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EXISTENCE OF PERIODIC SOLUTIONS OF 2α-ORDER NONLINEAR FUNCTIONAL DIFFERENCE EQUATIONS WITH $p-$LAPLACIAN

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Abstract. The existence of periodic solutions of a higher order nonlinear functional difference equation with p-Laplacian is studied. Sufficient conditions for the existence of periodic solutions of such equation are established. The result is based on Mawhin′s continuation theorem. The methods used to estimate the priori bound on periodic solutions are very technical.

1. Introduction

In recent years, there has been a large amount of attention paid to the study on the dynamic properties of solutions of the difference equations that arise from various applied problems [4, 5, 6, 7, 11, 12, 27, 16, 17, 20, 30, 31, 32, 33, 34] and [8, 9, 10, 36].

Consider the difference equation of the form

$$y_{n+1} - y_n + f(n, y_n, y_{n-1}, \cdots, y_{n-k}) = 0, \quad n \in Z. \tag{1}$$

Many authors discussed the properties, such as permanence, existence of periodic solutions, stability and oscillatory properties, of equation (1) or its special cases, see the text books [14, 18, 19, 21, 22, 23, 24, 26, 25].

In [27], Furumochi and Naito considered the following first order difference equation

$$x_{n+1} = f(n, x_n), n \in Z, \tag{2}$$

by using Schauder's fixed point theorem, sufficient conditions are obtained for (2) to have periodic solutions

In [11, 12], the authors studied the existence of periodic solutions for the (k+1)-th order difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), \quad n \in Z \tag{3}$$

and established the necessary and sufficient conditions that make all solutions of (3) are periodic.

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In [34], using Schaefer′s fixed-point theorem, Raffoul showed that if there is a priori bound on all possible T-periodic solutions of a Volterra-type difference equation

$$\Delta x(n) = \lambda \left(Dx(n) + \sum_{j=-\infty}^{n} (n-j)x(j) + g(n) \right) \text{ with } \sum_{j=0}^{\infty} |C(j)| < \infty,$$

then there is a T-periodic solution of the difference equations

$$\Delta x(n) = Dx(n) + \sum_{j=-\infty}^{n} (n-j)x(j) + g(n) \text{ with } \sum_{j=0}^{\infty} |C(j)| < \infty.$$

The priori bound of solutions of the first equation is established by means of a Lyapunov functional on which no bound is required.

It is well known that the bending of elastic beam can be described with some fourth-order p-Laplacian differential equations. Recently, in [5], The authors considered the functional difference equation

$$\Delta^2(r_{n-2}\Delta^2 x_{n-2}) + f(n, x_n) = 0,\ n \in Z, \tag{4}$$

where $f : Z \times R \to R$ is a continuous function in the second variable, $f(n+T, z) = f(n, z)$, $r_{n+T} = r_n$, for all $(n, z) \in Z \times R$, and T is a positive integer. equation (4) is a discrete form of the nonlinear elastic beam equation

$$[r(t)x''(t)]'' + f(t, x_t) = 0, t \in R.$$

By using linking theorem, the authors obtained some new criteria for the existence and multiplicity of periodic solutions of equation (4).

Then in [6], the authors obtained some new sufficient conditions for the existence of nontrivial m-periodic solutions of the following nonlinear difference equation

$$\Delta(p_n \Delta^\delta x_{n-1}) + f(n, x_n) = 0,\ n \in Z, \tag{5}$$

by using the critical point method, where $f : Z \times R \to R$ is continuous in the second variable, $m \geq 2$ is a given positive integer, $p_{n+m} = p_n$ for any $n \in Z$ and $f(t+m, z) = f(t, z)$ for any $(t, z) \in Z \times R$, $(-1)^\delta = -1$ and $\delta > 0$.

For more general higher order functional difference equation

$$\Delta^n(r_{n-t}\Delta^n x_{n-t}) + f(t, x_t) = 0,\ n \in Z(3),\ \ t \in Z, \tag{6}$$

where $f : Z \times R \to R$ is a continuous function in the second variable, $f(t+T, z) = f(t, z)$ for all $(t, z) \in Z \times R$, $r_{t+T} = r_t$ for all $t \in Z$, and T a given positive integer. By the Linking Theorem, in [7], some new criteria were obtained for the existence and multiplicity of periodic solutions of equation (6).

