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# NUMERICAL SOLUTION OF ONE–DIMENSIONAL ADVECTION–DIFFUSION EQUATION WITH VARIABLE COEFFICIENTS VIA LEGENDRE–GAUSS–LOBATTO TIME–SPACE PSEUDO–SPECTRAL METHOD

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Abstract. In this paper, we present a Legendre pseudo–spectral method based on Legendre–Gauss–Lobatto zeros with the aid of the Kronecker product formulation for solving one–dimensional parabolic advection–diffusion equation with variable coefficient subject to a given initial condition and boundary conditions. First, we introduce an approximation to the unknown function by using Legendre differentiation matrices and its derivatives with respect to space $x$ and time $t$. Secondly , we convert our problem to a linear system of equations to unknowns at the collocation points, and then solve it. Finally, two examples are given to illustrate the validity and applicability of the proposed technique with the aid of $L_\infty$-norm error and $L_2$-norm error to the exact solution.

## 1. Introduction

The combination of advection and diffusion is important for mass transport in fluids. It is well known that the volumetric concentration of a pollutant, $u(x,t)$, at a point $x(a \le x \le b)$ in a one-dimensional moving fluid with a constant speed $\beta$ and diffusion coefficient $\alpha$ in $x$ direction at time $t(t \ge 0)$ is given by the one–dimensional time–dependent advection–diffusion equation of the form

$$\frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} = \alpha \frac{\partial^2 u}{\partial x^2}, \quad a \le x \le b, t \ge 0, \tag{1}$$

with initial condition

$$u(x,0) = u_0(x), \quad x \in [a,b],$$

and the boundary conditions

$$\begin{aligned} u(a,t) &= g_1(t), \\ u(b,t) &= g_2(t), \quad t \in [0,T]. \end{aligned}$$

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Many authors deal with the equation (1) numerically. For example, in [1] the authors used cubic B-spline collocation method to find numerical solution to problem (1). The method of the fourth-order compact finite difference scheme was presented in [2].

In this paper we deal with the form

$$\frac{\partial u}{\partial t}+q(x)\frac{\partial u}{\partial x}-p(x)\frac{\partial^2 u}{\partial x^2}=f(x,t), \tag{2}$$

in which $q(x)$ represent a variable speed and diffusion coefficient $p(x)$ in $x$ direction at time $t$ ($t \geq 0$), with $u(x,t) \in [a,b] \times [0,T]$, subject to the initial condition

$$u(x,0)=u_0(x), \quad x \in [a,b], \tag{3}$$

and the boundary conditions represented by

$$\begin{aligned} u(a,t) &= g_1(t), \\ u(b,t) &= g_2(t), \quad t \in [0,T], \end{aligned} \tag{4}$$

where $f(x,t)$, $u_0(x)$, $g_1(t)$ and $g_2(t)$ are known functions and assumed to be smooth functions. Whereas $u$ is the unknown function. Note that $p(x)$ and $q(x)$ are considered to be positive and smooth functions quantifying the diffusion and advection processes, respectively.

One–dimensional version of the partial differential equations which describe advection–diffusion of quantities such as mass, heat, energy, vorticist, etc [3, 4]. Equation (2) has been used to describe heat transfer in a draining film [5], water transfer in soils [6], dispersion of tracers in porous media [7], the intrusion of salt water into fresh water aquifers, the spread of pollutants in rivers and streams [8], the dispersion of dissolved material in estuaries and coastal seas [9], contaminant dispersion in shallow lakes [10], the absorption of chemicals into beds [11], the spread of solute in a liquid flowing through a tube, long–range transport of pollutants in the atmosphere [12], forced cooling by fluids of solid material such as windings in turbo generators [13], thermal pollution in river systems [14], flow in porous media [15] and dispersion of dissolved salts in groundwater [16].

In recent years there has been a high level of interest of employing spectral methods for numerically solving many types of integral and differential equations, due to their ease of applying them for finite and infinite domains [17, 18, 19, 20, 21]. The speed of convergence is one of the great advantages of spectral method. Besides, spectral methods have exponential rates of convergence; they also have high level of accuracy. From the overview of spectral approximation to differential equations, the spectral methods have been divided to four types, namely, collocation [22, 23], tau [24, 25], Galerkin [26, 27], and Petrov Galerkin [28, 29] methods.

