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GLOBAL CHARACTER OF SYSTEMS OF RATIONAL DIFFERENCE EQUATIONS

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Abstract. This paper deals with the solutions, stability character and asymptotic behavior of the systems of difference equations

$$x_{n+1} = \frac{1}{1 \pm y_n},\ y_{n+1} = \frac{1}{1 \pm x_n}, n \in \mathbb{N}_0$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$and the initial conditions x_0 and y_0, are nonzero real numbers, such that their solutions are associated to Fibonacci numbers.

1. Introduction

There has been a great interest in studying difference equations and systems, see [1]-[13] and references cited therein. Difference equations usually describe the evolution of certain phenomena over the course of time. Indeed difference equations have been applied in several mathematical models in biology, economics, genetics, population dynamics, medicines and so forth.

In this paper and motivated by [7], We deal with the form of the solutions of the following systems of rational difference equations

$$x_{n+1} = \frac{1}{1 \pm y_n},\ y_{n+1} = \frac{1}{1 \pm x_n}$$

initials conditions are arbitrary nonzero real numbers.
Now, We review some results which will be useful in our investigation.

1.1. Fibonacci numbers.

Here we will give some information about Fibonacci numbers. Fibonacci sequence $\{F_n\}_{n \in \mathbb{N}}$ are defined by

$$F_{n+2} = F_{n+1} + F_n, n \in \mathbb{N}_0 \tag{1.1}$$

where $F_0 = 0, F_1 = 1$. The solution of equation (1.1) is given by following formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.2}$$

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which is called Binet formula of Fibonacci numbers, where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$. Also, it is obtained to extend the Fibonacci sequence backward as

$$F_{-n} = F_{-n+2} - F_{-n+1} = (-1)^{n+1} F_n.$$

More generally, we can give the following limit

$$\lim_{n\to\infty} \frac{F_{n+r}}{F_n} = \alpha^r, r \in \mathbb{Z}.$$

1.2. **Linearized stability.** Let f and g be two continuously differentiable functions:

$$f: I \times J \longrightarrow I,\ g: I \times J \longrightarrow J$$

where I, J are some interval of real numbers. Consider the system of difference equations

$$x_{n+1} = f\left(x_n, y_n\right),\ y_{n+1} = f\left(x_n, y_n\right), \tag{1.3}$$

where $n \in \mathbb{N}_0$, $x_0 \in I$ and $y_0 \in J$.
Define the map

$$H: I \times J \longrightarrow I \times J$$

by

$$H(V) = (f(V), g(V))$$

where $V = (u, v)^T$. Let $V_n = (x_n, y_n)^T$, then, we can easily see that system (1.3) is equivalent to the following system written in the vector form

$$V_{n+1} = H(V_n),\ n = 0, 1, \ldots, \tag{1.4}$$

Definition 1.

- *An equilibrium point point $(\overline{x}, \overline{y}) \in I \times J$ of system* (1.3) *is a solution of the system*
$$x = f\left(x, y\right), y = g\left(x, y\right).$$
- *An equilibrium point $\overline{V} \in I \times J$ of system* (1.4) *is a solution of the system*
$$V = H(V).$$

Remark 1. *The linearized system, associated to System* (1.3)*, about the equilibrium point $(\overline{x}, \overline{y})$ is given by*

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial f}{\partial y}(\overline{x}, \overline{y}) \\ \frac{\partial g}{\partial x}(\overline{x}, \overline{y}) & \frac{\partial g}{\partial y}(\overline{x}, \overline{y}) \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}.$$

Definition 2. *Let $\overline{V}$ be an equilibrium point of system* (1.4) *and $\| . \|$ any norm, for example the Euclidean norm.*

(1) *The equilibrium point $\overline{V}$ is called stable (or locally stable) if for every $\epsilon > 0$ there exist $\delta > 0$ such that $\| V_0 - \overline{V} \| < \delta$ implies $\| V_n - \overline{V} \| < \epsilon$ for $n \geq 0$.*

