

**COMMON FIXED POINT THEOREMS FOR SIX SELF-MAPS
SATISFYING COMMON(*E.A*) AND COMMON(*CLR*)
PROPERTIES IN COMPLEX VALUED METRIC SPACE**

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ABSTRACT. In this paper, using the concept of common (*E.A*) property and common (*CLR*) property fixed point results for six self-maps under rational type contractive condition in the context of complex valued metric space are established. Appropriate examples are also given for the existence of our results. The derived results extends well known results in the literature.

1. INTRODUCTION

Banach fixed point theorem[4] is one of the pivotal result in nonlinear analysis known as Banach's Contraction Principle. This principal is constructive in nature which explains the existence and uniqueness of fixed points of operators or mappings. This Principle has been obtained in several directions like 2-metric spaces, D-metric spaces, G-metric spaces etc. These generalization were made either by wakening the contractive condition or by imposing some additional conditions on ambient space.

Azam *et al.*[1] introduced the concept of complex-valued metric space and obtained fixed point theorems of contractive type mappings using the rational inequality in a complex-valued metric space. Sumit Chandok and Deepak Kumar[8] proved some common fixed point theorems for four self-maps having weakly compatibility satisfying a contractive condition of rational type using (*E.A*) property and (*CLR*) property in the context of complex valued metric spaces. Rahul Tiwari *et al.*[19] and Yogita R. Sharma[21] proved common fixed point theorem with six self-maps in the context of complex valued metric spaces.

The aim of this paper is to establish common fixed point theorems for six self-maps having weakly compatibility satisfying a contractive condition of rational type using common (*E.A*) property and common (*CLR*) property in the context of complex valued metric spaces.

We recall some definitions that will be used in our discussion.

Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C}

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as follows:

$z_1 \lesssim z_2$ if and only if $Re(z_1) \leq Re(z_2), Im(z_1) \leq Im(z_2)$

Consequently, one can say that $z_1 \lesssim z_2$ if one of the following conditions is satisfied:

- 1). $Re(z_1) = Re(z_2), Im(z_1) < Im(z_2),$
- 2). $Re(z_1) < Re(z_2), Im(z_1) = Im(z_2),$
- 3). $Re(z_1) < Re(z_2), Im(z_1) < Im(z_2),$
- 4). $Re(z_1) = Re(z_2), Im(z_1) = Im(z_2),$

In particular, we will write $z_1 \not\lesssim z_2$ if $z_1 \neq z_2$ and one of (1),(2) and (4) is satisfied and we will write $z_1 \prec z_2$ if only (3) is satisfied.

Note that

- i). $a, b \in \mathbb{R}$ and $a \leq b \Rightarrow az \lesssim bz$ for all $z \in \mathbb{C}$;
- ii). $0 \lesssim z_1 \not\lesssim z_2 \Rightarrow |z_1| < |z_2|$;
- iii). $z_1 \lesssim z_2$ and $z_2 \prec z_3 \Rightarrow z_1 \prec z_3$

Azam *et al.*[1] defined the complex-valued metric space (X, d) in the following way:

Definition 1 Let X be a nonempty set. Suppose that the mapping $d : X \times X \rightarrow \mathbb{C}$ satisfies:

- 1). $0 \lesssim d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
- 2). $d(x, y) = d(y, x)$ for all $x, y \in X$;
- 3). $d(x, y) \lesssim d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a complex valued metric on X and (X, d) is called a complex valued metric space.

Example 1 [18] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by

$$d(z_1, z_2) = e^{ik}|z_1 - z_2|,$$

where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space.

Definition 2 [1] Let (X, d) be a complex valued metric space. Then

- i). Any point $x \in X$ is called an interior point of a set $A \subseteq X$ if there exists $0 < r \in \mathbb{C}$ such that $B(x; r) = \{y \in X : d(x, y) < r\} \subseteq A$.
- ii). Any subset A of X is called open if each point of A is an interior point of A .
- iii). Any point $x \in X$ is called a limit point of A if for every $0 < r \in \mathbb{C}$, $B(x; r) \cap (A \setminus X) \neq \emptyset$.
- iv). Any subset $A \subseteq X$ is called closed if each limit point of A belongs to A .
- v). The family $F = \{B(x; r) : x \in X, 0 < r\}$ is sub-basis for Hausdorff topology on X .

Definition 3 [1] Let $\{x_n\}$ be a sequence in complex valued metric (X, d) and $x \in X$. Then

- i). x is called the limit of $\{x_n\}$ if for every $c \in \mathbb{C}$, with $0 < c$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) < c$ for all $n > n_0$ and we write $\lim_{n \rightarrow \infty} x_n = x$.
- ii). $\{x_n\}$ is called Cauchy sequence if for every $c \in \mathbb{C}$ with $0 < c$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x_{n+m}) < c$ for all $n > n_0$.
- iii). (X, d) is complete complex valued metric space if every Cauchy sequence is convergent in (X, d) .

Lemma 1 [1] Any sequence $\{x_n\}$ in complex valued metric space (X, d) converges to x if and only if $|d(x_n, x)| \rightarrow 0$ as $n \rightarrow \infty$.

Lemma 2 [1] Any sequence $\{x_n\}$ in complex valued metric space (X, d) is a Cauchy

sequence if and only if $|d(x_n, x_{n+m})| \rightarrow 0$ as $n \rightarrow \infty$, where $m \in \mathbb{N}$.

Definition 4 [16] Let S and T be self maps of a non empty set X . Then

- (i). Any point $x \in X$ is said to be fixed point of T if $Tx = x$.
- (ii). Any point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and we shall called $w = Sx = Tx$ that a point of coincidence of S and T .
- (iii). Any point $x \in X$ is said to be a common fixed point of S and T if $Sx = Tx = x$.

Jungck [11] define weakly compatible maps and redefine weakly compatible maps by Bhatt *et al.*[3] in the complex valued metric space.

Definition 5 [3] A pair of self-mapping $S, T : X \rightarrow X$ is weakly compatible if they commute at their coincidence points, that is if there exist a point $x \in X$ such that $STx = TSx$ whenever $Sx = Tx$.

Example 2 [20] Let $X = \mathbb{C}$. Define the mapping $d : X \times X \rightarrow \mathbb{C}$ by $d(z_1, z_2) = e^{ia}|z_1 - z_2|$ for all $z_1, z_2 \in X$, where $k \in \mathbb{R}$. Then (X, d) is a complex valued metric space. Suppose that self-mappings S and T is defined as

$$Sz = \begin{cases} 2e^{\frac{i\pi}{4}} & \text{if } \operatorname{Re}(z) \neq 0, \\ 3e^{\frac{i\pi}{3}} & \text{if } \operatorname{Re}(z) = 0 \end{cases}$$

and

$$Tz = \begin{cases} 2e^{\frac{i\pi}{4}} & \text{if } \operatorname{Re}(z) \neq 0, \\ 4e^{\frac{i\pi}{6}} & \text{if } \operatorname{Re}(z) = 0 \end{cases}$$

Clearly, S and T are weakly compatible self maps.

