1-\beta-\gamma e^{-x})\ln\left[\frac{cx}{x(1-b)-a}\right] - \alpha \tag{2.24}$$

Let z be a solution of $F(x) = 0$. We claim that

$$F'(z) < 0. \tag{2.25}$$

From (2.24) we see that

$$F'(z) = -\frac{a(1-\beta-\gamma e^{-z})}{z[z(1-b)-a]} + \gamma e^{-z}\ln\left[\frac{cz}{z(1-b)-a}\right]. \tag{2.26}$$

Since z satisfies equation $F(x) = 0$, then it follows that

$$\ln\left[\frac{cz}{z(1-b)-a}\right] = \frac{\alpha}{(1-\beta-\gamma e^{-z})} \tag{2.27}$$

Therefore, relations (2.26) and (2.27) imply that

$$F'(z) = -\frac{a(1-\beta-\gamma e^{-z})}{z[z(1-b)-a]} + \frac{\alpha\gamma e^{-z}}{1-\beta-\gamma e^{-z}}. \tag{2.28}$$

Using (2.28), to prove our claim (2.25), it suffices to prove that

$$H(z)-G(z)<0,\quad H(z)=\alpha\gamma z[z(1-b)-a],\quad G(z)=ae^{z}(1-\beta-\gamma e^{-z})^{2}. \tag{2.29}$$

From (2.29) we get

$$\begin{gathered}H'(z)=\alpha\gamma[2z(1-b)-a],\ G'(z)=(1-\beta)^{2}ae^{z}-a\gamma^{2}e^{-z},\ H''(z)=2\alpha\gamma(1-b),\\ G''(z)=(1-\beta)^{2}ae^{z}+a\gamma^{2}e^{-z},\ H'''(z)=0,\ G'''(z)=(1-\beta)^{2}ae^{z}-a\gamma^{2}e^{-z}.\end{gathered} \tag{2.30}$$

Now from (2.17), (2.18) and (2.30), we see that as $z>a$ we have

$$\begin{aligned}H'''(z)-G'''(z)=&-(1-\beta)^{2}ae^{z}+a\gamma^{2}e^{-z}=a\frac{[\gamma^{2}-(1-\beta)^{2}e^{2z}]}{e^{z}}\\ =&a\frac{[\gamma-(1-\beta)e^{z}].[\gamma+(1-\beta)e^{z}]}{e^{z}}\\ <&a\frac{[\gamma-(1-\beta)e^{a}].[\gamma+(1-\beta)e^{z}]}{e^{z}}<0.\end{aligned} \tag{2.31}$$

Since $z>a$, we take

$$H''(z)-G''(z)<H''(a)-G''(a). \tag{2.32}$$

Using (2.30) we get

$$\begin{aligned}H''(a)-G''(a)&=2\alpha\gamma(1-b)-(1-\beta)^{2}ae^{a}-a\gamma^{2}e^{-a}\\ &=-e^{-a}[a\gamma^{2}-2\alpha\gamma(1-b)e^{a}+(1-\beta)^{2}ae^{2a}]\end{aligned} \tag{2.33}$$

Moreover if $\alpha(1-b)\geq a(1-\beta)$, then from (2.17) it follows that

$$0<\gamma<e^{a}\frac{\alpha(1-b)-\sqrt{\alpha^{2}(1-b)^{2}-a^{2}(1-\beta)^{2}}}{a}$$

and we get

$$a\gamma^{2}-2\alpha\gamma(1-b)e^{a}+(1-\beta)^{2}ae^{2a}>0. \tag{2.34}$$

If $\alpha(1-b)\leq a(1-\beta)$ we can easily prove that (2.34) holds true. Then from (2.33) and (2.34) we get $H''(a)-G''(a)<0$ and so from (2.32) it follows that

$$H''(z)-G''(z)<0. \tag{2.35}$$

Therefore from (2.35) and since $z>a$ it follows

$$H'(z)-G'(z)<H'(a)-G'(a). \tag{2.36}$$

Hence using (2.30) we get

$$\begin{aligned}H'(a)-G'(a)&=a\alpha\gamma(1-2b)-(1-\beta)^{2}ae^{a}+a\gamma^{2}e^{-a}\\ &=ae^{-a}[\gamma^{2}+\alpha(1-2b)e^{a}\gamma-(1-\beta)^{2}e^{2a}].\end{aligned} \tag{2.37}$$