(2) *The equilibrium point $\overline{V}$ is called asymptotically stable (or locally asymptotically stable) if it is stable and there exist $\gamma > 0$ such that $\| V_0 - \overline{V} \| < \gamma$ implies*

$$\| V_n - \overline{V} \| \to 0,\ n \to +\infty.$$

(3) *The equilibrium point $\overline{V}$ is said to be global attractor (respectively global attractor with basin of attraction a set $G \subseteq I \times J$, if for every V_0 (respectively for every $V_0 \in G$)*

$$\| V_n - \overline{V} \| \to 0,\ n \to +\infty.$$

(4) *The equilibrium point $\overline{V}$ is called globally asymptotically stable (respectively globally asymptotically stable relative to G) if it is asymptotically stable, and if for every V_0 (respectively for every $V_0 \in G$),*

$$\| V_n - \overline{V} \| \to 0,\ n \to +\infty.$$

(5) *The equilibrium point $\overline{V}$ is called instable if it is not stable.*

Theorem 1. *(Linearized stability)*

(1) *If all the eigenvalues of the Jacobian matrix lie in the open unit disk $|\lambda| < 1$, then the equilibrium point $\overline{V}$ of system (1.4) is asymptotically stable.*
(2) *If at least one eigenvalue of the Jacobian matrix have absolute value greater than one, then the equilibrium point $\overline{V}$ of system (1.4) is unstable.*

2. First System

In this section, we study the solutions of the system of difference equations

$$x_{n+1} = \frac{1}{1+y_n},\ y_{n+1} = \frac{1}{1+x_n} \tag{2.1}$$

where the initial values are arbitrary real numbers with $x_0, y_0 \notin \left\{ -\frac{F_{2n}}{F_{2n-1}}, n = 1, 2, ...\right\} \cup \left\{ -\frac{F_{2n+1}}{F_{2n}}, n = 1, 2, ...\right\}$.

2.1. **Form of the solutions.** The following theorem describes the form of the solutions of system (2.1).

Theorem 2. *Let $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ be a solution of (2.1). Then for $n = 1, 2, ...$,*

$$x_{2n-1} = \frac{F_{2n-1} + F_{2n-2}y_0}{F_{2n} + F_{2n-1}y_0}, \quad x_{2n} = \frac{F_{2n} + F_{2n-1}x_0}{F_{2n+1} + F_{2n}x_0},$$
$$y_{2n-1} = \frac{F_{2n-1} + F_{2n-2}x_0}{F_{2n} + F_{2n-1}x_0}, \quad y_{2n} = \frac{F_{2n} + F_{2n-1}y_0}{F_{2n+1} + F_{2n}y_0}.$$

Proof. For $n = 0$ the result holds. Suppose that $n > 0$ and that our assumption holds for $n-1$. That is,

$$x_{2n-3} = \frac{F_{2n-3} + F_{2n-4}y_0}{F_{2n-2} + F_{2n-3}y_0}, \quad x_{2n-2} = \frac{F_{2n-2} + F_{2n-3}x_0}{F_{2n-1} + F_{2n-2}x_0},$$
$$y_{2n-3} = \frac{F_{2n-3} + F_{2n-2}x_0}{F_{2n-2} + F_{2n-3}x_0}, \quad y_{2n-2} = \frac{F_{2n-2} + F_{2n-3}y_0}{F_{2n-1} + F_{2n-2}y_0}.$$

Now it follows from system (2.1) that

$$\begin{aligned}
x_{2n-1} &= \frac{1}{1+y_{2n-2}} \\
&= \frac{1}{1+\dfrac{F_{2n-2}+F_{2n-3}y_0}{F_{2n-1}+F_{2n-2}y_0}} \\
&= \frac{1}{\dfrac{F_{2n-1}+F_{2n-2}y_0+F_{2n-2}+F_{2n-3}y_0}{F_{2n-1}+F_{2n-2}y_0}} \\
&= \frac{F_{2n-1}+F_{2n-2}y_0}{F_{2n-1}+F_{2n-2}+(F_{2n-2}+F_{2n-3})y_0}.
\end{aligned}$$