Aamri and Moutawakil [2] define the notion of $(E.A)$ property and redefine this property by R. K. Verma and H. K. Pathak [20] in a complex-valued metric space.

Definition 6 [20] Let $A, S : X \rightarrow X$ be two self-maps of a complex-valued metric space (X, d) . The pair (A, S) is said to satisfy property $(E.A)$, if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = t \text{ for some } t \in X$$

Note that weakly compatibility and $(E.A.)$ property are independent of each other (see [14]).

Sintunavarat and Kumam [16] defined the notion of (CLR) -property and redefine this property by R. K. Verma and H. K. Pathak [20] in a complex-valued metric space.

Definition 7 [16] Let (X, d) be a complex valued metric space and $T, S : X \rightarrow X$. Then T and S are said to satisfy the common limit in the range of S property, denoted by (CLR_S) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sx_n = Sx \text{ for some } x \in X$$

Example of the definition(6) and (7) is presented as follows

Example 3 Let $X = \mathbb{C}$. Define $T, S : X \rightarrow X$ by $Tz = 3z - i$ and $Sz = (z + 1)^2$ for all $z \in X$. Let $\{z_n\} = \{i + \frac{1}{n}\}_{n \geq 1}$ be the sequence in X . Then

$$\lim_{n \rightarrow \infty} Tz_n = \lim_{n \rightarrow \infty} (3i + \frac{3}{n} - i) = 2i \text{ and } \lim_{n \rightarrow \infty} Sz_n = \lim_{n \rightarrow \infty} (i + \frac{1}{n} + 1)^2 = 2i$$

That is there exists a sequence $\{z_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Tz_n = \lim_{n \rightarrow \infty} Sz_n = 2i \in X$$

Hence T and S satisfies (E.A)property.
Further, since

$$\lim_{n \rightarrow \infty} Tz_n = \lim_{n \rightarrow \infty} Sz_n = 2i = S(0 + i)$$

so that (T,S) satisfies (CLR_S) property with $z = 0 + i \in X$.

Yincheng Liu *et al*[13] introduce common (E.A) property which is the extension of (E.A) property. We redefine common (E.A) property in the complex valued metric space as follows.

Definition 8 Let (X, d) be a complex valued metric space and $A, B, S, T : X \rightarrow X$ be four self maps. The pairs (A,S) and (B,T) satisfy the common (E.A) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = t \in X.$$

We obtain definition(6), if we put $B = A$ and $S = T$ in definition(8).

Now we present an example of the above definition in the complex-valued metric space as follows;

Example 4 Let $X = \mathbb{C}$ and d be a complex valued metric space. Define $A, B, S, T : X \rightarrow X$ by

$$Az = 2 - iz, Bz = i - 2z^2, Sz = i - 2z, Tz = 2 + (z - 2i)^3$$

Let $\{z_n\} = \{-1 + \frac{i}{n}\}_{n \geq 1}$ and $\{w_n\} = \{\frac{1}{n} + i\}_{n \geq 1}$ be the two sequences in X . Then

$$\lim_{n \rightarrow \infty} Az_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Bw_n = \lim_{n \rightarrow \infty} Tw_n = 2 + i \in X$$

Hence the pairs (A, S) and (B, T) satisfy common (E.A) property.

Muhammad Imdad *et al*[10] introduce common (CLR) property which is the extension of (CLR) property. We redefine common (CLR) property in the complex valued metric space as follows.

Definition 9 Let (X, d) be a complex valued metric space and $A, B, S, T : X \rightarrow X$ be four self maps. The pairs (A,S) and (B,T) satisfy the common limit range property with respect to mapping S and T , denoted by (CLR_{ST}) if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} By_n = \lim_{n \rightarrow \infty} Ty_n = t \in S(X) \cap T(X).$$

Using the (CLR) property and (E.A) property fixed point theorems have been proved by various researchers (see [5]-[10],[12],[15],[16], [17],[20]).

2. MAIN RESULTS

In this section, we shall prove common fixed point theorems for weakly compatible mappings using common (E.A.) property and common (CLR) property in the complex valued metric spaces.

Theorem 1 Let (X,d) be a complex valued metric space and $A, B, S, T, P, Q : X \rightarrow X$ be six self mapping satisfying the following conditions

$$I \quad A(X) \subseteq T(X), A(X) \subseteq P(X), B(X) \subseteq S(X) \text{ and } B(X) \subseteq Q(X);$$

II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, By) \lesssim k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right\};$$

III the pairs (A, S) , (B, T) , (A, Q) and (B, P) are weakly compatible;

IV either both the pairs (A, S) and (A, Q) satisfies common $(E.A)$ property or both the pairs (B, T) and (B, P) satisfies common $(E.A)$ property;

V either both $T(X)$ and $P(X)$ are closed subspaces of X or both $S(X)$ and $Q(X)$ are closed subspaces of X .

Then the mapping A, B, S, T, P and Q have unique common fixed point in X .

Proof. Suppose that the pairs (B, T) and (B, P) satisfies common $(E.A)$ property. Then, by definition(8) there exist two sequences $\{x_n\}$ and $\{x_n^*\}$ in X such that

$$\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Bx_n^* = \lim_{n \rightarrow \infty} Px_n^* = t \text{ for some } t \in X. \quad (1)$$

Since $B(X) \subseteq S(X)$, so there exist two sequences $\{y_n\}$ and $\{y_n^*\}$ in X such that $Bx_n = Sy_n$ and $Bx_n^* = Sy_n^*$. Using equation (1), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} Bx_n &= \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = \lim_{n \rightarrow \infty} Bx_n^* \\ &= \lim_{n \rightarrow \infty} Px_n^* = \lim_{n \rightarrow \infty} Sx_n^* = t \text{ for some } t \in X. \end{aligned} \quad (2)$$

Since $B(X) \subseteq Q(X)$, so there exist two sequences $\{z_n\}$ and $\{z_n^*\}$ in X such that $Bx_n = Qz_n$ and $Bx_n^* = Qz_n^*$. Using equation (2), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} Bx_n &= \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = \lim_{n \rightarrow \infty} Qz_n \\ &= \lim_{n \rightarrow \infty} Bx_n^* = \lim_{n \rightarrow \infty} Px_n^* = \lim_{n \rightarrow \infty} Sy_n^* = \lim_{n \rightarrow \infty} Qz_n^* = t. \end{aligned} \quad (3)$$

Next, we claim that $\lim_{n \rightarrow \infty} Ay_n = t$, if $\lim_{n \rightarrow \infty} Ay_n = w \neq t$, then putting $x = y_n$ and $y = x_n$ in condition(II), we have

$$d(Ay_n, Bx_n) \lesssim k \left\{ \frac{d(Sy_n, Ay_n)d(Qy_n, Ay_n)d(Sy_n, Bx_n)d(Qy_n, Bx_n)}{1 + d(Sy_n, Bx_n)d(Qy_n, Bx_n) + d(Tx_n, Ay_n)d(Px_n, Ay_n)} + \frac{d(Tx_n, Bx_n)d(Px_n, Bx_n)d(Tx_n, Ay_n)d(Px_n, Ay_n)}{1 + d(Sy_n, Bx_n)d(Qy_n, Bx_n) + d(Tx_n, Ay_n)d(Px_n, Ay_n)} \right\}$$