In the present contribution, we construct the solution using the pseudo–spectral techniques [30, 31] with Legendre basis. Pseudo–spectral methods are powerful approach for numerical solution of partial differential equations [32, 33, 34], which can be traced back to 1970s [35]. In pseudo–spectral methods [36], there are basically two steps to obtaining a numerical approximation to a solution of differential equation. First, an appropriate finite or discrete representation of the solution must be chosen. This may be done by polynomial interpolation of the solution based on some suitable nodes. In fact, as the number of collocation points increases, interplant polynomials typically diverge. This poor behavior of the polynomial interpolation can be avoided for smoothly differentiable functions by removing the restriction to equally spaced collocation points. Good results are obtained by relating the collocation points to the structure of classical orthogonal polynomials, such as the well-known Legendre-Gauss-Lobatto points. The second step is to obtain a system of algebraic equations from discretization of the original equation. In the case of differential equations, this second step involves finding an approximation for the differential operator (see [35]).

Many authors have considered this technique to solve many problems. In [37, 38], pseudo–spectral scheme to approximate the optimal control problems. Also, a Legendre pseudo–spectral Penalty scheme used for solving time–domain Maxwells equations [39]. The method of Hermite pseudo–spectral scheme is used for Dirac equation [40], and nonlinear partial differential equations [41], respectively. In [42], multidomain pseudo–spectral method for nonlinear convection diffusion equations was presented. Nonlinear Schrödinger equation was discussed in [43] by Time Space pseudo–spectral method with Chebyshev basis. Finally, [44] pseudo–spectral methods used in Quantum and Statistical Mechanics.

The organization of this article is as follows. In Section 2, we present some preliminaries about Legendre polynomials and drive some tools for discretizing the introduced problem. In section 3, we summarize the application of Legendre pseudo–spectral method to the solution of the problem (2)–(4). As a result a set of algebraic linear equations are formed and a solution of the considered problem is discussed. In Section 4, we present some numerical examples to demonstrate the effectiveness of the proposed method.

## 2. Preliminaries and Notations

In this section, we give some notations about most commonly used set of orthogonal polynomials, Legendre polynomials [45, 46] which are defined on the interval [-1,1] and can be determined with the aid of the following.

The Legendre polynomials $L_n(z)$, $n = 0, 1, ...,$ are the Eigenfunctions of the singular Sturm–Liouville problem

$$\frac{d}{dz}\left((1-z^2)\frac{dL_n(z)}{dz}\right) + n(n+1)L_n(z) = 0,$$

they are mutually orthogonal with respect to $L^2_\omega$ inner product on the interval $[-1, 1]$ with the weight function $\omega(x) = 1$, this imply to

$$\int_{-1}^{1} L_n(z)L_m(z)dz = \frac{2}{2n+1}\delta_{nm},$$

where $\delta_{nm}$ is the Kronecker delta. The Legendre polynomials satisfy the following three–term recurrence relations

$$\begin{aligned} L_0(z) &= 1, \quad L_1(z) = z, \\ L_{i+1}(z) &= \frac{2i+1}{i+1} z L_i(z) - \frac{i}{i+1} L_{i-1}(z), \quad i \geq 1, \end{aligned} \tag{5}$$

and

$$L_n(z) = \frac{1}{2(n+1)}\left(L'_{n+1}(z) - L'_{n-1}(z)\right), \quad n \geq 1.$$

The Rodrigues' formula for Legendre polynomials is obtained directly by;

$$L_n(z) = \frac{(-1)^n}{2^n(n!)}\frac{d^n}{dz^n}\left\{(1-z^2)^n\right\}.$$

Let $L_N(z)$ denote the Legendre polynomial of order $N$, then the Legendre–Gauss–Lobatto nodes (**LGL**) nodes will be $z_0^{(N)}, ..., z_N^{(N)}$, where these nodes defined by $z_0^{(N)} = -1, z_N^{(N)} = 1$ and for $\{z_i^{(N)}\}_{i=1}^{N-1}$ are the zeros of $L'_N(z)$. Unfortunately, there are no explicit formulas for the **LGL** nodes is known. However, they can be computed numerically [47].

Let $\{\phi_i^{(N)}(z)\}_{i=0}^{N}$ be the Lagrange polynomials based on **LGL** nodes, that are expressed as [48, 49]:

$$\phi_j^{(N)}(z) = \prod_{i=0, i\neq j}^{N} \frac{z - z_i^{(N)}}{z_j^{(N)} - z_i^{(N)}}, \, j = 0, ..., N, \tag{6}$$

with the Kronecker property

$$\phi_j^{(N)}(z_k^N) = \delta_{jk} = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases}$$

It is more convenient to consider an alternative expression [48, 49], for $j = 0, ..., N$,

$$\phi_j^{(N)}(z_k^N) = \frac{1}{N(N+1)L_N(z_j^{(N)})} \frac{(1-z^2)L'_N(z)}{z - z_j^{(N)}} \tag{7}$$