So, we have

$$x_{2n-1} = \frac{F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}.$$

Also, it follow from System (2.1) that

$$\begin{aligned}
y_{2n-1} &= \frac{1}{1+x_{2n-2}} \\
&= \frac{1}{1+\dfrac{F_{2n-2}+F_{2n-3}x_0}{F_{2n-1}+F_{2n-2}x_0}} \\
&= \frac{1}{\dfrac{F_{2n-1}+F_{2n-2}x_0+F_{2n-2}+F_{2n-3}x_0}{F_{2n-1}+F_{2n-2}x_0}} \\
&= \frac{F_{2n-1}+F_{2n-2}x_0}{F_{2n-1}+F_{2n-2}+(F_{2n-2}+F_{2n-3})x_0}.
\end{aligned}$$

Hence we have

$$y_{2n-1} = \frac{F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}.$$

Similarly, it follow from System (2.1) that

$$\begin{aligned}
x_{2n} &= \frac{1}{1+y_{2n-1}} \\
&= \frac{1}{1+\dfrac{F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}} \\
&= \frac{1}{\dfrac{F_{2n}+F_{2n-1}x_0+F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}} \\
&= \frac{F_{2n}+F_{2n-1}x_0}{F_{2n}+F_{2n-1}+(F_{2n-1}+F_{2n-2})x_0}.
\end{aligned}$$

So, we have

$$x_{2n} = \frac{F_{2n}+F_{2n-1}x_0}{F_{2n+1}+F_{2n}x_0}.$$

Also, it follow from System (2.1) that

$$\begin{aligned} y_{2n} &= \frac{1}{1+x_{2n-1}} \\ &= \frac{1}{1+\dfrac{F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}} \\ &= \frac{1}{\dfrac{F_{2n}+F_{2n-1}y_0+F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}} \\ &= \frac{F_{2n}+F_{2n-1}y_0}{F_{2n}+F_{2n-1}+(F_{2n-1}+F_{2n-2})y_0}. \end{aligned}$$

So we get

$$y_{2n} = \frac{F_{2n}+F_{2n-1}y_0}{F_{2n+1}+F_{2n}y_0}.$$

□

2.2. **Global stability of positive solutions.** Our aim in this section is to study the asymptotic behavior of positive solutions of the system (2.1). Let $I = J = (0, +\infty)$, and consider the functions

$$f: I \times J \longrightarrow I, \; g: I \times \longrightarrow J$$

defined by

$$f(x,y) = \frac{1}{1+y}, \quad g(x,y) = \frac{1}{1+x}.$$

Corollary 1. *System* (2.1) *has a unique equilibrium point in* $I \times J$*, namely*

$$E = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right).$$

Proof. Clearly the system

$$\overline{x} = \frac{1}{1+\overline{y}}, \; \overline{y} = \frac{1}{1+\overline{x}},$$

has a unique solution in $I \times J$ which is

$$E = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right).$$

□

Theorem 3. *The equilibrium point E is locally asymptotically stable.*

Proof. The linearized system, associated to System (2.1), about the equilibrium E is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{-3+\sqrt{5}}{2} \\ \frac{-3+\sqrt{5}}{2} & 0 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}. \tag{2.2}$$

The characteristic polynomial of the System (2.2) about E is given by

$$P(\lambda) = \lambda^2 - \left(\frac{-3+\sqrt{5}}{2}\right)^2$$

Consider the two functions defined by

$$a(\lambda) = \lambda^2,\ b(\lambda) = \left(\frac{-3+\sqrt{5}}{2}\right)^2.$$

We have

$$\left|\frac{-3+\sqrt{5}}{2}\right| < 1.$$

Then

$$|b(\lambda)| < |a(\lambda)|, \forall \lambda : |\lambda| = 1$$

Thus, by Rouche's theorem, all zeros of $P(\lambda) = a(\lambda) - b(\lambda) = 0$ lie in $|\lambda| < 1$. So, by theorem (1) we get that E is locally asymptotically stable. □