The motivation of this paper also comes from papers [4, 33, 20, 16, 29]. In paper [4], Atici and Guseinov investigated the problem

$$\begin{cases} -\Delta(p(n-1)\Delta y(n-1)) + q(n)y(n) = f(n, y(n)), & n \in [1, N], \\ y(0) = y(N),\ p(0)\Delta y(0) = p(N)\Delta y(N), \end{cases} \tag{7}$$

by using a fixed point theorem in cones in Banach space, the existence results for positive solutions of BVP(7) were established.

In [16], by using the dual least action principle, the authors proved some existence theorems for periodic solutions of second order discrete convex systems involving the p-Laplacian

$$\Delta[\phi_p(\Delta x(t-1))] + \nabla F(t, x(t)) = 0, t \in Z,$$

where ϕ_p is p-Laplacian operator, i.e.,

$$\phi_p(x) = |x|^{p-2}x = \left(\sqrt{\sum_{i=1}^{N} x_i^2}\right)^{p-2} (x_1, , x_2, \cdots , x_N)^\tau,$$

$x \in R^N$, $p > 1$, τ stands for the transpose of a vector or a matrix, $F : Z \times R^N \to R$, $F(t, x))$ is continuously differentiable and convex in $x \in R^N$ for every $t \in Z$ and T-periodic in t for all $x \in R^N$, $\nabla F(t, x(t))$ denotes the gradient of $F(t, x)$ in x.

In [20], the author concerned with the existence of at least one T-periodic solution of nonlinear functional difference equation

$$\Delta x(n) + a(n)x(n) = f(n, x(n), x(\tau_1(n)), \cdots , x(\tau_m(n))), n \in Z$$

with $\prod_{j=0}^{T-1}(1 - a(j)) \neq 1$. Sufficient conditions for the existence of T-periodic solution of above equation was established.

Motivated by [1, 2, 3, 15, 28, 35], in what follows we seek to enrich the discussion found in the above cited literature by exploring the existence of periodic soltions of the discrete functional difference equations heretofore not considered. We study the higher order nonlinear functional difference equations with p-Laplacian

$$\Delta^\alpha[p(n)\phi(\Delta^\alpha x(n))] = (-1)^\alpha f(n, x(n + \alpha), x(\tau_1(n)), \cdots , x(\tau_m(n))),\ n \in Z, \quad (8)$$

where $\alpha \in Z(1) = \{1, 2, 3, \cdots \}$, Z is the integers set, $p(n)$ is a positive T-periodic sequence, $\phi : R \to R$ with $\phi(x) = |x|^{r-2}x$ for $x \neq 0$ and $\phi(0) = 0$, its inverse defined by ϕ^{-1} with $\phi^{-1}(x) = |x|^{t-2}x$ for $x \neq 0$ and $\phi^{-1}(0) = 0$, where $r > 1, t > 1$ with $1/r + 1/t = 1$, $\tau_i(i = 1, \cdots , m)$ are T-periodic sequences, $f(n, u)$ is T-periodic in n and continuous in $u = (x_0, \cdots , x_m)$.

The purpose of this paper is to establish sufficient conditions for the existence of at least one $T-$periodic solution of equation (8) by using coincidence degree theory of Mawhin. Equation (8) is more general than equations (4), (5), (6) and (7), respectively. The methods in this paper are motivated by paper [34] and are different from those used in papers [4, 5, 6, 7], the priori bound of solutions of (8) is established by means of a new way that is extensively different from the Lyapunov functional methods used in [34]. It is interesting that we allow that f to be sublinear, at most linear or superlinear.

This paper is organized as follows. In Section 2, we give the main results, and in Section 3, examples to illustrate the main results will be presented.

2. Main Results

To get existence results for solutions of equation (8), we need the following fixed point theorem.

Let X and Y be Banach spaces, $L :\ D(L) \subset X \to Y$ be a Fredholm operator of index zero, $P :\ X \to X$, $Q :\ Y \to Y$ be projectors such that

$$\text{Im } P = \text{Ker } L,\ \text{Ker } Q = \text{Im } L,\ \ X = \text{Ker } L \oplus \text{Ker } P,\ \ Y = \text{Im } L \oplus \text{Im } Q.$$

It follows that

$$L|_{D(L)\cap \text{Ker } P} :\ D(L) \cap \text{Ker } P \to \text{Im } L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L)\cap\overline{\Omega}\neq\emptyset$, the map $N:\ X\to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega}\to X$ is compact.