Now observe that from (2.17), (2.18) we have

$$0<\gamma<e^{a}\frac{-\alpha(1-2b)+\sqrt{\alpha^{2}(1-2b)^{2}+4(1-\beta)^{2}}}{2}$$

and so

$$\gamma^{2}+\alpha(1-2b)e^{a}\gamma-(1-\beta)^{2}e^{2a}<0 \tag{2.38}$$

Therefore relations (2.37) and (2.38) imply that $H'(a)-G'(a)<0$ and so from (2.36) it follows that

$$H'(z)-G'(z)<0. \tag{2.39}$$

Hence from (2.39) and as $z > a$, we get

$$H(z) - G(z) < H(a) - G(a) = -a^2 b\alpha\gamma - ae^a(1-\beta-\gamma e^{-a})^2 < 0. \tag{2.40}$$

Thus from (2.40), we get $H(z) - G(z) < 0$ which implies that (2.25) is true. Since (2.25) holds, it is known that there exists an ϵ such that for $x \in (z-\epsilon,\ z+\epsilon)$

$$F'(x) < 0. \tag{2.41}$$

Therefore from (2.41) the function F is decreasing in the interval $(z-\epsilon,\ z+\epsilon)$. Suppose that F has roots greater than the root z. Let z_1 be the smallest root of F such that $z_1 > z$. From the argument above, we can show that there exists an ϵ_1 such that F is decreasing in the interval $(z-\epsilon_1,\ z+\epsilon_1)$. Since $F(z+\epsilon) < 0,\ F(z_1-\epsilon) > 0$ and F is continuous, we see that F must have a root in the interval $(z+\epsilon,\ z_1-\epsilon)$. This is clearly a contradiction since z_1 is the smallest root of F such that $z_1 > z$. Similarly we can prove that F has no solutions in $(0,\ z)$. Therefore equation $F(x) = 0$ must have a unique solution. Hence from (2.23) and (2.24) $m,\ M$ are the solutions of the equation $F(x) = 0$. Thus we see that $m = M$. Similarly if we set

$$G(x) = (1-b-ce^{-x})\ln\left[\frac{\gamma x}{x(1-\beta)-\alpha}\right] - a$$

and using (2.17), (2.18) we can show that equation $G(x) = 0$ has a unique solution. Also as $r,\ R$ are the solutions of equation $G(x) = 0$, it follows that $r = R$. Therefore from Lemma 2.1 the proof of the theorem is complete. □

Theorem 2.4. *Consider system* (1.1) *and suppose that the constants* $a,\ b,\ c,\ \alpha,\ \beta,\ \gamma$ *satisfy the following relations:*

$$\begin{aligned} &\gamma < e^a \min\left\{\frac{(1-\beta)(1-b-ce^{-\alpha})}{1+a-b-ce^{-\alpha}},\ \frac{(1-b)(1-\beta)-c(1+\alpha-\beta)e^{-\alpha}}{1-b-ce^{-\alpha}}\right\};\\ &c < e^{\alpha}\frac{(1-b)(1-\beta)}{1+\alpha-\beta}. \end{aligned} \tag{2.42}$$

Then system (1.1) *has unique positive equilibrium* $(\bar{x},\ \bar{y})$ *such that*

$$\bar{x} \in \left(a,\ \frac{a}{1-b-ce^{-\alpha}}\right),\ \bar{y} \in \left(\alpha,\ \frac{\alpha}{1-\beta-\gamma e^{-a}}\right). \tag{2.43}$$

Moreover every positive solution of (1.1) *tends to the unique positive equilibrium* $(\bar{x},\ \bar{y})$ *as* $n \to \infty$.

Proof. First we prove that (1.1) has a positive equilibrium such that relations (2.43) hold. First we consider the following system of algebraic equations

$$x = a + bx + cxe^{-y},\quad y = \alpha + \beta y + \gamma ye^{-x} \tag{2.44}$$

Observe that system (2.44) is equivalent to the following system

$$x = \frac{a}{1-b-ce^{-y}},\quad y = \frac{\alpha}{1-\beta-\gamma e^{-x}}. \tag{2.45}$$

So we set

$$F(x) = \frac{a}{1-b-ce^{-f(x)}},\quad f(x) = \frac{\alpha}{1-\beta-\gamma e^{-x}},\quad x \in \left(a,\ \frac{a}{1-b-ce^{-\alpha}}\right). \tag{2.46}$$

Then from (2.46) we get

$$F(a)=\frac{a[b+ce^{-f(a)}]}{1-b-ce^{-f(a)}}>0$$
$$F\left(\frac{a}{1-b-ce^{-\alpha}}\right)=\frac{ac\left[e^{-f\left(\frac{a}{1-b-ce^{-\alpha}}\right)}-e^{-\alpha}\right]}{\left[1-b-ce^{-f\left(\frac{a}{1-b-ce^{-\alpha}}\right)}\right](1-b-ce^{-\alpha})}<0 \tag{2.47}$$

Therefore from (2.47) equation $F(x)=0$ has a solution $\bar{x}\in\left(a,\ \frac{a}{1-b-ce^{-\alpha}}\right)$.