Taking limit as $n \rightarrow \infty$, we get

$$d(w, t) \lesssim 0 \text{ or } |d(w, t)| \leq 0, \text{ which is a contradiction, thus } \lim_{n \rightarrow \infty} Ay_n = t.$$

Hence equation(3) becomes

$$\begin{aligned} \lim_{n \rightarrow \infty} Bx_n &= \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = \lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Qz_n \\ &= \lim_{n \rightarrow \infty} Bx_n^* = \lim_{n \rightarrow \infty} Px_n^* = \lim_{n \rightarrow \infty} Sy_n^* = \lim_{n \rightarrow \infty} Qz_n^* = t. \end{aligned} \quad (4)$$

Also, we show that $\lim_{n \rightarrow \infty} Ay_n^* = t$. If $\lim_{n \rightarrow \infty} Ay_n^* = z \neq t$, then putting $x = y_n^*$,

$y = x_n^*$ in condition (II) and repeating the same procedure as above, we get $|d(z, t)| \leq 0$, which is a contradiction, thus $\lim_{n \rightarrow \infty} Ay_n^* = t$. Hence equation(4) becomes

$$\begin{aligned} \lim_{n \rightarrow \infty} Bx_n &= \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = \lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Qz_n \\ &= \lim_{n \rightarrow \infty} Bx_n^* = \lim_{n \rightarrow \infty} Px_n^* = \lim_{n \rightarrow \infty} Sy_n^* = \lim_{n \rightarrow \infty} Ay_n^* = \lim_{n \rightarrow \infty} QZ_n^* = t. \end{aligned} \quad (5)$$

Now, suppose that $S(X)$ is closed subspace of X , so there exists $u_1 \in X$ such that $Su_1 = t$ and hence equation(5) becomes;

$$\begin{aligned} \lim_{n \rightarrow \infty} Bx_n &= \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = \lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Qz_n \\ &= \lim_{n \rightarrow \infty} Bx_n^* = \lim_{n \rightarrow \infty} Px_n^* = \lim_{n \rightarrow \infty} Sy_n^* = \lim_{n \rightarrow \infty} Ay_n^* = \lim_{n \rightarrow \infty} QZ_n^* = t = Su_1. \end{aligned} \quad (6)$$

We show that $Au_1 = Su_1$. Putting $x = u_1$ and $y = x_n$ in condition(II), we have

$$\begin{aligned} d(Au_1, Bx_n) &\lesssim k \left\{ \frac{d(Su_1, Au_1)d(Qu_1, Au_1)d(Su_1, Bx_n)d(Qu_1, Bx_n)}{1 + d(Su_1, Bx_n)d(Qu_1, Bx_n) + d(Tx_n, Au_1)d(Px_n, Au_1)} \right. \\ &\quad \left. + \frac{d(Tx_n, Bx_n)d(Px_n, Bx_n)d(Tx_n, Au_1)d(Px_n, Au_1)}{1 + d(Su_1, Bx_n)d(Qu_1, Bx_n) + d(Tx_n, Au_1)d(Px_n, Au_1)} \right\}. \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\begin{aligned} d(Au_1, t) &\lesssim k \left\{ \frac{d(t, Au_1)d(Qu_1, Au_1)d(t, t)d(Qu_1, t)}{1 + d(t, t)d(Qu_1, t) + d(t, Au_1)d(t_1, Au_1)} \right. \\ &\quad \left. + \frac{d(t, t)d(t_1, t)d(t, Au_1)d(t_1, Au_1)}{1 + d(t, t)d(Qu_1, t) + d(t, Au_1)d(t_1, Au_1)} \right\} \end{aligned}$$

or $d(Au_1, t) \lesssim 0$ or $|d(Au_1, t)| \leq 0$ or $Au_1 = t$, thus $Au_1 = Su_1 = t$.

But, since $A(X) \subseteq T(X)$, so there exist $v_1 \in X$ such that $Au_1 = Tv_1$ and hence

$$Au_1 = Su_1 = Tv_1 = t.$$

Next, we show that $Bv_1 = Tv_1$. Putting $x = u_1$ and $y = v_1$ in condition(II), we have

$$\begin{aligned} d(Au_1, Bv_1) &\lesssim k \left\{ \frac{d(Su_1, Au_1)d(Qu_1, Au_1)d(Su_1, Bv_1)d(Qu_1, Bv_1)}{1 + d(Su_1, Bv_1)d(Qu_1, Bv_1) + d(Tv_1, Au_1)d(Pv_1, Au_1)} \right. \\ &\quad \left. + \frac{d(Tv_1, Bv_1)d(Pv_1, Bv_1)d(Tv_1, Au_1)d(Pv_1, Au_1)}{1 + d(Su_1, Bv_1)d(Qu_1, Bv_1) + d(Tv_1, Au_1)d(Pv_1, Au_1)} \right\}, \end{aligned}$$

or

$$\begin{aligned} d(t, Bv_1) &\lesssim k \left\{ \frac{d(t, t)d(Qu_1, t)d(t, Bv_1)d(Qu_1, Bv_1)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right. \\ &\quad \left. + \frac{d(t, Bv_1)d(Pv_1, Bv_1)d(t, t)d(Pv_1, t)}{1 + d(t, Bv_1)d(Qu_1, Bv_1) + d(t, t)d(Pv_1, t)} \right\} \end{aligned}$$

or $d(t, Bv_1) \lesssim 0$ or $|d(t, Bv_1)| \leq 0$ or $Bv_1 = t$, thus $Bv_1 = Tv_1$.

Hence

$$Au_1 = Su_1 = Tv_1 = Bv_1 = t. \quad (7)$$

Now, suppose that $Q(X)$ is closed subspace of X , so there exist $u_2 \in X$ such that $Qu_2 = t$ and hence from(5), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} Bx_n &= \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Sy_n = \lim_{n \rightarrow \infty} Ay_n = \lim_{n \rightarrow \infty} Qz_n \\ &= \lim_{n \rightarrow \infty} Bx_n^* = \lim_{n \rightarrow \infty} Px_n^* = \lim_{n \rightarrow \infty} Sy_n^* = \lim_{n \rightarrow \infty} Ay_n^* = \lim_{n \rightarrow \infty} Qz_n^* = t = Qu_2. \end{aligned} \quad (8)$$