Theorem 2.1 [13]**.** Let L be a Fredholm operator of index zero and let N be L-compact on nonempty open bounded subset Ω of X centered at zero. Assume that the following conditions are satisfied:

(i) $Lx\neq\lambda Nx$ for every $(x,\lambda)\in[(D(L)\setminus \mathrm{Ker}L)\cap\partial\Omega]\times(0,1)$;

(ii) $Nx\notin \mathrm{Im}L$ for every $x\in \mathrm{Ker}L\cap\partial\Omega$;

(iii) $\deg(\wedge QN\big|_{\mathrm{Ker}L},\ \Omega\cap\mathrm{Ker}L,0)\neq 0$, where $\wedge:\ Y/\mathrm{Im}L\to\mathrm{Ker}L$ is the isomorphism.

Then the equation $Lx=Nx$ has at least one solution in $D(L)\cap\overline{\Omega}$.

Let X_1 be the set of all T-periodic sequences. Choose $X=X_1\times X_1=Y$ endowed with the norm

$$||(x,y)||=\max\left\{||x||=:\max_{n\in Z}|x(n)|,\ ||y||=:\max_{n\in Z}|y(n)|\right\}\text{ for all }(x,y)\in X.$$

It is easy to see that X is a Banach space. Let $L:X\to Y$, be defined by

$$L\begin{pmatrix}x(n)\\ y(n)\end{pmatrix}=\begin{pmatrix}\Delta^\alpha x(n)\\ \Delta^\alpha y(n)\end{pmatrix},$$

and $N:X\to Y$ by

$$N\begin{pmatrix}x(n)\\ y(n)\end{pmatrix}=\begin{pmatrix}\phi^{-1}\left(\frac{y(n)}{p(n)}\right)\\ (-1)^\alpha f(n,x(n+\alpha),x(\tau_1(n)),\cdots,x(\tau_m(n)))\end{pmatrix}$$

for all $(x,y)\in X$.

Theorem 2.2. It holds that

(i) $\mathrm{Ker}L=\{(x,y)\in X \text{ with } x(n)=c,\ y(n)=d \text{ for all } n\in Z\}$.

(ii) $\mathrm{Im}L=\left\{(u,v)\in Y:\sum_{n=0}^{T-1}u(n)=\sum_{n=0}^{T-1}v(n)=0\right\}$.

(iii) L is a Fredholm operator of index zero.

(iv) Let $\Omega\subset X$ be an open bounded subset with $\overline{\Omega}\cap D(L)\neq\emptyset$, then N is L-compact on $\overline{\Omega}$.

(v) There exist projectors $P:X\to X$ and $Q:Y\to Y$ such that $\mathrm{Ker}L=\mathrm{Im}P$, $\mathrm{Ker}Q=\mathrm{Im}L$.

(vi) If $(x,y)\in X$ is a solution of the operator equation $L(x,y)=N(x,y)$, then x is a solution of problem (8).

Proof. In fact, it is easy to show (i), (ii), (iii) and (v). Define the projectors $Q:\ Y\to Y$ and $P:\ X\to X$ by

$$P\begin{pmatrix}x(k)\\ y(k)\end{pmatrix}=\begin{pmatrix}\frac{1}{T}\sum_{k=0}^{T-1}x(k)\\ \frac{1}{T}\sum_{k=0}^{T-1}y(k)\end{pmatrix},\ \text{ for }(x,y)\in X,$$

and

$$Q\begin{pmatrix}u(k)\\ v(k)\end{pmatrix}=\begin{pmatrix}\frac{1}{T}\sum_{k=0}^{T-1}u(k)\\ \frac{1}{T}\sum_{k=0}^{T-1}v(k)\end{pmatrix}\text{ for }(u,v)\in Y,$$

respectively. it is easy to prove that Ker$L$ = ImP and Im L = Ker$Q$. For a T-periodic sequence $u \in X_1$ with $\sum_{n=0}^{T-1} y(n) = 0$, let

$$c_{\alpha-1}(y) = -\tfrac{1}{T-1}\textstyle\sum_{s=0}^{T-1}(T-s)y(s),$$

$$c_{\alpha-2}(y) = -\tfrac{1}{T-1}\left(\textstyle\sum_{s=0}^{T-1}\tfrac{(T+1-s)(T-s)}{2!}y(s) + \tfrac{(T+1)(T+2)}{2!}c_{\alpha-1}(y) - \tfrac{3\cdot 2}{2!}c_{\alpha-1}(y)\right),$$