Theorem 4. *The equilibrium point E is globally asymptotically stable.*

Proof. Let $\{x_n, y_n\}_{n\geq 0}$ be a solution of system (2.1). By theorem (3) we need only to prove that E is global attractor, that is

$$\lim_{n\to\infty}(x_n, y_n) = E.$$

From Theorem (2), We have

$$\begin{aligned}
\lim_{n\to\infty} x_{2n} &= \lim_{n\to\infty} \frac{F_{2n} + F_{2n-1}x_0}{F_{2n+1} + F_{2n}x_0} \\
&= \lim_{n\to\infty} \frac{1 + x_0\frac{F_{2n-1}}{F_{2n}}}{\frac{F_{2n+1}}{F_{2n}} + x_0} \\
&= \frac{1 + x_0\frac{1}{\alpha}}{\alpha + x_0} \\
&= \frac{-1+\sqrt{5}}{2},
\end{aligned}$$

and

$$\begin{aligned}
\lim_{n\to\infty} x_{2n-1} &= \lim_{n\to\infty} \frac{F_{2n-1} + F_{2n-2}y_0}{F_{2n} + F_{2n-1}y_0} \\
&= \lim_{n\to\infty} \frac{1 + x_0\frac{F_{2n-2}}{F_{2n-1}}}{\frac{F_{2n}}{F_{2n-1}} + x_0} \\
&= \frac{1 + x_0\frac{1}{\alpha}}{\alpha + x_0} \\
&= \frac{-1+\sqrt{5}}{2}.
\end{aligned}$$

Then $\lim_{n\to\infty} x_n = \frac{-1+\sqrt{5}}{2}$. Similarly, we obtain $\lim_{n\to\infty} y_n = \frac{-1+\sqrt{5}}{2}$. Hence we have

$$\lim_{n\to\infty}(x_n, y_n) = E.$$

□

2.3. **Numerical example.** For confirming the results of this section, we consider the following numerical example.

Example 1. *We assume* $x_0 = 0.4$ *and* $y_0 = 3.2$ *(See Fig.* (1)*).*

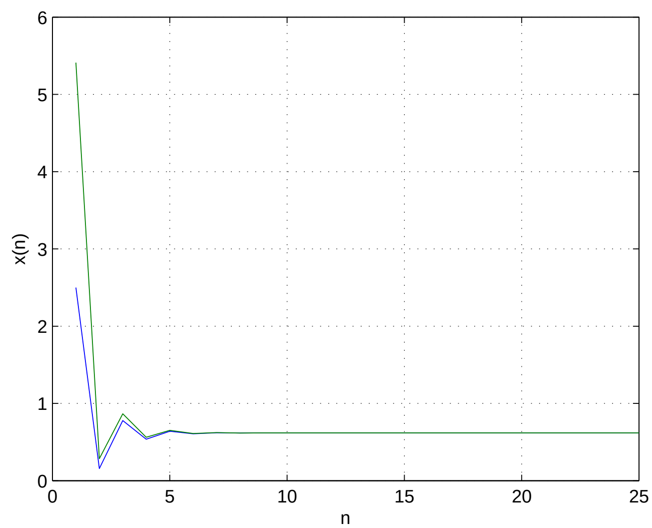

Figure 1. This figure shows that the solution of the system (2.1) is global attractor.

3. Second System

In this section, we study the solutions of the system of difference equations

$$x_{n+1} = \frac{1}{1-y_n},\ y_{n+1} = \frac{1}{1-x_n} \tag{3.1}$$

where the initial values are arbitrary real numbers with $x_0, y_0 \notin \{0,1\}$.

3.1. **Periodicity of solutions.**

Lemma 1. *Let* $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ *be a solution of* (3.1). *Then for* $n \geq 0$ *we have*

$$x_{n+6} = x_n,$$

$$y_{n+6} = y_n,$$

that is $\{x_n\}_{n\geq 0}$ *and* $\{y_n\}_{n\geq 0}$ *are periodic with periods six.*

Proof.