We show that $Au_2 = Qu_2$. Using triangular inequality, we have

$$d(Au_2, Qz_n^*) \lesssim d(Au_2, Bx_n^*) + d(Bx_n^*, Qz_n^*),$$

putting $x = u_2$ and $y = x_n^*$ in condition(II), we have from above

$$\begin{aligned} d(Au_2, Qz_n^*) &\lesssim k \left\{ \frac{d(Su_2, Au_2)d(Qu_2, Au_2)d(Su_2, Bx_n^*)d(Qu_2, Bx_n^*)}{1 + d(Su_2, Bx_n^*)d(Qu_2, Bx_n^*) + d(Tx_n^*, Au_2)d(Px_n^*, Au_2)} \right. \\ &\quad \left. + \frac{d(Tx_n^*, Bx_n^*)d(Px_n^*, Bx_n^*)d(Tx_n^*, Au_2)d(Px_n^*, Au_2)}{1 + d(Su_2, Bx_n^*)d(Qu_2, Bx_n^*) + d(Tx_n^*, Au_2)d(Px_n^*, Au_2)} \right\} + d(Bx_n^*, Qz_n^*). \end{aligned}$$

Taking limit as $n \rightarrow \infty$, we get

$$\begin{aligned} d(Au_2, t) &\lesssim k \left\{ \frac{d(Su_2, Au_2)d(t, Au_2)d(Su_2, t)d(t, t)}{1 + d(Su_2, t)d(t, t) + d(t_1^*, Au_2)d(t, Au_2)} \right. \\ &\quad \left. + \frac{d(t_1^*, t)d(t, t)d(t_1^*, Au_2)d(t, Au_2)}{1 + d(Su_2, t)d(t, t) + d(t_1^*, Au_2)d(t, Au_2)} \right\} + d(t, t) \end{aligned}$$

or $d(Au_2, t) \lesssim 0$ or $|d(Au_2, t)| \leq 0$ or $Au_2 = t$, that is $Au_2 = Qu_2 = t$.

But, since $A(X) \subseteq P(X)$ so there exists $v_2 \in X$ such that $Au_2 = Pv_2$ and hence

$$Au_2 = Qu_2 = Pv_2 = t.$$

Next, we show that $Bv_2 = Pv_2$. Putting $x = u_2$ and $y = v_2$ in condition(II), we have

$$\begin{aligned} d(Au_2, Bv_2) &\lesssim k \left\{ \frac{d(Su_2, Au_2)d(Qu_2, Au_2)d(Su_2, Bv_2)d(Qu_2, Bv_2)}{1 + d(Su_2, Bv_2)d(Qu_2, Bv_2) + d(Tv_2, Au_2)d(Pv_2, Au_2)} \right. \\ &\quad \left. + \frac{d(Tv_2, Bv_2)d(Pv_2, Bv_2)d(Tv_2, Au_2)d(Pv_2, Au_2)}{1 + d(Su_2, Bv_2)d(Qu_2, Bv_2) + d(Tv_2, Au_2)d(Pv_2, Au_2)} \right\} \end{aligned}$$

or $d(t, Bv_2) \lesssim 0$ or $|d(t, Bv_2)| \leq 0$ or $d(t, Bv_2) = 0$ or $Bv_2 = t$.

Hence

$$Au_2 = Qu_2 = Pv_2 = Bv_2 = t. \quad (9)$$

Therefor from(7) and (9), we get

$$Au_1 = Su_1 = Tv_1 = Bv_1 = Au_2 = Qu_2 = Pv_2 = Bv_2 = t. \quad (10)$$

That is t is common coincident point of A, B, S, T, P and Q .

Now we show that t is the common fixed point of A, B, S, T, P and Q . For this, using the weak compatibility of the pairs (A, S) , (B, T) , (A, Q) , (B, P) and equation(10) we have

$$Au_1 = Su_1 \Rightarrow ASu_1 = SAu_1 \Rightarrow At = St. \quad (11)$$

$$Tv_1 = Bv_1 \Rightarrow BTv_1 = TBv_1 \Rightarrow Bt = Tt. \quad (12)$$

$$Au_2 = Qu_2 \Rightarrow AQu_2 = QAu_2 \Rightarrow At = Qt. \quad (13)$$

$$Pv_2 = Bv_2 \Rightarrow BPv_2 = PBv_2 \Rightarrow Bt = Pt. \quad (14)$$

Now, putting $x = u_1$ and $y = t$ in condition(II), we have

$$d(Au_1, Bt) \lesssim k \left\{ \frac{d(Su_1, Au_1)d(Qu_1, Au_1)d(Su_1, Bt)d(Qu_1, Bt)}{1 + d(Su_1, Bt)d(Qu_1, Bt) + d(Tt, Au_1)d(Pt, Au_1)} + \frac{d(Tt, Bt)d(Pt, Bt)d(Tt, Au_1)d(Pt, Au_1)}{1 + d(Su_1, Bt)d(Qu_1, Bt) + d(Tt, Au_1)d(Pt, Au_1)} \right\}$$

or $d(t, Bt) \lesssim 0$ or $|d(t, Bt)| \leq 0$ or $d(t, Bt) = 0$ or $Bt = t$ and hence from (12) and (14), we get

$$Bt = Tt = Pt = t \quad (15)$$

Similarly, putting $x = t$ and $y = v_1$ in condition(II) we obtained $At = t$ and hence from (11) and (13), we get

$$At = St = Qt = t \quad (16)$$

Therefor from (15) and (16), we get

$$At = Bt = St = Tt = Pt = Qt = t. \quad (17)$$

That is t is the common fixed point of A, B, S, T, P and Q .

Similar, if we assume that the pairs (A, S) and (A, Q) satisfies common $(E.A)$ property and that both $T(X)$ and $P(X)$ are closed subspaces of X . Then we can prove that t is the common fixed point of A, B, S, T, P and Q in the same lines as above.

Uniqueness: Assume that t^* be another common fixed point of A, B, S, T, P and Q . Then

$$d(t, t^*) = d(At, Bt^*)$$

$$d(t, t^*) \lesssim k \left\{ \frac{d(St, At)d(Qt, At)d(St, Bt^*)d(Qt, Bt^*)}{1 + d(St, Bt^*)d(Qx, Bt^*) + d(Tt^*, At)d(Pt^*, At)} + \frac{d(Tt^*, Bt^*)d(Pt^*, Bt^*)d(Tt^*, At)d(Pt^*, At)}{1 + d(St, Bt^*)d(Qx, Bt^*) + d(Tt^*, At)d(Pt^*, At)} \right\}$$

$$\text{or } d(t, t^*) \lesssim 0 \text{ or } |d(t, t^*)| \leq 0 \text{ or } d(t, t^*) = 0 \text{ or } t = t^*$$

Hence t is unique common fixed point of A, B, S, T, P and Q .

From above theorem we deduce the following corollaries.

Corollary 1 Let (X, d) be a complex valued metric space and $A, S, T, P, Q : X \rightarrow X$ be five self mapping satisfying the following conditions

- I $A(X) \subseteq T(X), A(X) \subseteq P(X), A(X) \subseteq S(X)$ and $A(X) \subseteq Q(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, Ay) \lesssim k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, Ay)d(Qx, Ay)}{1 + d(Sx, Ay)d(Qx, Ay) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, Ay)d(Py, Ay)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, Ay)d(Qx, Ay) + d(Ty, Ax)d(Py, Ax)} \right\}$$

- III the pairs $(A, S), (A, T), (A, Q)$ and (A, P) are weakly compatible;
- IV either both the pairs (A, S) and (A, Q) satisfies common $(E.A)$ property or both the pairs (A, T) and (A, P) satisfies common $(E.A)$ property;
- V either both $T(X)$ and $P(X)$ are closed subspaces of X or both $S(X)$ and $Q(X)$ are closed subspaces of X .