$$c_{\alpha-3}(y) = -\tfrac{1}{T-1}\left(\textstyle\sum_{s=0}^{T-1}\tfrac{(T+2-s)(T+1-s)(T-s)}{3!}y(s) + \tfrac{(T+1)(T+2)(T+3)}{3!}c_{\alpha-1}(y)\right.$$

$$\left. - \tfrac{4\cdot 3\cdot 2}{3!}c_{\alpha-1}(y) + \tfrac{(T+1)(T+2)}{2!}c_{\alpha-2}(y) - \tfrac{3\cdot 2}{2!}c_{\alpha-2}(y)\right),$$

$$\cdots\cdots,$$

$$c_1(y) = -\tfrac{1}{T-1}\left(\textstyle\sum_{s=0}^{T-1}\tfrac{\prod_{i=0}^{\alpha-3}(T+i-s)}{(\alpha-3)!}y(s) + \sum_{j=2}^{\alpha-1}\tfrac{\prod_{s=1}^{j}(T+s)}{j!}c_j(y)\right.$$

$$\left. - \textstyle\sum_{j=2}^{\alpha-1}\tfrac{\prod_{s=2}^{j+1}s}{j!}c_j(y)\right),$$

$$c_0(y) = -\tfrac{1}{T-1}\left(\textstyle\sum_{s=0}^{T-1}\tfrac{\prod_{i=0}^{\alpha-2}(T+i-s)}{(\alpha-1)!}y(s) + \sum_{j=1}^{\alpha-1}\tfrac{\prod_{s=1}^{j+1}(T+s)}{(j+1)!}c_j(y)\right.$$

$$\left. - \textstyle\sum_{j=1}^{\alpha-1}\tfrac{\prod_{s=2}^{j+2}s}{(j+1)!}c_j(y)\right).$$

Then the inverse $K_p:\ \mathrm{Im}L \to D(L)\cap \mathrm{Ker}P$ of the map $L:\ D(L)\cap\mathrm{Ker}P \to \mathrm{Im}L$ can be written by

$$K_p\begin{pmatrix} u(k) \\ v(k)\end{pmatrix} = \begin{pmatrix} x(k) \\ y(k)\end{pmatrix}$$

$$\begin{aligned}
x(k) &= \sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}x(s) \\
&\quad + \sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(x) - \frac{1}{T}\sum_{k=0}^{T-1}\left(\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(x)\right. \\
&\quad \left. + \sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}x(s)\right), \\
y(k) &= \sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}y(s) \\
&\quad + \sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(y) - \frac{1}{T}\sum_{k=0}^{T-1}\left(\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(y)\right. \\
&\quad \left. + \sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}y(s)\right).
\end{aligned}$$

In fact, for $(u,v)\in \text{Im } L$, we have $(LK_p)\begin{pmatrix} u(k)\\ v(k)\end{pmatrix}=\begin{pmatrix} u(k)\\ v(k)\end{pmatrix}$. On the other hand, for $x\in \text{Ker}P\cap D(L)$, it follows that $(K_pL)\begin{pmatrix} x(k)\\ y(k)\end{pmatrix}=\begin{pmatrix} x(k)\\ y(k)\end{pmatrix}$. Furthermore, let $\wedge:\ \text{Ker}L\to R^2$ be the isomophism with $\wedge(a,b)=(b,a)$. Set