Any function $f$ defined on the interval $[-1, 1]$ may be approximated by Lagrange polynomials as

$$f(z) \simeq \sum_{i=0}^{N} c_i \phi_i^{(N)}(z), \tag{8}$$

where $c_i = \{f(z_i^{(N)})\}_{i=0}^{N}$. Equation (8) will be exact when $f$ is a polynomial of degree at most $N$. Equation (8) can be expressed in the following matrix form

$$f(z) \simeq \Phi^{(N)}\mathbf{F},$$

where $\Phi^{(N)} = \left[\phi_0^{(N)}(z), ..., \phi_N^{(N)}(z)\right]$ and $\mathbf{F} = [f(z_0^{(N)}), ..., f(z_N^{(N)})]^T$. The first derivative to equation (8) can be expressed as

$$f'(z) \simeq \sum_{i=0}^{N} c_i \phi_i'^{(N)}(z), \tag{9}$$

where $\phi_i'^{(N)}(z)$ is a polynomial of degree $N-1$, which can be written as

$$\phi_i'^{(N)}(z) = \sum_{k=0}^{N} \phi_i'^{(N)}(z_k^{(N)})\phi_k^{(N)}(z), \quad i = 0, ..., N. \tag{10}$$

Equation (10) can be expressed in the following matrix form:

$$\frac{d}{dz}\Phi^{(N)}(z) = \Phi^{(N)}(z)\mathbf{D}_{N+1}, \tag{11}$$

where $\mathbf{D}_{N+1}$ is the so–called differentiation matrix with dimension $N+1$. From the last two equations (10,11) we get $[\mathbf{D}_{N+1}]_{i,k} = \phi_i'^{(N)}(z_k^{(N)})$. The entries of the differentiation matrix can be defined for **LGL** points (cf. [49]) as the following

$$[\mathbf{D}_{N+1}]_{i,k} = \begin{cases} \dfrac{L_N(z_i^{(N)})}{L_N(z_k^{(N)})}\dfrac{1}{z_i^{(N)} - z_k^{(N)}}, & i \neq k, \\ -\dfrac{N(N+1)}{4}, & i = k = 0, \\ \dfrac{N(N+1)}{4}, & i = k = N, \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

Now, we introduce the second order differentiation matrix as $\mathbf{D}^2_{N+1}$ which is the derivative to differentiation matrix $\mathbf{D}_{N+1}$. The entries to the second order differentiation matrix can be defined for **LGL** points (cf. [50]) as the following

$$[\mathbf{D}^2_{N+1}]_{i,j} = \begin{cases} 2[\mathbf{D}_{N+1}]_{i,k}\Big([\mathbf{D}_{N+1}]_{i,i} - \dfrac{1}{z_i^{(N)} - z_k^{(N)}}\Big), & i \neq k \\ -\displaystyle\sum_{i=0, i\neq k}^{N} [\mathbf{D}^2_{N+1}]_{i,k}, & i = k. \end{cases} \tag{13}$$

Also, any defined function $h(x)$ on an arbitrary interval $[a, b]$ may be approximated by making transformation from $z \in [-1, 1]$ to $x \in [a, b]$ as:

$$h(x) \simeq \sum_{i=0}^{N} h(x_i^{(N)})\phi_i^{(N)}(\frac{2}{b-a}(x-a) - 1), \tag{14}$$

where $x_i^{(N)} = \{\frac{b-a}{2}(z_i^{(N)} + 1) + a\}_{i=0}^{N}$ are the shifted **LGL** nodes associated with interval $[a, b]$. Equation (14) can be expressed in the following matrix form:

$$h(x) \simeq \Phi^{(N)}_{[a,b]}(x)\mathbf{H}. \tag{15}$$

In view of equations (11) and (14), we conclude that

$$\frac{d^i}{dx^i}\Phi^{(N)}_{[a,b]}(x) = (\frac{2}{b-a})^i\Phi^{(N)}_{[a,b]}(x)\mathbf{D}^i_{N+1}, \tag{16}$$

For an arbitrary $N$ and $M$, any function of two variables $u : [a, b] \times [c, d] \to \mathbf{R}$ may be approximated by

$$u(x, y) \simeq \sum_{i=0}^{N}\sum_{j=0}^{M} U_{i,j}\phi_i^{(N)}(\frac{2}{b-a}(x-a) - 1)\phi_j^{(M)}(\frac{2}{d-c}(y-c) - 1), \tag{17}$$

where

$$U_{i,j} = u\Big(\frac{b-a}{2}(z_i^{(N)} + 1) + a, \frac{d-c}{2}(z_j^{(M)} + 1) + c\Big). \tag{18}$$