Then the mapping A, S, T, P and Q have unique common fixed point in X .

Proof. The proof follows from Theorem(1) by taking $A = B$.

Corollary 2 Let (X, d) be a complex valued metric space and $A, B, S, T : X \rightarrow X$ be four self mapping satisfying the following conditions

- I $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, By) \lesssim k \frac{[d(Sx, Ax)d(Sx, By)]^2 + [d(Ty, By)d(Ty, Ax)]^2}{1 + [d(Sx, By)]^2 + [d(Ty, Ax)]^2};$$

- III the pairs (A, S) and (B, T) are weakly compatible;
- IV either (A, S) or (B, T) satisfies $(E.A)$ property;
- V either $T(X)$ or $S(X)$ is closed subspaces of X .

Then the mapping A, B, S and T have unique common fixed point in X .

Proof. The proof follows from Theorem(1) by taking $P = T$ and $Q = S$.

Corollary 3 Let (X, d) be a complex valued metric space and $A, T, P : X \rightarrow X$ be three self mapping satisfying the following conditions

- I $A(X) \subseteq T(X)$, $A(X) \subseteq P(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, Ay) \lesssim k \left\{ \frac{d(Tx, Ax)d(Px, Ax)d(Tx, Ay)d(Px, Ay)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, Ay)d(Py, Ay)d(Ty, Ax)d(Py, Ax)}{1 + d(Tx, Ay)d(Px, Ay) + d(Ty, Ax)d(Py, Ax)} \right\};$$

- III the pairs (A, T) and (A, P) are weakly compatible;
- IV the pairs (A, T) and (A, P) satisfies common $(E.A)$ property;
- V both $T(X)$ and $P(X)$ are closed subspaces of X .

Then the mapping A, T and P have unique common fixed point in X .

Proof. The proof follows from Theorem(1) by taking $A = B, T = S$ and $P = Q$.

Corollary 4 Let (X, d) be a complex valued metric space and $A, T : X \rightarrow X$ be two self mapping satisfying the following conditions

- I $A(X) \subseteq T(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, Ay) \lesssim k \frac{[d(Tx, Ax)d(Tx, Ay)]^2 + [d(Ty, Ay)d(Ty, Ax)]^2}{1 + [d(Tx, Ay)]^2 + [d(Ty, Ax)]^2};$$

- III the pair (A, T) is weakly compatible;
- IV the pair (A, T) satisfies $(E.A)$ property;
- V $T(X)$ is closed subspace of X .

Then the mapping A and T have unique common fixed point in X .

Proof. The proof follows from Theorem(1) by taking $A = B$ and $T = S = P = Q$.

Theorem 2 Let (X, d) be a complex valued metric space and $A, B, S, T, P, Q : X \rightarrow X$ be six self mapping satisfying the following conditions

- I $A(X) \subseteq T(X)$, $A(X) \subseteq P(X)$, $B(X) \subseteq S(X)$ and $B(X) \subseteq Q(X)$;

II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, By) \lesssim \begin{cases} k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{d(Sx, By)d(Qx, By)+d(Ty, Ax)d(Py, Ax)} \right. \\ \left. + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{d(Sx, By)d(Qx, By)+d(Ty, Ax)d(Py, Ax)} \right\}, & \text{if } D \neq 0; \\ 0, & \text{if } D=0., \end{cases}$$

where $D = d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)$;

- III the pairs $(A, S), (B, T), (A, Q)$ and (B, P) are weakly compatible;
- IV either both the pairs (A, S) and (A, Q) satisfies common $(E.A)$ property or both the pairs (B, T) and (B, P) satisfies common $(E.A)$ property;
- V either both $T(X)$ and $P(X)$ are closed subspaces of X or both $S(X)$ and $Q(X)$ are closed subspaces of X .

Then the mapping A, B, S, T, P and Q have unique common fixed point in X .

Proof. We can easily prove the Theorem using the same reasoning as in Theorem(1).

Theorem 3 Let (X, d) be a complex valued metric space and $A, B, S, T, P, Q : X \rightarrow X$ be six self mapping satisfying the following conditions

- I $A(X) \subseteq T(X), A(X) \subseteq P(X), B(X) \subseteq S(X)$ and $B(X) \subseteq Q(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, By) \lesssim k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right\};$$

- III the pairs $(A, S), (B, T), (A, Q)$ and (B, P) are weakly compatible;
- IV either both the pairs (A, S) and (A, Q) satisfies common (CLR_A) property or both the pairs (B, T) and (B, P) satisfies common (CLR_B) property.

Then the mapping A, B, S, T, P and Q have unique common fixed point in X .

Proof. Suppose that the pairs (B, T) and (B, P) satisfies common (CLR_B) property. Then by definition(9) there exist two sequences $\{x_n\}$ and $\{x_n^*\}$ in X such that

$$\lim_{n \rightarrow \infty} Bx_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} Bx_n^* = \lim_{n \rightarrow \infty} Px_n^* = t \in B(X). \tag{18}$$

Since $B(X) \subseteq S(X)$, so $Su_1 = t$ for some $u_1 \in X$ and $B(X) \subseteq Q(X)$, so $Qu_2 = t$ for some $u_2 \in X$.

We show that $Au_1 = Su_1$. For this putting $x = u_1$ and $y = x_n$ in condition(II), we have

$$d(Au_1, Bx_n) \lesssim k \left\{ \frac{d(Su_1, Au_1)d(Qu_1, Au_1)d(Su_1, Bx_n)d(Qu_1, Bx_n)}{1 + d(Su_1, Bx_n)d(Qu_1, Bx_n) + d(Tx_n, Au_1)d(Px_n, Au_1)} + \frac{d(Tx_n, Bx_n)d(Px_n, Bx_n)d(Tx_n, Au_1)d(Px_n, Au_1)}{1 + d(Su_1, Bx_n)d(Qu_1, Bx_n) + d(Tx_n, Au_1)d(Px_n, Au_1)} \right\}.$$

Taking limit as $n \rightarrow \infty$, we get

$$d(Au_1, t) \lesssim 0 \text{ or } |d(Au_1, t)| \leq 0 \text{ or } Au_1 = t \text{ and hence } Au_1 = Su_1 = t.$$

Further, since $A(X) \subseteq T(X)$, so there exists $v_1 \in X$ such that $Au_1 = Tv_1$ and hence $Au_1 = Su_1 = Tv_1 = t$.