$$f_x(n)=f(n,x(n+\alpha),x(\tau_1(n)),\cdots,x(\tau_m(n)))$$

for $x\in X_1$. One has

$$\begin{aligned}
QN\begin{pmatrix} x(k)\\ y(k)\end{pmatrix} &= Q\begin{pmatrix} \phi^{-1}\left(\frac{y(n)}{p(n)}\right)\\ (-1)^\alpha f(n,x(n+\alpha),x(\tau_1(n)),\cdots,x(\tau_m(n)))\end{pmatrix}\\
&= \frac{1}{T}\begin{pmatrix} \sum_{n=0}^{T-1}\phi^{-1}\left(\frac{y(n)}{p(n)}\right)\\ (-1)^\alpha \sum_{n=0}^{T-1} f(n,x(n+\alpha),x(\tau_1(n)),\cdots,x(\tau_m(n)))\end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&K_p(I-Q)N\begin{pmatrix} x(k)\\ y(k)\end{pmatrix}\\
=\ & K_p(I-Q)\begin{pmatrix} \phi^{-1}\left(\frac{y(n)}{p(n)}\right)\\ (-1)^\alpha f(n,x(n+\alpha),x(\tau_1(n)),\cdots,x(\tau_m(n)))\end{pmatrix}\\
=\ & K_p(I-Q)\begin{pmatrix} \phi^{-1}\left(\frac{y(n)}{p(n)}\right)\\ (-1)^\alpha f_x(n)\end{pmatrix}=\begin{pmatrix} x_0(k)\\ y_0(k)\end{pmatrix},
\end{aligned}$$

$$\begin{aligned}
x_0(k) &= \sum_{s=0}^{k-2m}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}f_x(k)\\
&\quad+\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(f_x)-\frac{1}{T}\sum_{k=0}^{T-1}\left(\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(f_x)\right.\\
&\quad\left.+\sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}f_x(s)\right)\\
&\quad+\frac{1}{T}\left(\sum_{k=0}^{T-1}f_x(k)\right)\sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}\\
&\quad+\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(f_x)-\frac{1}{T}\sum_{k=0}^{T-1}\left(\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(f_x)\right.\\
&\quad\left.+\frac{1}{T}\left(\sum_{k=0}^{T-1}f_x(k)\right)\sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}\right),\\
y_0(k) &= \sum_{s=0}^{k-2m}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}f_y(k)\\
&\quad+\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(f_y)-\frac{1}{T}\sum_{k=0}^{T-1}\left(\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(f_y)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}f_y(s)\Bigg)\\
&+\frac{1}{T}\left(\sum_{k=0}^{T-1}f_y(k)\right)\sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}\\
&+\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(f_y)-\frac{1}{T}\sum_{k=0}^{T-1}\left(\sum_{i=1}^{\alpha-1}\frac{\prod_{s=1}^{i}(k+s-1)}{i!}c_i(f_y)\right.\\
&\left.+\frac{1}{T}\left(\sum_{k=0}^{T-1}f_y(k)\right)\sum_{s=0}^{k-\alpha}\frac{(k-1-s)(k-2-s)\cdots(k-(\alpha-1)-s)}{(\alpha-1)!}\right).
\end{aligned}
$$

Since f is continuous, using the Ascoli-Arzela theorem, we can prove that $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega}\to X$ is compact, thus N is $L-$compact on $\overline{\Omega}$. Then (iv) holds.

Theorem 2.3. Suppose that

(A) there exist numbers $\beta>0$, $\theta>1$, nonnegative sequences $p_i(n), r(n)(i=0,\cdots,m)$, functions $g(n,x_0,\cdots,x_m)$, $h(n,x_0,\cdots,x_m)$ such that

$$f(n,x_0,\cdots,x_m)=g(n,x_0,\cdots,x_m)+h(n,x_0,\cdots,x_m) \tag{9}$$

$$g(n,x_0,x_1,\cdots,x_m)x_0\le-\beta|x_0|^{\theta+1}, \tag{10}$$

and

$$|h(n,x_0,\cdots,x_m)|\le\sum_{s=0}^{m}p_i(n)|x_i|^{\theta}+r(n), \tag{11}$$

for all $n\in\{1,\cdots,T\}$, $(x_0,x_1,\cdots,x_m)\in R^{m+1}$.

(B) there exists a constant $M>0$ such that

$$(-1)^{\alpha}c\left[\sum_{n=0}^{T-1}f(n,c,c,\cdots,c)\right]>0 \tag{12}$$

for all $|c|>M$ or

$$(-1)^{\alpha}c\left[\sum_{n=0}^{T-1}f(n,c,c,\cdots,c)\right]<0 \tag{13}$$

for all $|c|>M$.

Then equation (8) has at least one solution if

$$||p_0||+T^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||<\beta. \tag{14}$$

Proof. To obtain a solution x of equation (8), it suffices to get a solution (x,y) of the operator equation $L(x,y)=N(x,y)$ in X. It follows from Theorem 2.2 that L is a Fredholm operator of index zero and N is L-compact on each nonempty open bounded subset Ω of X centered at zero. We need to get a nonempty open bounded subset Ω of X centered at zero such that (i), (ii) and (iii) in Theorem 2.1 hold. This is done by dividing into three steps.