Now we will prove that $\bar{x}$ is the unique solution of $F(x)=0$. From (2.42) and (2.46) it follows that

$$F'(x)=\frac{ac\alpha\gamma e^{-f(x)-x}}{[1-b-ce^{-f(x)}]^2(1-\beta-\gamma e^{-x})^2}-1<\frac{ac\alpha\gamma e^{-a-\alpha}}{(1-b-ce^{-\alpha})^2(1-\beta-\gamma e^{-a})^2}-1 \tag{2.48}$$

Moreover from (2.42) we get,

$$\frac{a\gamma e^{-a}}{(1-b-ce^{-\alpha})(1-\beta-\gamma e^{-a})}<1,\ \frac{c\alpha e^{-\alpha}}{(1-b-ce^{-\alpha})(1-\beta-\gamma e^{-a})}<1. \tag{2.49}$$

Hence relations (2.48) and (2.49) imply that $F'(x)<0$ which implies that equation $F(x)=0$ has a unique solution $\bar{x}\in\left(a,\ \frac{a}{1-b-ce^{-\alpha}}\right)$. Then from (2.45) and (2.46) system (2.44) has a unique solution $(\bar{x},\ \bar{y})$ such that (2.43) holds.

Let $(x_n,\ y_n)$ be an arbitrary solution of (1.1). Using relations (2.42) and Theorem 2.1, we see that (2.11) hold which also imply that

$$\begin{aligned}L_1\le\frac{a}{1-b-ce^{-l_2}},\ l_1\ge\frac{a}{1-b-ce^{-L_2}},\\ L_2\le\frac{\alpha}{1-\beta-\gamma e^{-l_1}},\ l_2\ge\frac{\alpha}{1-\beta-\gamma e^{-L_1}}.\end{aligned} \tag{2.50}$$

From (2.50) we get

$$\begin{aligned}L_1l_2\le\frac{al_2}{1-b-ce^{-l_2}},\ l_1L_2\ge\frac{aL_2}{1-b-ce^{-L_2}},\\ L_2l_1\le\frac{\alpha l_1}{1-\beta-\gamma e^{-l_1}},\ l_2L_1\ge\frac{\alpha L_1}{1-\beta-\gamma e^{-L_1}}.\end{aligned}$$

and so we see that

$$\frac{\alpha L_1}{1-\beta-\gamma e^{-L_1}}\le\frac{al_2}{1-b-ce^{-l_2}},\ \frac{aL_2}{1-b-ce^{-L_2}}\le\frac{\alpha l_1}{1-\beta-\gamma e^{-l_1}}. \tag{2.51}$$

Now we consider the functions

$$\begin{aligned}f(x)=\frac{\alpha x}{1-\beta-\gamma e^{-x}},\ g(y)=\frac{ay}{1-b-ce^{-y}},\\ x\in\left(a,\ \frac{a}{1-b-ce^{-\alpha}}\right),\ y\in\left(\alpha,\ \frac{\alpha}{1-\beta-\gamma e^{-a}}\right).\end{aligned} \tag{2.52}$$

Then from (2.52) it follows that

$$f'(x) = \frac{\alpha[1-\beta-\gamma(1+x)e^{-x}]}{(1-\beta-\gamma e^{-x})^2},\ g'(y) = \frac{a[1-b-c(1+y)e^{-y}]}{(1-b-ce^{-y})^2} \tag{2.53}$$

From (2.42) and (2.49), consider $x \in \left(a,\ \dfrac{a}{1-b-ce^{-\alpha}}\right),\ y \in \left(\alpha,\ \dfrac{\alpha}{1-\beta-\gamma e^{-a}}\right)$ we have:

$$\begin{aligned} 1-\beta-\gamma(1+x)e^{-x} &> 1-\beta-\gamma(1+\frac{a}{1-b-ce^{-\alpha}})e^{-a} > 0 \\ 1-b-c(1+y)e^{-y} &> 1-b-c(1+\frac{\alpha}{1-\beta-\gamma e^{-a}})e^{-\alpha} > 0 \end{aligned} \tag{2.54}$$