Next, we show that $Tv_1 = Bv_1$. For this putting $x = u_1$ and $y = v_1$ in condition(II), we have

$$d(Au_1, Bv_1) \lesssim k \left\{ \frac{d(Su_1, Au_1)d(Qu_1, Au_1)d(Su_1, Bv_1)d(Qu_1, Bv_1)}{1 + d(Su_1, Bv_1)d(Qu_1, Bv_1) + d(Tv_1, Au_1)d(Pv_1, Au_1)} \right. \\ \left. + \frac{d(Tv_1, Bv_1)d(Pv_1, Bv_1)d(Tv_1, Au_1)d(Pv_1, Au_1)}{1 + d(Su_1, Bv_1)d(Qu_1, Bv_1) + d(Tv_1, Au_1)d(Pv_1, Au_1)} \right\}$$

$$\text{or } d(t, Bv_1) \lesssim 0 \text{ or } |d(t, Bv_1)| \leq 0 \text{ or } d(t, Bv_1) = 0 \text{ or } Bv_1 = t.$$

Hence

$$Au_1 = Su_1 = Tv_1 = Bv_1 = t. \quad (19)$$

Similarly, we claim that $Au_2 = Qu_2$. For this using triangular inequality, we have $d(Au_2, t) \lesssim d(Au_2, Bx_n^*) + d(Bx_n^*, t)$

$$\text{or } d(Au_2, t) \lesssim k \left\{ \frac{d(Su_2, Au_2)d(Qu_2, Au_2)d(Su_2, Bx_n^*)d(Qu_2, Bx_n^*)}{1 + d(Su_2, Bx_n^*)d(Qu_2, Bx_n^*) + d(Tx_n^*, Au_2)d(Px_n^*, Au_2)} \right. \\ \left. + \frac{d(Tx_n^*, Bx_n^*)d(Px_n^*, Bx_n^*)d(Tx_n^*, Au_2)d(Px_n^*, Au_2)}{1 + d(Su_2, Bx_n^*)d(Qu_2, Bx_n^*) + d(Tx_n^*, Au_2)d(Px_n^*, Au_2)} \right\} + d(Bx_n^*, t).$$

Taking limit as $n \rightarrow \infty$, we get

$$d(Au_2, t) \lesssim 0 \text{ or } |d(Au_2, Bt)| \leq 0 \text{ or } Au_2 = t \text{ and hence } Au_2 = Qu_2 = t.$$

Further, since $A(X) \subseteq P(X)$, so there exists $v_2 \in X$ such that $Au_2 = Pv_2$ and hence $Au_2 = Qu_2 = Pv_2 = t$.

Next, we claim that $Pv_2 = Bv_2$. For this using triangular inequality, we have $d(Pv_2, Bv_2) = d(Au_2, Bv_2)$

$$\text{or } d(Pv_2, Bv_2) \lesssim k \left\{ \frac{d(Su_2, Au_2)d(Qu_2, Au_2)d(Su_2, Bv_2)d(Qu_2, Bv_2)}{1 + d(Su_2, Bv_2)d(Qu_2, Bv_2) + d(Tv_2, Au_2)d(Pv_2, Au_2)} \right. \\ \left. + \frac{d(Tv_2, Bv_2)d(Pv_2, Bv_2)d(Tv_2, Au_2)d(Pv_2, Au_2)}{1 + d(Su_2, Bv_2)d(Qu_2, Bv_2) + d(Tv_2, Au_2)d(Pv_2, Au_2)} \right\}$$

$$\text{or } d(Pv_2, Bv_2) \lesssim 0 \text{ or } |d(Pv_2, Bv_2)| \leq 0 \text{ or } d(Pv_2, Bv_2) = 0 \text{ or } Pv_2 = Bv_2.$$

Hence

$$Au_2 = Qu_2 = Pv_2 = Bv_2 = t. \quad (20)$$

Therefor from (19) and (20), we get

$$Au_1 = Su_1 = Tv_1 = Bv_1 = Au_2 = Qu_2 = Pv_2 = Bv_2 = t. \quad (21)$$

That is t is the common coincident point of A, B, S, T, P and Q .

Now we show that t is the common fixed point of A, B, S, T, P and Q . For this, using the weak compatibility of the pairs (A, S) , (B, T) , (A, Q) , (B, P) and equation(21) we have

$$Au_1 = Su_1 \Rightarrow ASu_1 = SAu_1 \Rightarrow At = St. \quad (22)$$

$$Tv_1 = Bv_1 \Rightarrow BTv_1 = TBv_1 \Rightarrow Bt = Tt. \quad (23)$$

$$Au_2 = Qu_2 \Rightarrow AQu_2 = QAu_2 \Rightarrow At = Qt. \quad (24)$$

$$Pv_2 = Bv_2 \Rightarrow BPv_2 = PBv_2 \Rightarrow Bt = Pt. \quad (25)$$

Now, putting $x = u_1$ and $y = t$ in condition(II), we have

$$d(Au_1, Bt) \lesssim k \left\{ \frac{d(Su_1, Au_1)d(Qu_1, Au_1)d(Su_1, Bt)d(Qu_1, Bt)}{1 + d(Su_1, Bt)d(Qu_1, Bt) + d(Tt, Au_1)d(Pt, Au_1)} + \frac{d(Tt, Bt)d(Pt, Bt)d(Tt, Au_1)d(Pt, Au_1)}{1 + d(Su_1, Bt)d(Qu_1, Bt) + d(Tt, Au_1)d(Pt, Au_1)} \right\}$$

or $d(t, Bt) \lesssim 0$ or $|d(t, Bt)| \leq 0$ or $d(t, Bt) = 0$ or $Bt = t$ and hence from equations (23) and (25), we get

$$Bt = Tt = Pt = t. \quad (26)$$

Similarly, putting $x = t$ and $y = v_1$ in condition(II), we obtained that $At = t$ and hence from equations (22) and (24), we get

$$At = St = Qt = t. \quad (27)$$

Therefor from (26) and (27), we get

$$At = Bt = St = Tt = Pt = Qt = t. \quad (28)$$

That is t is the common fixed point of A, B, S, T, P and Q .

Similar, if we assume that the pairs (A, S) and (A, Q) satisfies common (CLR_A) property, then we can prove that t is the common fixed point of A, B, S, T, P and Q in the same lines as above.

Uniqueness: Assume that t^* be another common fixed point of A, B, S, T, P and Q . Then

$$d(t, t^*) = d(At, Bt^*)$$

$$\text{or } d(t, t^*) \lesssim k \left\{ \frac{d(St, At)d(Qt, At)d(St, Bt^*)d(Qt, Bt^*)}{1 + d(St, Bt^*)d(Qx, Bt^*) + d(Tt^*, At)d(Pt^*, At)} + \frac{d(Tt^*, Bt^*)d(Pt^*, Bt^*)d(Tt^*, At)d(Pt^*, At)}{1 + d(St, Bt^*)d(Qx, Bt^*) + d(Tt^*, At)d(Pt^*, At)} \right\}$$

$$\text{or } d(t, t^*) \lesssim 0 \text{ or } |d(t, t^*)| \leq 0 \text{ or } d(t, t^*) = 0 \text{ or } t = t^*.$$

Hence t is unique common fixed point of A, B, S, T, P and Q .