Equation (17) can be expressed based on Kronecker product in the following matrix form:

$$u(x, y) \simeq \Big(\Phi^{(N)}_{[a,b]}(x) \otimes \Phi^{(M)}_{[c,d]}(y)\Big)\mathbf{U}, \tag{19}$$

where $\mathbf{U}$ is the $(N + 1)(M + 1)$ vector as the following form:

$$\mathbf{U} = [U_{0,0}, ..., U_{0,M} \mid U_{1,0}, ..., U_{1,M} \mid ... \mid U_{N,0}, ..., U_{N,M}]^T \tag{20}$$

The previous representations that are based on Kronecker product, provide some simplification in calculations when we deal with our original problem. Also by using first and second differentiation matrices we can approximate relative derivatives of any function from its expansion as we can see next. For example let $u$ be approximated as in (19), now we can write the first derivative to $u$ with respect to $x$ as the following:

$$\begin{aligned} u_x(x,y) &\simeq \Big(\frac{d}{dx}\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[c,d]}(y)\Big)\mathbf{U} \\ &= \frac{2}{b-a}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}_{N+1}\otimes\Phi^{(M)}_{[c,d]}(y)\Big)\mathbf{U} \\ &= \frac{2}{b-a}\Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[c,d]}(y)\Big)\Big(\mathbf{D}_{N+1}\otimes\mathbf{I}_{M+1}\Big)\mathbf{U}. \end{aligned} \tag{21}$$

In a similar way, we can conclude that the first derivative to $u$ with respect to $y$ as the following:

$$u_y(x,y) \simeq \frac{2}{d-c}\Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[c,d]}(y)\Big)\Big(\mathbf{I}_{N+1}\otimes\mathbf{D}_{M+1}\Big)\mathbf{U}. \tag{22}$$

## 3. Legendre Pseudo–spectral Approximation

In order to solve problem (2)–(4), we approximate $u(x,t)$ as:

$$u(x,t) \simeq \Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[0,T]}(t)\Big)\mathbf{U}, \tag{23}$$

where the positive and integer numbers $N$ and $M$ are discretization parameters corresponding to space and time dimensions, respectively. Also we will consider $\{x_i\}_{i=0}^N$ and $\{t_j\}_{j=0}^M$ as the **LGL** nodes corresponding to the intervals $[a,b]$ and $[0,T]$, respectively.

By using (23) and differentiation matrices, we can write the derivatives to $u(x,t)$ as the following

$$u_x(x,t) \simeq \frac{2}{b-a}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}_{N+1}\otimes\Phi^{(M)}_{[0,T]}(t)\Big)\mathbf{U}, \tag{24}$$

$$u_{xx}(x,t) \simeq \frac{4}{(b-a)^2}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}^2_{N+1}\otimes\Phi^{(M)}_{[0,T]}(t)\Big)\mathbf{U}, \tag{25}$$

$$u_t(x,t) \simeq \frac{2}{T}\Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[0,T]}(t)\mathbf{D}_{M+1}\Big)\mathbf{U}. \tag{26}$$

Now, by substituting from the previous equations in equation (2), we obtain

$$\begin{aligned} &\Big[\frac{2}{T}\Big(\Phi^{(N)}_{[a,b]}(x)\otimes\Phi^{(M)}_{[0,T]}(t)\mathbf{D}_{M+1}\Big) + q(x)\frac{2}{b-a}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}_{N+1}\otimes\Phi^{(M)}_{[0,T]}(t)\Big) \\ &-p(x)\frac{4}{(b-a)^2}\Big(\Phi^{(N)}_{[a,b]}(x)\mathbf{D}^2_{N+1}\otimes\Phi^{(M)}_{[0,T]}(t)\Big)\Big]\mathbf{U} = f(x,t). \end{aligned} \tag{27}$$

Now, for $1<i<N-1$ and $1<j<M$, we collocate the above equation at the collocation points $\{(x_i,t_j)\}_{i,j}$. Note that these collocation points are the interior points not lie in initial or boundary conditions. After collocating, equation (27) becomes:

$$\begin{aligned} &\Big[\frac{2}{T}\Big(e^{N+1}_{i+1}\otimes e^{M+1}_{j+1}\mathbf{D}_{M+1}\Big) + q(x_i)\frac{2}{b-a}\Big(e^{N+1}_{i+1}\mathbf{D}_{N+1}\otimes e^{M+1}_{j+1}\Big) \\ &-p(x_i)\frac{4}{(b-a)^2}\Big(e^{N+1}_{i+1}\mathbf{D}^2_{N+1}\otimes e^{M+1}_{j+1}\Big)\Big]\mathbf{U}_1 = f(x_i,t_j), \\ &\qquad i=1,\cdots,N-1, \quad j=1,\cdots,M, \end{aligned} \tag{28}$$