$$\begin{aligned}
x_{n+6} &= x_{(n+5)+1} = \frac{1}{1-y_{n-5}} \\
&= \frac{1}{1-\dfrac{1}{1-x_{n+4}}} \\
&= \frac{-1+x_{n+4}}{x_{n+4}} \\
&= \frac{1-\dfrac{1}{-1+y_{n+3}}}{\dfrac{1}{1-y_{n+3}}} = y_{n+3} \\
&= \frac{1}{1-x_{n+2}} = \frac{1}{1-\dfrac{1}{1-y_{n+1}}} \\
&= \frac{-1+y_{n+1}}{y_{n+1}} = \frac{-1+\dfrac{1}{1-x_n}}{\dfrac{1}{1-x_n}}.
\end{aligned}$$

Hence we have

$$x_{n+6} = x_n, \, n \geq 0.$$

Similarly, we have

$$\begin{aligned}
y_{n+6} &= y_{(n+5)+1} = \frac{1}{1-x_{n-5}} \\
&= \frac{1}{1-\dfrac{1}{1-y_{n+4}}} \\
&= \frac{-1+y_{n+4}}{y_{n+4}} \\
&= \frac{1-\dfrac{1}{-1+x_{n+3}}}{\dfrac{1}{1-x_{n+3}}} = x_{n+3} \\
&= \frac{1}{1-y_{n+2}} = \frac{1}{1-\dfrac{1}{1-x_{n+1}}} \\
&= \frac{-1+x_{n+1}}{x_{n+1}} = \frac{-1+\dfrac{1}{1-y_n}}{\dfrac{1}{1-y_n}}.
\end{aligned}$$

So we have

$$y_{n+6} = y_n, \, n \geq 0.$$

□

3.2. **Form of the solutions.** In the following theorem we give explicit formulas for the solutions of system (3.1).

Theorem 5. *Let* $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$ *be a solution of* (3.1). *Then for* $n = 1, 2, ...,$

$$\begin{aligned}
x_{6n-5} &= \frac{1}{1-y_0}, & y_{6n-5} &= \frac{1}{1-x_0},\\
x_{6n-4} &= \frac{-1+x_0}{x_0}, & y_{6n-4} &= \frac{-1+y_0}{y_0},\\
x_{6n-3} &= y_0, & y_{6n-3} &= x_0,\\
x_{6n-2} &= \frac{1}{1-x_0}, & y_{6n-2} &= \frac{1}{1-y_0},\\
x_{6n-1} &= \frac{-1+y_0}{y_0}, & y_{6n-1} &= \frac{-1+x_0}{x_0},\\
x_{6n} &= x_0, & y_{6n} &= y_0.
\end{aligned}$$

Proof. For $n = 0$ the result holds. Suppose that $n > 0$ and that our assumption holds for $n - 1$. That is,

$$\begin{aligned}
x_{6n-11} &= \frac{1}{1-y_0}, & y_{6n-11} &= \frac{1}{1-x_0},\\
x_{6n-10} &= \frac{-1+x_0}{x_0}, & y_{6n-10} &= \frac{-1+y_0}{y_0},\\
x_{6n-9} &= y_0, & y_{6n-9} &= x_0,\\
x_{6n-8} &= \frac{1}{1-x_0}, & y_{6n-8} &= \frac{1}{1-y_0},\\
x_{6n-7} &= \frac{-1+y_0}{y_0}, & y_{6n-7} &= \frac{-1+x_0}{x_0},\\
x_{6n-6} &= x_0, & y_{6n-6} &= y_0.
\end{aligned}$$

Now it follows from system (3.1) that

$$x_{6n-5} = \frac{1}{1-y_{6n-6}} = \frac{1}{1-y_0}$$

and

$$y_{6n-5} = \frac{1}{1-x_{6n-6}} = \frac{1}{1-x_0}.$$

Also, it follow from System (3.1) that

$$\begin{aligned}
x_{6n-4} &= \frac{1}{1-y_{6n-5}}\\
&= \frac{1}{1-\dfrac{1}{1-x_0}}.
\end{aligned}$$

So we get

$$x_{6n-4} = \frac{-1+x_0}{x_0}.$$

Again from System (3.1)

$$
\begin{aligned}
y_{6n-4} &= \frac{1}{1-x_{6n-5}} \\
&= \frac{1}{1-\dfrac{1}{1-y_0}}.
\end{aligned}
$$

So we have

$$
y_{6n-4} = \frac{-1+y_0}{y_0}.
$$

Similarly, one can easily prove the other relations. Thus, the proof is complete. □

3.3. **Numerical example.** For confirming the results of this section, we consider the following numerical example.