From above theorem we deduce the following corollaries.

Corollary 5 Let (X, d) be a complex valued metric space and $A, S, T, P, Q : X \rightarrow X$ be five self mapping satisfying the following conditions

- I $A(X) \subseteq T(X), A(X) \subseteq P(X), A(X) \subseteq S(X)$ and $A(X) \subseteq Q(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, Ay) \lesssim k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, Ay)d(Qx, Ay)}{1 + d(Sx, Ay)d(Qx, Ay) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, Ay)d(Py, Ay)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, Ay)d(Qx, Ay) + d(Ty, Ax)d(Py, Ax)} \right\};$$

- III the pairs $(A, S), (A, T), (A, Q)$ and (A, P) are weakly compatible;
- IV either the pairs (A, S) and (A, Q) satisfies common (CLR_A) property or the pairs (A, T) and (A, P) satisfies common (CLR_A) property.

Then the mapping A, S, T, P and Q have unique common fixed point in X .

Proof. The proof follows from Theorem(3) by taking $A = B$.

Corollary 6 Let (X, d) be a complex valued metric space and $A, B, S, T : X \rightarrow X$ be four self mapping satisfying the following conditions

- I $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, By) \lesssim k \frac{[d(Sx, Ax)d(Sx, By)]^2 + [d(Ty, By)d(Ty, Ax)]^2}{1 + [d(Sx, By)]^2 + [d(Ty, Ax)]^2};$$

- III the pairs (A, S) and (B, T) are weakly compatible;
- IV either (A, S) satisfies (CLR_A) property or (B, T) satisfies (CLR_B) property.

Then the mapping A, B, S and T have unique common fixed point in X .

Proof. The proof follows from Theorem(3) by taking $P = T$ and $Q = S$.

Corollary 7 Let (X, d) be a complex valued metric space and $A, T, P : X \rightarrow X$ be three self mapping satisfying the following conditions

- I $A(X) \subseteq T(X)$, $A(X) \subseteq P(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, Ay) \lesssim k \left\{ \frac{d(Tx, Ax)d(Px, Ax)d(Tx, Ay)d(Px, Ay)}{1 + d(Tx, Ay)d(Px, Ay) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, Ay)d(Py, Ay)d(Ty, Ax)d(Py, Ax)}{1 + d(Tx, Ay)d(Px, Ay) + d(Ty, Ax)d(Py, Ax)} \right\};$$

- III the pairs (A, T) and (A, P) are weakly compatible;
- IV the pairs (A, T) and (A, P) satisfies common (CLR_A) property.

Then the mapping A, T and P have unique common fixed point in X .

Proof. The proof follows from Theorem(3) by taking $A = B, T = S$ and $P = Q$

Corollary 8 Let (X, d) be a complex valued metric space and $A, T : X \rightarrow X$ be two self mapping satisfying the following conditions

- I $A(X) \subseteq T(X)$;
- II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, Ay) \lesssim k \frac{[d(Tx, Ax)d(Tx, Ay)]^2 + [d(Ty, Ay)d(Ty, Ax)]^2}{1 + [d(Tx, Ay)]^2 + [d(Ty, Ax)]^2};$$

- III the pair (A, T) is weakly compatible;
- IV the pair (A, T) satisfies (CLR_A) property.

Then the mapping A and T have Theorem(3) unique common fixed point in X .

Proof. The proof follows from Theorem(3) by taking $A = B$ and $T = S = P = Q$.

Theorem 4 Let (X, d) be a complex valued metric space and $A, B, S, T, P, Q : X \rightarrow X$ be six self mapping satisfying the following conditions

- I $A(X) \subseteq T(X)$, $A(X) \subseteq P(X)$, $B(X) \subseteq S(X)$ and $B(X) \subseteq Q(X)$;

II for all $x, y \in X$ and $0 < k < 1$,

$$d(Ax, By) \lesssim \begin{cases} k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{d(Sx, By)d(Qx, By)+d(Ty, Ax)d(Py, Ax)} \right. \\ \left. + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{d(Sx, By)d(Qx, By)+d(Ty, Ax)d(Py, Ax)} \right\}, & \text{if } D \neq 0, \\ 0, & \text{if } D=0, \end{cases}$$

where $D = d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)$;

III the pairs $(A, S), (B, T), (A, Q)$ and (B, P) are weakly compatible;

IV the pairs (A, S) and (A, Q) satisfies common (CLR_A) property or the pairs (B, T) and (B, P) satisfies common (CLR_B) property.

Then the mapping A, B, S, T, P and Q have unique common fixed point in X .

Proof. We can easily prove the Theorem using the same reasoning as in Theorem(3).

At the end of this section we give examples in support of Theorem(1) and Theorem(3)

Example 1 Let $X = \{\frac{-1}{3}\} \cup (0, 3]$ be a metric space with metric $d = e^{\frac{\pi}{8}}|x - y|$, where $x, y \in X$ and A, B, S, T, P, Q be self-maps of X , defined by:

$$\begin{aligned} Ax &= \begin{cases} 1 & \text{if } x \in \{\frac{-1}{3}\} \cup [1, 3] \\ \frac{1}{3} & \text{if } x \in (0, 1) \end{cases} & \text{and } Bx &= \begin{cases} 1 & \text{if } x \in \{\frac{-1}{3}\} \cup [1, 3] \\ \frac{1}{2} & \text{if } x \in (0, 1) \end{cases} \\ Tx &= \begin{cases} 1 & \text{if } x = 1 \\ \frac{4}{3} & \text{if } x \in (0, 1) \\ \frac{x+1}{2} & \text{if } x \in \{\frac{-1}{3}\} \cup (1, 3] \end{cases} & \text{and } Sx &= \begin{cases} 1 & \text{if } x = 1 \\ \frac{3}{2} & \text{if } x \in (0, 1) \\ \frac{x-1}{2} & \text{if } x \in \{\frac{-1}{3}\} \cup (1, 3] \end{cases} \\ Px &= \begin{cases} 1 & \text{if } x = 1 \\ \frac{2}{3} & \text{if } x \in (0, 1) \\ \frac{x-1}{3} & \text{if } x \in \{\frac{-1}{3}\} \cup (1, 3] \end{cases} & \text{and } Qx &= \begin{cases} 1 & \text{if } x = 1 \\ \frac{3}{2} & \text{if } x \in (0, 1) \\ \frac{x+0.4}{3} & \text{if } x \in \{\frac{-1}{3}\} \cup (1, 3] \end{cases} \end{aligned}$$

Then $A(X) = \{1, \frac{1}{3}\}, B(X) = \{1, \frac{1}{2}\}, S(X) = \{\frac{-2}{3}, \frac{3}{2}\} \cup (0, 1],$