where $e_k^p$ is the $k^{\text{th}}$ row of $p \times p$ identity matrix. Equation (28) can be represented in the following matrix form using identity matrix:

$$\Big[\frac{2}{T}\Big([\mathbf{I}]_2^N \otimes [\mathbf{I}]_2^{M+1}\mathbf{D}_{M+1}\Big) + q(x_i)\frac{2}{b-a}\Big([\mathbf{I}]_2^N\mathbf{D}_{N+1} \otimes [\mathbf{I}]_2^{M+2}\Big) -p(x_i)\frac{4}{(b-a)^2}\Big([\mathbf{I}]_2^N\mathbf{D}_{N+1}^2 \otimes [\mathbf{I}]_2^{M+1}\Big)\Big]\mathbf{U}_1 = \mathbf{F}_1, \tag{29}$$

which can be formed as

$$\mathbf{A}_1\mathbf{U}_1 = \mathbf{F}_1, \tag{30}$$

where $\mathbf{F}_1$ and $\mathbf{U}_1$ are the $(N-1)(M)$ vectors they take the following forms:

$$\begin{aligned}\mathbf{F}_1 &= [f_{1,1}, ..., f_{1,M} \mid \cdots \mid f_{N-1,1}, ..., f_{N-1,M}]^T,\\ \mathbf{U}_1 &= [U_{1,1}, ..., U_{1,M} \mid \cdots \mid U_{N-1,1}, ..., U_{N-1,M}]^T,\end{aligned}$$

and $\mathbf{A}_1$ is a matrix of dimension $N(N-1) \times (M+1)^2$, that can be defined as

$$\begin{aligned}\mathbf{A}_1 = &\Big[\frac{2}{T}\Big([\mathbf{I}]_2^N \otimes [\mathbf{I}]_2^{M+1}\mathbf{D}_{M+1}\Big) + q(x_i)\frac{2}{b-a}\Big([\mathbf{I}]_2^N\mathbf{D}_{N+1} \otimes [\mathbf{I}]_2^{M+2}\Big)\\ &-p(x_i)\frac{4}{(b-a)^2}\Big([\mathbf{I}]_2^N\mathbf{D}_{N+1}^2 \otimes [\mathbf{I}]_2^{M+1}\Big)\Big].\end{aligned}$$

For discretization the initial condition, we substitute (27) in (3) getting the following

$$\Big(\Phi_{[a,b]}^{(N)}(x) \otimes \Phi_{[0,T]}^{(M)}(0)\Big)\mathbf{U} = u_0(x), \quad a \le x \le b,$$

Now, for $0 < i < N$, we collocate the above equation at the collocation points $\{(x_i, 0)\}$. After collocating, the previous equation becomes:

$$\Big(e_{i+1}^{N+1} \otimes e_1^{M+1}\Big)\mathbf{U}_2 = u_0(x_i), \tag{31}$$

then in matrix form using identity matrix

$$\Big([\mathbf{I}]_1^{N+1} \otimes e_1^{M+1}\Big)\mathbf{U}_2 = \mathbf{U}_0, \tag{32}$$

which can be formed as

$$\mathbf{A}_2\mathbf{U}_2 = \mathbf{U}_0, \tag{33}$$

where $\mathbf{U}_0$ and $\mathbf{U}_2$ are the $(N+1)$ vectors, they can be described as the following forms:

$$\begin{aligned}\mathbf{U}_0 &= [u_0(x_0), ..., u_0(x_N)]^T,\\ \mathbf{U}_2 &= [U_{0,0}, ..., U_{N,0}]^T,\end{aligned}$$

and $\mathbf{A}_2$ is a matrix of dimension $(N+1) \times (N+1)^2$, that has the following form

$$\mathbf{A}_2 = \Big([\mathbf{I}]_1^{N+1} \otimes e_1^{M+1}\Big).$$

Finally, to discrete the boundary conditions, we substitute (27) in (4). First, we deal with the left boundary to find the reduced form, then doing the same with the right boundary. Equation (4) will be

$$\Big(\Phi_{[a,b]}^{(N)}(a) \otimes \Phi_{[0,T]}^{(M)}(t)\Big)\mathbf{U} = g_1(t), \tag{34}$$