Step 1. Let $\Omega_1=\{(x,y):L(x,y)=\lambda N(x,y),\ ((x,y),\lambda)\in[(D(L)\setminus \mathrm{Ker}L)]\times(0,1)\}$, we prove that Ω_1 is bounded.

For $(x,y)\in\Omega_1$, we have $L(x,y)=\lambda N(x,y)$, $\lambda\in(0,1)$, so

$$\begin{cases}\Delta^\alpha x(n)=\lambda\phi^{-1}\left(\frac{y(n)}{p(n)}\right)\\ \Delta^\alpha y(n)=(-1)^\alpha\lambda f(n,x(n+\alpha),x(\tau_1(n)),\cdots,x(\tau_m(n))),\\ x(n+T)=x(n),\\ y(n+T)=y(n)\end{cases}\tag{15}$$

hold for all $n\in Z$. It follows from the first and second equation in (13) that

$$\Delta^\alpha\left[p(n)\phi\left(\frac{\Delta^\alpha x(n)}{\lambda}\right)\right]=(-1)^\alpha\lambda f(n,x(n+\alpha),x(\tau_1(n)),\cdots,x(\tau_m(n))).$$

Then

$$\Delta^\alpha\left[p(n)\phi\left(\ \Delta^\alpha x(n)\right)\right]=(-1)^\alpha\lambda\phi(\lambda)f(n,x(n+\alpha),x(\tau_1(n)),\cdots,x(\tau_m(n))).$$

It is easy to see that

$$\begin{aligned}
&(-1)^\alpha\sum_{s=n}^{n+T-1}\Delta^\alpha\left[p(s)\phi\left(\ \Delta^\alpha x(s)\right)\right]x(s+\alpha)\\
=\ &\sum_{s=n}^{n+T-1}\left\{\Delta^{\alpha-1}\left[p(s+1)\phi\left(\ \Delta^\alpha x(s+1)\right)\right]-\Delta^{\alpha-1}\left[p(s)\phi\left(\ \Delta^\alpha x(s)\right)\right]\right\}\times\\
&[x(s+\alpha+1)-\Delta x(s+\alpha)]\\
=\ &(-1)^\alpha\sum_{s=n}^{n+T-1}\left\{\Delta^{\alpha-1}\left[p(s+1)\phi\left(\ \Delta^\alpha x(s+1)\right)\right]x(s+\alpha+1)\right.\\
&\left.-\Delta^{\alpha-1}\left[p(s)\phi\left(\ \Delta^\alpha x(s)\right)\right]x(s+\alpha)\right\}\\
&-\sum_{s=n}^{n+T-1}\Delta^{\alpha-1}\left[p(s+1)\phi\left(\ \Delta^\alpha x(s+1)\right)\right]\Delta x(s+\alpha)\\
=\ &-(-1)^\alpha\sum_{s=n}^{n+T-1}\Delta^{\alpha-1}\left[p(s+1)\phi\left(\ \Delta^\alpha x(s+1)\right)\right]\Delta x(s+\alpha)\\
=\ &-\sum_{s=n}^{n+T-1}\left\{\Delta^{\alpha-2}\left[p(s+2)\phi\left(\ \Delta^\alpha x(s+2)\right)\right]-\Delta^{\alpha-2}\left[p(s+1)\phi\left(\ \Delta^\beta x(s+1)\right)\right]\right\}\times\\
&[\Delta x(s+\alpha+1)-\Delta^2x(s+\alpha)]\\
=\ &-(-1)^\alpha\sum_{s=n}^{n+T-1}\left\{\Delta^{\alpha-2}\left[p(s+2)\phi\left(\ \Delta^\alpha x(s+2)\right)\right]\Delta x(s+\alpha+1)\right.\\
&\left.-\Delta^{\alpha-2}\left[p(s+1)\phi\left(\ \Delta^\alpha x(s+1)\right)\right]\Delta x(s+\alpha)\right\}\\
&+\sum_{s=n}^{n+T-1}\Delta^{\alpha-2}\left[p(s+2)\phi\left(\Delta^\beta x(s+2)\right)\right]\Delta^2x(s+\alpha)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{\alpha-2}\sum_{s=n}^{n+T-1}\Delta^{\alpha-2}\Big[p(s+2)\phi\Big(\,\Delta^{\alpha}x(s+2)\Big)\Big]\,\Delta^{2}x(s+\alpha)\\
&= \cdots\cdots\\
&= \sum_{s=n}^{n+T-1}p(s+\alpha)\phi\Big(\Delta^{\alpha}x(s+\alpha)\,\Big)\,\Delta^{\alpha}x(s+\alpha).
\end{aligned}
$$