Therefore from (2.53) and (2.54) we see that

$$f'(x) > 0,\ \ g'(y) > 0,\ \ x \in \left(a,\ \frac{a}{1-b-ce^{-\alpha}}\right),\ y \in \left(\alpha,\ \frac{\alpha}{1-\beta-\gamma e^{-a}}\right).$$

Hence, f, g are increasing function and this, together with (2.51) implies that $l_1 = L_1$. Then, from (2.51) again, we see that $l_2 = L_2$. Therefore, this completes the proof of this theorem. □

In the last theorem of this section, we will study the global asymptotic stability of the positive equilibrium of (1.1).

Theorem 2.5. *Consider system* (1.1) *such that either* (2.17) *and* (2.18) *hold or* (2.42) *holds. Also suppose that the following relation holds true:*

$$b + ce^{-\alpha} + \beta + \gamma e^{-a} + (b+ce^{-\alpha})(\beta+\gamma e^{-a}) + \frac{ac\alpha\gamma e^{-a-\alpha}}{(1-b-ce^{-\alpha})(1-\beta-\gamma e^{-a})} < 1. \tag{2.55}$$

Then the unique positive equilibrium $(\bar{x},\ \bar{y})$ *of* (1.1) *is globally asymptotically stable.*

Proof. First we will prove that $(\bar{x},\ \bar{y})$ is locally asymptotically stable. The linearized system of (1.1) about $(\bar{x},\ \bar{y})$ is

$$\begin{aligned} x_{n+1} &= (b+ce^{-\bar{y}})x_{n-1} - c\bar{x}e^{-\bar{y}}y_n \\ y_{n+1} &= (\beta+\gamma e^{-\bar{x}})y_{n-1} - \gamma\bar{y}e^{-\bar{x}}x_n \end{aligned} \tag{2.56}$$

We clearly see that (2.56) is equivalent to the system

$$w_{n+1} = Aw_n,\ \ A = \begin{pmatrix} 0 & b_1 & a_1 & 0 \\ d_1 & 0 & 0 & c_1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},\ \ w_n = \begin{pmatrix} x_n \\ y_n \\ x_{n-1} \\ y_{n-1} \end{pmatrix},$$

$$a_1 = b + ce^{-\bar{y}},\ \ b_1 = -c\bar{x}e^{-\bar{y}},\ \ c_1 = \beta + \gamma e^{-\bar{x}},\ \ d_1 = -\gamma\bar{y}e^{-\bar{x}}.$$

Then the characteristic equation of A is

$$\lambda^4 - (a_1 + c_1 + b_1 d_1)\lambda + a_1 c_1 = 0 \tag{2.57}$$

Since $\bar{x},\ \bar{y}$ satisfy (2.44) it is obvious that $\bar{x} > a,\ \bar{y} > \alpha$. Hence, from (2.55) and as $\bar{x},\ \bar{y}$ satisfy (2.45) we get

$$
\begin{aligned}
&|a_1| + |c_1| + |b_1 d_1| + |a_1 c_1| = b + ce^{-\bar{y}} + \beta + \gamma e^{-\bar{x}} + c\bar{x}e^{-\bar{y}}\gamma\bar{y}e^{-\bar{x}} + (b + ce^{-\bar{y}})(\beta + \gamma e^{-\bar{x}}) \\
&= b + ce^{-\bar{y}} + \beta + \gamma e^{-\bar{x}} + (b + ce^{-\bar{y}})(\beta + \gamma e^{-\bar{x}}) + \frac{ac\alpha\gamma e^{-\bar{x}-\bar{y}}}{(1 - b - ce^{-\bar{y}})(1 - \beta - \gamma e^{-\bar{x}})} \\
&< b + ce^{-\alpha} + \beta + \gamma e^{-a} + (b + ce^{-\alpha})(\beta + \gamma e^{-a}) + \frac{ac\alpha\gamma e^{-a-\alpha}}{(1 - b - ce^{-\alpha})(1 - \beta - \gamma e^{-a})} < 1.
\end{aligned}
\tag{2.58}
$$

Therefore, from (2.58) and Remark 1.3.1 of [9] all the roots (2.57) have the modulus are less than 1 which implies that $(\bar{x},\ \bar{y})$ is locally asymptotically stable. Using Theorem 2.3 $(\bar{x},\ \bar{y})$ is globally asymptotically stable. This completes the proof of the theorem. □

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