Now, for $1 < j < M$, we collocate the above equation at the collocation points $\{(a, t_j)\}$ for the first boundary condition. After collocating, the previous equation becomes:

$$\Big(e_1^{N+1} \otimes e_{j+1}^{M+1}\Big)\mathbf{U}_3 = g_1(t_j), \tag{35}$$

then in matrix form using identity matrix

$$\left(e_1^{N+1} \otimes [\mathbf{I}]_2^{M+1}\right)\mathbf{U}_3 = \mathbf{G}_1, \tag{36}$$

which can be formed as

$$\mathbf{A}_3\mathbf{U}_3 = \mathbf{G}_1, \tag{37}$$

where $\mathbf{G}_1$ and $\mathbf{U}_3$ are the $(M)$ vectors, they can be described as the following forms:

$$\begin{aligned}\mathbf{G}_1 &= [g_1(t_1), ..., g_1(t_M)]^T,\\ \mathbf{U}_3 &= [U_{0,1}, ..., U_{0,M}]^T,\end{aligned}$$

and $\mathbf{A}_3$ is a matrix of dimension $(M) \times (M+1)^2$, that has the following form

$$\mathbf{A}_3 = \left(e_1^{N+1} \otimes [\mathbf{I}]_2^{M+1}\right).$$

Similarly, we can write the equation of the second boundary condition as the following form

$$\left(e_{N+1}^{N+1} \otimes [\mathbf{I}]_2^{M+1}\right)\mathbf{U}_4 = \mathbf{G}_2, \tag{38}$$

which can be formed as

$$\mathbf{A}_4\mathbf{U}_4 = \mathbf{G}_2, \tag{39}$$

where $\mathbf{G}_2$ and $\mathbf{U}_4$ are the $(M)$ vectors, they can be described as the following forms:

$$\begin{aligned}\mathbf{G}_2 &= [g_2(t_1), ..., g_2(t_M)]^T,\\ \mathbf{U}_4 &= [U_{N,1}, ..., U_{N,M}]^T,\end{aligned}$$

and $\mathbf{A}_4$ is a matrix of dimension $(M) \times (M+1)^2$, that has the following form

$$\mathbf{A}_4 = \left(e_{N+1}^{N+1} \otimes [\mathbf{I}]_2^{M+1}\right).$$

The resulting system of equations can be described, from collecting equations (30), (33), (37) and (39), as the following

$$\mathbf{A}\mathbf{U} = \mathbf{F}, \tag{40}$$

where $\mathbf{A}$ is a matrix of dimension $(N+1)^2 \times (M+1)^2$, that has the form $\mathbf{A} = [\mathbf{A}_1 \mid \mathbf{A}_2 \mid \mathbf{A}_3 \mid \mathbf{A}_4]$. For $\mathbf{U}$ and $\mathbf{F}$, each one is a vector with dimension $(M+1)^2$, and take the following form

$$\begin{aligned}\mathbf{U} &= [\mathbf{U}_1 \mid \mathbf{U}_2 \mid \mathbf{U}_3 \mid \mathbf{U}_4]^T,\\ \mathbf{F} &= [\mathbf{F}_1 \mid \mathbf{U}_0 \mid \mathbf{G}_1 \mid \mathbf{G}_2]^T.\end{aligned}$$

After solving the linear system described in (40), we can find the approximated solution to our problem (2).

## 4. Numerical Examples

In order to test the utility of the proposed method, we apply the new scheme to the following examples whose exact solutions are provided in each case. For both examples, we take $N = M$ and to show the efficiency of the presented method for our problems in comparison with the exact solution. Also, to study the convergence behavior of the presented method, we applied the following laws for different values of $N$ and for $t = T$:

- The $\|E\|_\infty$ error norm of the solution which is defined by

$$\|E\|_\infty = \|U(x,t) - u(x,t)\|_\infty = \max_{1\le i\le N-1} \left|U_{i,M} - u(x_i, t_M)\right|,$$

- The $\|E\|_2$ error norm of the solution which is defined by

$$\|E\|_2 = \|U(x,t) - u(x,t)\|_2 = \left[\sum_{i=1}^{N-1} \Big(U_{i,M} - u(x_i, t_M)\Big)^2\right]^{1/2},$$

- The condition number $K_g(\mathbf{A})$ of the coefficient matrix $\mathbf{A}$ is given by

$$K_g(\mathbf{A}) = \|\mathbf{A}\|_g \|\mathbf{A}^{-1}\|_g, \qquad g = 2, \infty.$$

Finally, we compare our presented method with B-spline finite difference method presented in [51].

All the computations are carried out in double precision arithmetic using Matlab 7.9.0 (R2009b). To obtain sufficient accurate calculations, variable arithmetic precision (vpa) is employed with digit being assigned to be 32. The code was executed on a second generation Intel Core i52410M, 2.3 Ghz Laptop. Finally, the CPU time indicates the time for all calculations of operations in the solution of the entire problem is presented.

