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CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY INTEGRAL OPERATORS

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Abstract. The object of the present paper is to prove an interesting results for certain analytic functions belonging to the classes $\mathcal{R}_{\beta}^{\alpha+1}(\delta)$ and $\mathcal{T}^{\alpha+1}(\delta)$.

1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \geq 0). \tag{1}$$

Let $\mathcal{S}(\delta)$ be starlike function of order δ of the form

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta \quad (0 \leq \delta < 1). \tag{2}$$

Jung et al. [3] introduced the following one-parameter families integral operators

$$Q_{\beta}^{\alpha} f(z) = \binom{\alpha+\beta}{\beta} z^{\frac{\alpha}{\beta}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)\, dt \quad (\alpha \geq 0; \beta > -1) \tag{3}$$

$$= \begin{cases} z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{k=2}^{\infty} \frac{\Gamma(\beta+k)}{\Gamma(\alpha+\beta+k)} a_k z^k & (\alpha > 0; \beta > -1) \\ f(z) & (\alpha = 0; \beta > -1) \end{cases}, \tag{4}$$

and

$$I^{\alpha} f(z) = \frac{2^{\alpha}}{z\Gamma(\alpha)} \int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} f(t)\, dt \quad (\alpha \geq 0) \tag{5}$$

$$= \begin{cases} z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^{\alpha} a_k z^k & (\alpha > 0) \\ f(z) & (\alpha = 0) \end{cases}. \tag{6}$$

Using (5) and (6), it easily verify the following identities:

$$z\left(Q_{\beta}^{\alpha+1} f(z)\right)' = (\alpha+\beta+1)\, Q_{\beta}^{\alpha} f(z) - (\alpha+\beta)\, Q_{\beta}^{\alpha+1} f(z) \quad (\alpha > 0; \beta > -1) \tag{7}$$

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and

$$z\left(I^{\alpha+1}f(z)\right)^{'}=2I^{\alpha}f(z)-I^{\alpha+1}f(z)\quad(\alpha>0)\,. \tag{8}$$

We note that

$$Q_c^1 f(z)=F_c(f)(z)=\frac{c+1}{z^c}\int t^{c-1}f(t)\,dt\quad(c>-1)\,, \tag{9}$$

where F_c is the Bernardi-libera-livingston integral operator (see [1, 2, 5]).

Definition 1. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}_{\beta}^{\alpha+1}(\delta)$ if it satisfies inequality

$$Re\left(\frac{Q_{\beta}^{\alpha}f(z)}{Q_{\beta}^{\alpha+1}f(z)}\right)>\frac{\alpha+\beta+\delta}{\alpha+\beta+1}\quad(\alpha>0;\beta>-1;0\le\delta<1;z\in\mathbb{U})\,, \tag{10}$$

we write that

$$\mathcal{R}_{\beta}^{\alpha+1}(0)=\mathcal{R}_{\beta}^{\alpha+1}. \tag{11}$$

Definition 2. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{T}^{\alpha+1}(\delta)$ if it satisfies inquelity

$$Re\left\{\frac{I^{\ \alpha}f(z)}{I^{\ \alpha+1}f(z)}\right\}>\frac{\delta+1}{2}\quad(z\in\mathbb{U};\ 0\le\delta<1)\,, \tag{12}$$

we write that

$$\mathcal{T}^{\alpha+1}(0)=\mathcal{T}^{\alpha+1}. \tag{13}$$

In this paper, we obtain some inclusion relations of the classes $\mathcal{R}_{\beta}^{\alpha+1}(\delta)$ and $\mathcal{T}^{\alpha+1}(\delta)$.

2. Main results

Unless otherwise mentioned, we shell assume in the reminder of this paper that, the parameters $\alpha\ge0$, $\beta>-1$, $0\le\delta<1$, $z\in\mathbb{U}$. We begin with the statement of the following lemma due to Miller and Mocanu [4].

Lemma 1 [4]. Let $\phi(u,v)$ be acomplex valued function,

$$\phi:D\to\mathbb{C}\ ,D\subset\mathbb{C}\times\mathbb{C}\quad(\mathbb{C}\text{ is complex plane}),$$

and let $u=u_1+i\ u_2\ ,v=v_1+iv_2$. Suppose that $\phi(u,v)$ satisfies the following conditions:

(i) $\phi(u,v)$ is continuous in D ;

(ii) $(1,0)\in D$ and $Re\{\phi(1,0)\}>0$;

(iii) for all $(iu_2,v_1)\in D$ such that $v_1\le\frac{-\left(1+u_2^2\right)}{2}$, $Re\{\phi(iu_2,v_1)\}\le0$.