$T(X) = [1, 2], P(X) = \{\frac{-4}{9}, 1\} \cup [0, \frac{2}{3}], Q(X) = (0.4666, 1.1333] \cup \{\frac{1}{45}, \frac{3}{2}\}.$

and

(I) $A(X) \subseteq T(X), A(X) \subseteq P(X), B(X) \subseteq S(X)$ and $B(X) \subseteq Q(X).$

(II) To check the condition(II), we have the following cases

Case(1). let $x, y \in (0, 1)$, then

$$Ax = \frac{1}{3}, By = \frac{1}{2}, Sx = \frac{3}{2}, Ty = \frac{4}{3}, Py = \frac{2}{3} \text{ and } Qx = \frac{3}{2}.$$

Now,

$$\begin{aligned} d(Ax, By) &= \frac{1}{6}(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8}) \lesssim 0.71071562 + 1.45581366i \\ &= \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right. \\ &\quad \left. + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right\}. \end{aligned}$$

Hence for $k \in (0, \frac{1}{4})$ one can verify that

$$d(Ax, By) \lesssim k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right\};$$

Case(2). let $x, y \in \{\frac{-1}{3}\} \cup [1, 3]$, then
for $x = y = 1$,

$$Ax = By = Sx = Ty = Py = Qx = 1,$$

for $x \neq 1$ and $y = 1$,

$$Ax = By = Ty = Py = 1, \quad Sx = \frac{x-1}{2}, \quad \text{and} \quad Qx = \frac{x+0.4}{3},$$

for $x = 1$ and $y \neq 1$,

$$Ax = Sx = Qx = By = 1, \quad Ty = \frac{y+1}{2}, \quad \text{and} \quad Py = \frac{y-1}{3},$$

for $x \neq 1$ and $y \neq 1$,

$$Ax = By = 1, \quad Sx = \frac{x-1}{2}, \quad Ty = \frac{y+1}{2}, \quad Py = \frac{y-1}{3} \quad \text{and} \quad Qx = \frac{x+0.4}{3}.$$

Hence in all the above cases and for all $k \in (0, 1)$, we get

$$d(Ax, By) = 0 \lesssim k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right\}.$$

Therefore from the above two cases we obtained that for $k \in (0, \frac{1}{4})$

$$d(Ax, By) \lesssim k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right\};$$

(III) the pairs (A, S) , (B, T) , (A, Q) and (B, P) are weakly compatible;

(IV) let $\{x_n\} = \{3 - \frac{1}{n}\}_{n \geq 1}$ and $\{x_n^*\} = \{2.6 - \frac{1}{n}\}_{n \geq 1}$ be two sequences in X . Then

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ax_n^* = \lim_{n \rightarrow \infty} Qx_n^* = 1 \in X.$$

Hence (A, S) and (A, Q) satisfies common $(E.A)$ property;

(V) $T(X)$ and $P(X)$ are closed subspaces of X .

Therefore from Theorem(3), 1 is a unique common fixed point of A, B, S, T, P and Q .

Example 2 Let $X = (0, 3]$ be a metric space with metric $d = e^{i \frac{\pi}{8}} |x - y|$, where $x, y \in X$ and A, B, S, T, P, Q be self-maps of X , defined by:

$$Ax = \begin{cases} 1 & \text{if } x \in [1, 3] \\ \frac{2}{3} & \text{if } x \in (0, 1) \end{cases} \quad \text{and} \quad Bx = \begin{cases} 1 & \text{if } x \in [1, 3] \\ \frac{1}{2} & \text{if } x \in (0, 1) \end{cases}$$

$$Tx = \begin{cases} 1 & \text{if } x = 1 \\ \frac{4}{3} & \text{if } x \in (0, 1) \\ \frac{x}{2} & \text{if } x \in (1, 3] \end{cases} \quad \text{and} \quad Sx = \begin{cases} 1 & \text{if } x = 1 \\ \frac{3}{2} & \text{if } x \in (0, 1) \\ \frac{x}{3} & \text{if } x \in (1, 3] \end{cases}$$

$$Px = \begin{cases} 1 & \text{if } x = 1 \\ \frac{5}{3} & \text{if } x \in (0, 1) \\ \frac{x}{4} & \text{if } x \in (1, 3] \end{cases} \quad \text{and} \quad Qx = \begin{cases} 1 & \text{if } x = 1 \\ \frac{5}{2} & \text{if } x \in (0, 1) \\ \frac{x}{3} & \text{if } x \in (1, 3] \end{cases}$$

Then

$$A(X) = \left\{1, \frac{2}{3}\right\}, \quad B(X) = \left\{1, \frac{1}{2}\right\}, \quad S(X) = \left(\frac{1}{3}, 1\right] \cup \left\{\frac{3}{2}\right\},$$

$$T(X) = \left(\frac{1}{2}, \frac{3}{2}\right], \quad P(X) = \left(\frac{1}{4}, \frac{3}{4}\right] \cup \left\{1, \frac{5}{3}\right\}, \quad Q(X) = \left(\frac{1}{5}, \frac{3}{5}\right] \cup \left\{1, \frac{5}{2}\right\}.$$

and

- (I) $A(X) \subseteq T(X)$, $A(X) \subseteq P(X)$, $B(X) \subseteq S(X)$ and $B(X) \subseteq Q(X)$;
 (II) for $k \in (0, \frac{7}{26})$ one can verify that

$$d(Ax, By) \lesssim k \left\{ \frac{d(Sx, Ax)d(Qx, Ax)d(Sx, By)d(Qx, By)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} + \frac{d(Ty, By)d(Py, By)d(Ty, Ax)d(Py, Ax)}{1 + d(Sx, By)d(Qx, By) + d(Ty, Ax)d(Py, Ax)} \right\};$$

- (III) the pairs (A, S) , (B, T) , (A, Q) and (B, P) are weakly compatible;
 (IV) let $\{x_n\} = \{3 - \frac{1}{n}\}_{n \geq 1}$ and $\{x_n^*\} = \{3 - \frac{1}{n+1}\}_{n \geq 1}$ be two sequences in X . Then

$$\lim_{n \rightarrow \infty} Ax_n = \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} Ax_n^* = \lim_{n \rightarrow \infty} Qx_n^* = 1 \in A(X).$$

That is (A, S) and (A, Q) satisfies common (CLR_A) property;

Hence from theorem Theorem(3), 1 is a unique common fixed point of A, B, S, T, P and Q .

Note that Theorem(1) is not applicable to example 2 as neither $T(X)$ and $P(X)$ nor $S(X)$ and $Q(X)$ are closed subspaces of X .

Remark

1. We can deduce corollaries from Theorem(2) and Theorem(4) in the same way as obtained from Theorem(1) and Theorem(3).
2. Continuity of mappings and completeness of the whole space are relaxed in Theorems 1, 2, 3 and 4.
3. The common $(E.A)$ and common (CLR) properties does not need the completeness of the whole space and continuity of maps. However, $(E.A)$ property requires the condition of closedness of subspace, while (CLR) property never requires any condition on closedness of subspace. So, (CLR) property is an attractive helping tool for the existence of a common fixed point.
4. Composition of mapping is relaxed in Theorems 1, 2, 3 and 4.

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