**Example 4.1.** [1] *Consider the problem (2)–(4) with the initial condition* $u(x,0) = \sin(\pi x), \quad 0 \le x \le 1$, *and the boundary conditions are given as*

$$\begin{cases} u(0,t) = 0, \\ u(1,t) = 0, \end{cases} \qquad 0 \le t \le 1,$$

*and the exact solution* $u(x,t) = \sin(\pi x)e^{-\pi^2 t}$, *with* $p(x) = x/(1+x^2)$ *and* $q(x) = e^x$, *in this case the forcing function will be*

$$f(x,t) = e^{-\pi^2 t}\Big[\pi^2 \sin(\pi x)\big(p(x) - 1\big) + q(x)\pi\cos(\pi x)\Big].$$

*In Comparing with B-Spline finite difference method* [?], *the maximum error was*

Table 1. $\|E\|_\infty$ error, $\|E\|_2$ error, condition number of $g = \infty, g = 2$ with different values of $N$ for Example 4.1.

| *N* | $\|E\|_\infty$ | $K_\infty(\mathbf{A})$ | $\|E\|_2$ | $K_2(\mathbf{A})$ | *CPU(s)* |
|---|---|---|---|---|---|
| *6* | *2.58855E-04* | *8.277e+2* | *3.6454E-04* | *3.181e+2* | *1.9054* |
| *8* | *1.08734E-05* | *2.335e+3* | *1.7818E-05* | *9.208e+2* | *1.9290* |
| *10* | *4.64040E-07* | *5.497e+3* | *3.8690E-07* | *2.209e+3* | *2.0593* |
| *12* | *1.72755E-08* | *1.134e+4* | *3.6828E-08* | *4.635e+3* | *2.9796* |
| *14* | *5.06475E-10* | *2.145e+4* | *1.1478E-09* | *8.794e+3* | *4.0006* |
| *16* | *1.17347E-11* | *3.806e+4* | *2.7774E-11* | *1.544e+4* | *6.5005* |
| *18* | *2.17164E-13* | *6.350e+4* | *5.3401E-13* | *2.550e+4* | *13.592* |
| *20* | *1.38658E-14* | *1.007e+5* | *2.0433E-14* | *4.009e+4* | *40.912* |

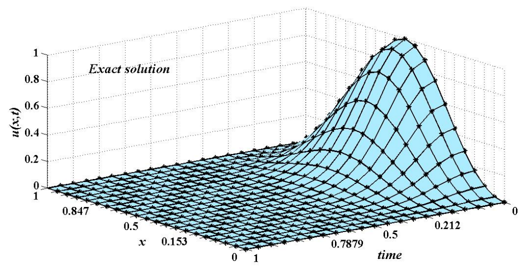
$4.44E-05$ *at* $T = 1$ *for* $\triangle x = 0.01$ *and* $\triangle t = 0.001$, *making CPU-time equal to* 12.3226459 *sec.*

**Example 4.2.** [1, 2] *Consider the problem (2)–(4) with the initial condition*

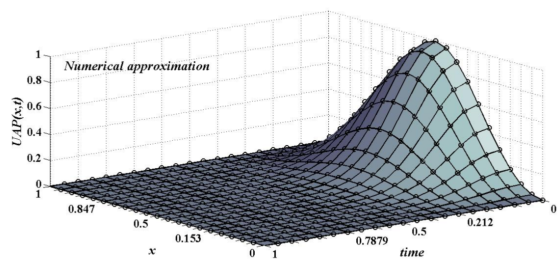
$$u(x,0) = e^{5x}\Big(\cos(\frac{\pi}{2}x) + 0.25\sin(\frac{\pi}{2}x)\Big), \quad 0 \le x \le 1,$$

*and the boundary conditions given by*

$$\begin{cases} u(0,t) = e^{-C_0 t}, \\ u(1,t) = 0.25e^{5-C_0 t}, \end{cases} \qquad 0 \le t \le 2,$$




(a) Exact solution




(b) Numerical solution

FIGURE 1. Exact and Numerical solutions for introduced $p(x)$, $q(x)$ with $x \in [0,1]$ and $t \in [0,1]$ at $N = 20$ for Example 4.1

*and the exact solution*

$$u(x,t) = e^{5x - C_0 t}\Big(\cos(\frac{\pi}{2}x) + 0.25\sin(\frac{\pi}{2}x)\Big),$$