Let $p(z)=1+p_1z+p_2z^2+\ldots.$be regular in the unit disc $\mathbb{U}$. Such that $\left(p(z),zp^{'}(z)\right)\in D$ for all $z\in\mathbb{U}$.If

$$Re\left\{\phi\left(p(z),zp^{'}(z)\right)\right\}>0\quad(z\in\mathbb{U})\,,$$

then $Re\{p(z)\}>0,\ z\in\mathbb{U}$.

Theorem 1. If the function $f\in\mathcal{R}_{\beta}^{\alpha}(\delta)\,,\,0\le\delta<\frac{1}{2}$, then

$$Re\left\{\frac{Q_{\beta}^{\alpha}f(z)}{Q_{\beta}^{\alpha+1}f(z)}\right\}>\gamma\quad(z\in\mathbb{U})\,, \tag{14}$$

where

$$\gamma = \frac{(2\alpha+2\beta+2\delta-1)+\sqrt{(2\alpha+2\beta+2\delta-1)^2+8(\alpha+\beta+1)}}{4(\alpha+\beta+1)}. \tag{15}$$

Proof. We define the function $p(z)$ by

$$\frac{Q_\beta^\alpha f(z)}{Q_\beta^{\alpha+1} f(z)} = \gamma + (1-\gamma)\, p(z), \tag{16}$$

where $\gamma = \gamma(\alpha,\beta,\delta)$ is given by (15). Then $p(z) = 1 + p_1 z + p_2 z^2 + ...$ is regular in the unit disc $\mathbb{U}$. Making use of the logarithmic differentiations of (16) and using (7), we obtain

$$\frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} = \frac{1}{\alpha+\beta}\left\{-1+(\alpha+\beta+1)\left[\gamma+(1-\gamma)p(z)\right]+\frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\right\}, \tag{17}$$

or

$$Re\left\{\frac{Q_\beta^{\alpha-1} f(z)}{Q_\beta^\alpha f(z)} - \frac{\alpha+\beta+\delta-1}{\alpha+\beta}\right\} \tag{2.5}$$

$$= Re\frac{1}{\alpha+\beta}\left\{-(\alpha+\beta+\delta)+(\alpha+\beta+1)\left[\gamma+(1-\gamma)p(z)\right]+\frac{(1-\gamma)zp'(z)}{\gamma+(1-\gamma)p(z)}\right\} > 0. \tag{18}$$

Letting

$$\phi(u,v) = \frac{1}{\alpha+\beta}\left\{-(\alpha+\beta+\delta)+(\alpha+\beta+1)\left[\gamma+(1-\gamma)u\right]+\frac{(1-\gamma)v}{\gamma+(1-\gamma)u}\right\}, \tag{19}$$

we see that

(i) $\phi(u,v)$ is continuous in $D = \left[\mathbb{C} - \left\{\frac{\gamma}{\gamma-1}\right\}\right] \times \mathbb{C}$;

(ii) $(1,0) \in D$ and $Re\{\phi(1,0)\} = \frac{1-\delta}{\alpha+\beta} > 0$;

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \le \frac{-\left(1+u_2^2\right)}{2}$,

$$Re\{\phi(iu_2,v_1)\} = \frac{1}{\alpha+\beta}\left\{-(\alpha+\beta+\delta)+\gamma(\alpha+\beta+1)+\frac{\gamma(1-\gamma)v_1}{\gamma^2+(1-\gamma)^2u_2^2}\right\}$$

$$\le \frac{1}{\alpha+\beta}\left\{-(\alpha+\beta+\delta)+\gamma(\alpha+\beta+1)-\frac{\gamma(1-\gamma)\left(1+u_2^2\right)}{2\left[\gamma^2+(1-\gamma)^2u_2^2\right]}\right\} \le 0$$

$$= \frac{1}{\alpha+\beta}\left\{\frac{\left[2\gamma^3(\alpha+\beta+1)-2\gamma^2\left(\alpha+\beta+\delta-\frac{1}{2}\right)-\gamma\right]+\left\{\left[2\gamma(\alpha+\beta+1)-2(\alpha+\beta+\delta)\right](1-\gamma)^2-\gamma(1-\gamma)\right\}u_2^2}{2\left[\gamma^2+(1-\gamma)^2u_2^2\right]}\right\} \le 0,$$