Example 2. *We assume* $x_0 = 0.2$ *and* $y_0 = 1.1$ *(See Fig. (2)).*

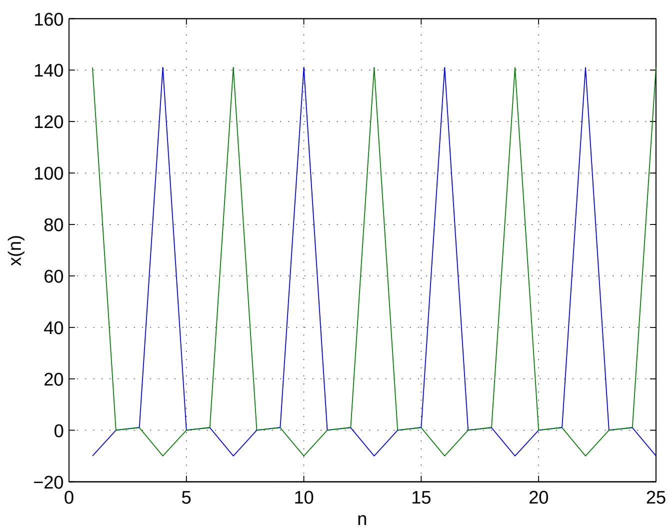

Figure 2. This figure shows the periodicity the solution of the system (3.1).

4. Conclusion

In this study, we mainly obtained the relation between the solutions of system of difference equations (2.1) and Fibonacci numbers. We also presented that the solutions of equations in (2.1) actually converge to $\left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$. We prove also that the solutions of system of difference equations (3.1) are periodic with period six and so this solutions are unstable.

References

[1] Q. Din, M. Qureshi and A. Q. Khan, *Dynamics of a fourth-order system of rational difference equations*, Advances in Difference Equations, (2012), Article ID 215.

[2] E. M. Elsayed, M. Mansour and M. M. El-Dessoky, *Solutions of fractional systems of difference equations*, Ars Combinatoria, **110**(2013), 469-479.

[3] E. M. Elabbasy, H. El-Metwally and E.M.Elsayed, *On the solutions of a class of difference equations systems*, Demonstratio Mathematica, **41(1)**(2008), 109-122.

[4] E. M. Elsayed, *On the solutions of a rational system of difference equations*, Fasciculi Mathematici, **45**(2010), 25-36.

[5] E. M. Elsayed, *Solutions of rational difference system of order two*, Mathematical and Computer Modelling, **55**(2012), 378-384.

[6] T. F. Ibrahim and N. Touafek, *On a third order rational difference equation with variable coefficients*, Dynamics of Continuous, Discrete & Impulsive Systems. Series B: Applications & Algorithms **20**(2013), 251-264.

[7] D. T. Tollu, Y. Yalzik and N. Taskara, *On the solutions of two special type of Riccati difference equation via fibonacci numbers*, Advance in defference equations, (2013) Article ID 174.

[8] D. T. Tollu, Y. Yalzik and N. Taskara, *On fourteen solvable systems of difference equations*, Applied Mathematics and Computation, **233**(2014), 310-319.

[9] N. Touafek, *On a second order rational difference equation*, Hacettepe Journal of Mathematics and Statistics, **41(6)**(2012), 867-874.

[10] N. Touafek and Y. Halim, *Global Attractivity of a Rational Difference Equation*, Mathematical Sciences Letters, **2(3)**(2013), 161-165.

[11] N. Touafek and Y. Halim, *On Max Type Difference Equations: Expressions of Solutions*, International Journal of Nonlinear Science, **11**(2011), 396-402.

[12] N. Touafek and E. M Elsayed, *On the periodicity of some systems of nonlinear difference equations*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie, **55(103)**(2012), 217-224.

[13] N. Touafek and E. M Elsayed, *On the solutions of systems of rational difference equations*, Mathematical and Computer Modelling, **55**(2012), 1987-1997.

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