*we take* $p(x) = xe^{-x}/(1+x^2)$ *and* $q(x) = e^{x}/(1+x^2)$*, in this case the forcing function will be*

$$\begin{aligned} f(x,t) &= \Big\{\cos(\frac{\pi}{2}x)\big[-C_0 + C_1 q(x) - p(x)(5C_1 + \frac{\pi}{2}C_1)\big] \\ &+ \sin(\frac{\pi}{2}x)\big[\frac{-C_0}{4} + C_2 q(x) - p(x)(5C_2 - \frac{\pi}{2}C_1)\big]\Big\}.e^{5x - C_0 t}, \end{aligned}$$

*where*

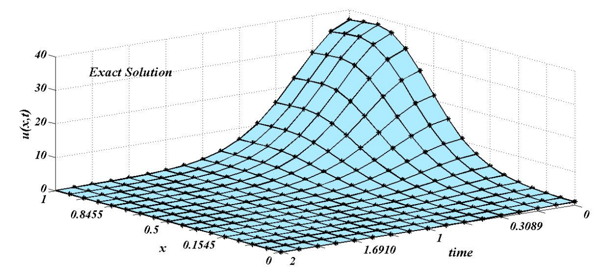
$$C_0 = \frac{\pi^2}{2} + \frac{5}{2}, \quad C_1 = 5 + \frac{\pi}{8}, \quad C_2 = \frac{5}{4} - \frac{\pi}{2}.$$

*In Comparing with B-Spline finite difference method* [**?**]*, the maximum error was* $1.451977E-03$ *at* $T = 2$ *for* $\triangle x = 0.01$ *and* $\triangle t = 0.001$*, making CPU-time equal to* $25.721$ *sec.*

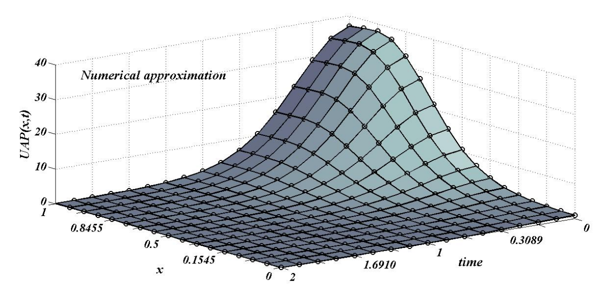
In Tables " 1 and 2", shows the absolute (*Error*) between the exact and numerical solutions, $\|E\|_\infty$-error, $K_\infty(\mathbf{A})$, $\|E\|_2$-error, $K_2(\mathbf{A})$ and CPU-time in some points of the interval $(0,1)$ and $T = 1$ for $6 \le N \le 20$. These tables indicates that as $N$ increases, the Error decreases more rapidly (exponentially). From Tables " 1

TABLE 2. $\|E\|_\infty$ error, $\|E\|_2$ error, condition number of $g = \infty, g = 2$ with different values of $N$ for Example 4.2.

| $N$ | $\|E\|_\infty$ | $K_\infty(\mathbf{A})$ | $\|E\|_2$ | $K_2(\mathbf{A})$ | *CPU(s)* |
|---|---|---|---|---|---|
| *6* | *1.77173E-03* | *4.210e+2* | *2.0244E-03* | *1.476e+2* | *2.288* |
| *8* | *4.51964E-05* | *1.120e+3* | *5.6853E-05* | *3.992e+2* | *2.540* |
| *10* | *7.02708E-07* | *2.472e+3* | *9.0166E-07* | *9.264e+2* | *3.091* |
| *12* | *7.59213E-09* | *4.959e+3* | *1.0370E-08* | *1.906e+3* | *4.273* |
| *14* | *6.93690E-11* | *9.167e+3* | *9.8183E-11* | *3.573e+3* | *6.013* |
| *16* | *4.93855E-13* | *1.583e+4* | *1.1083E-12* | *6.223e+3* | *8.476* |




(a) Exact solution




(b) Numerical solution

FIGURE 2. Exact and Numerical solutions for introduced $p(x)$, $q(x)$ with $x \in [0, 1]$ and $t \in [0, 2]$ at $N = 16$ for Example 4.2

and 2", it can be observed that the accuracy increases with the increase of number of collocation points.

## 5. Conclusion

In this work, we applied Legendre Pseudo-spectral method for one-dimensional advection-diffusion equation with variable coefficients on Legendre-Gauss-Lobatto nodes. The differentiation matrices are used to represent the unknown functions. Two examples are introduced in this article to show that the proposed numerical procedure is efficient and provides very accurate results even with using a small number of collocation points. The stability of the resulting system was proved by utility of the condition number $K_g(\mathbf{A})$. Finally, The Pseudo-spectral scheme is a powerful approach for the numerical solution of parabolic advection-diffusion equation.

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