because

$$2(\alpha+\beta+1)\gamma^2 - 2\left(\alpha+\beta+\delta-\frac{1}{2}\right)\gamma - 1 = 0,$$

and under the condition $0 \le \delta < \frac{1}{2}$, we have

$$0 < \gamma \le 1 - \frac{1-2\delta}{2(\alpha+\beta+1)} < 1.$$

Therefore, the function $\phi(u,v)$ satisfies the conditions in Lemma 1. Since $Re\{p(z)\} > 0$, $z \in \mathbb{U}$, we conclude that

$$Re\left\{\frac{Q_{\beta}^{\alpha} f(z)}{Q_{\beta}^{\alpha+1} f(z)}\right\} > \gamma = \frac{(2\alpha+2\beta+2\delta-1)+\sqrt{(2\alpha+2\beta+2\delta-1)^2+8(\alpha+\beta+1)}}{4(\alpha+\beta+1)}.$$

This completed the proof of Theorem 1. □

Since

$$\begin{aligned} Re\left\{\frac{Q_{\beta}^{\alpha} f(z)}{Q_{\beta}^{\alpha+1} f(z)}\right\} &> \gamma = \frac{(2\alpha+2\beta+2\delta-1)+\sqrt{(2\alpha+2\beta+2\delta-1)^2+8(\alpha+\beta+1)}}{4(\alpha+\beta+1)} \\ &> \frac{\alpha+\beta+\delta}{\alpha+\beta+1}, \end{aligned}$$

we obtain the following corollary.

Corollary 1. For $0 \leq \delta < \frac{1}{2}$, we have

$$\mathcal{R}_{\beta}^{\alpha}(\delta) \subset \mathcal{R}_{\beta}^{\alpha+1}(\delta). \tag{20}$$

Putting $\delta = 0$ in Theorem 1, we obtain the following corollary.

Corollary 2. For $\alpha > 0$ and $\beta > -1$, we have

$$\mathcal{R}_{\beta}^{\alpha} \subset \mathcal{R}_{\beta}^{\alpha+1}. \tag{21}$$

Theorem 2. If the function $f \in \mathcal{T}^{\alpha}(\delta), 0 \leq \delta < \frac{1}{2}$, then

$$Re\left\{\frac{I^{\ \alpha} f(z)}{I^{\ \alpha+1} f(z)}\right\} > \gamma \quad (z \in \mathbb{U}), \tag{22}$$

where

$$\gamma = \frac{(1+2\delta)+\sqrt{(1+2\delta)^2+16}}{8}. \tag{23}$$

Proof. We define the function $p(z)$ by

$$\frac{I^{\ \alpha} f(z)}{I^{\ \alpha+1} f(z)} = \gamma + (1-\gamma)\, p(z), \tag{24}$$

thus the proof of Theorem 2 is the same manner as the above proof of Theorem 1 by using the same Lemma 1. □

Since

$$\begin{aligned} Re\left\{\frac{I^{\ \alpha} f(z)}{I^{\ \alpha+1} f(z)}\right\} &> \gamma = \frac{(1+2\delta)+\sqrt{(1+2\delta)^2+16}}{8} \\ &> \frac{\delta+1}{2}, \end{aligned}$$

we obtain the following corollary.

Corollary 3. For $0 \leq \delta < \frac{1}{2}$, we have

$$\mathcal{T}^{\alpha}(\delta) \subset \mathcal{T}^{\alpha+1}(\delta). \tag{25}$$

Putting $\delta = 0$ in Theorem 2, we obtain the following corollary.

Corollary 4. For $\alpha > 0$ and $\beta > -1$, we have

$$\mathcal{T}^{\alpha} \subset \mathcal{T}^{\alpha+1}. \tag{26}$$

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References

[1] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl., 276 (2002), 432-445.

[2] R. M. Goel and N. S. Sohi, A new criterion for p-valent functions, Proc. Amer. Math. Soc., 78(1980), 353-357.

[3] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators, J. Math. Anal. Appl., 176 (1993), 138-147.

[4] S. S. Miller and P.T. Mocanu, Second order differential inequalities in the complex plane, J. Math. Anal. Appl., 65 (1978), 289-305.

[5] S. Owa and H. M. Srivastava, Some applications of generalized Libera integral operator, Proc. Japan Acad. Ser. A, 62 (1986), 125-128.

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