Since $x\phi(x)\geq 0$ for all $x\in R$ and $p(n)>0$ for all $n\in Z$, we get

$$
\begin{aligned}
&(-1)^{\alpha}\sum_{s=n}^{n+T-1}\Delta^{\alpha}\left[p(s)\phi\left(\,\Delta^{\alpha}x(s)\right)\right]x(s+\alpha)\\
&=\sum_{s=n}^{n+T-1}p(s+\alpha)\phi\left(\Delta^{\alpha}x(s+\alpha)\,\right)\Delta^{\alpha}x(s+\alpha)\geq 0.
\end{aligned}
\tag{16}
$$

Then

$$
\sum_{s=n}^{n+T-1}f(s,x(s+\alpha),x(s-\tau_1(s)),\cdots,x(s-\tau_m(s))x(s+\alpha)\geq 0.
$$

It follows from (9), (10) and (11) that

$$
\begin{aligned}
&\beta\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}=\beta\sum_{s=n}^{n+T-1}|x(s+\alpha)|^{\theta+1}\\
\leq\;& -\sum_{s=n}^{n+T-1}g(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))x(s+\alpha)\\
\leq\;& \sum_{s=n}^{n+T-1}h(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))x(s+\alpha)\\
\leq\;& \sum_{s=n}^{n+T-1}|h(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))|\,|x(s+\alpha)|\\
\leq\;& \sum_{s=n}^{n+T-1}p_0(s)|x(s+\alpha)|^{\theta+1}+\sum_{i=1}^{m}\sum_{s=n}^{n+T-1}p_i(s)|x(\tau_i(s))|^{\theta}|x(s+\alpha)|\\
&+\sum_{s=n}^{n+T-1}r(s)|x(s+\alpha)|\\
\leq\;& ||p_0||\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}+\sum_{i=1}^{m}||p_i||\sum_{n=0}^{T-1}|x(\tau_i(n))|^{\theta}|x(n+\alpha)|\\
&+||r||\sum_{n=0}^{T-1}|x(n+\alpha)|.
\end{aligned}
$$

For $x_i\geq 0, y_i\geq 0$, Holder′s inequality implies

$$
\sum_{i=1}^{s}x_iy_i\leq\left(\sum_{i=1}^{s}x_i^p\right)^{1/p}\left(\sum_{i=1}^{s}y_i^q\right)^{1/q},\;\;1/p+1/q=1,\;\;q>0,\;p>0.
$$

It follows that

$$
\begin{aligned}
&\beta \sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1} \\
\leq\;& ||p_0|| \sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
&+ \sum_{i=1}^{m} ||p_i|| \left[\sum_{n=0}^{T-1} |x(\tau_i(n))|^{\theta+1}\right]^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
=\;& ||p_0|| \sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1} + ||r|| T^{\theta/(\theta+1)} \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
&+ \sum_{i=1}^{m} ||p_i|| \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \left[\sum_{u\in\{\tau_i(n)-\alpha:\ n=0,\cdots,T-1\}} |x(u+\alpha)|^{\theta+1}\right]^{\theta/(\theta+1)} \\
\leq\;& ||p_0|| \sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
&+ \sum_{i=1}^{m} ||p_i|| \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \left[T \sum_{u\in[0,T-1]} |x(u+\alpha)|^{\theta+1}\right]^{\theta/(\theta+1)} \\
=\;& ||p_0|| \sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
&+ T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{\theta/(\theta+1)} \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
=\;& ||p_0|| \sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1} + ||r|| T^{\frac{\theta}{\theta+1}} \left[\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
&+ T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i|| \sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}.
\end{aligned}
$$

One gets that

$$
\left(\beta - ||p_0|| - T^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i||\right) \sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1} \leq ||r|| T^{\frac{\theta}{\theta+1}} \left(\sum_{n=0}^{T-1} |x(n+\alpha)|^{\theta+1}\right)^{1/(\theta+1)}.
$$

It follows from (12) that there is $M_1 > 0$ such that $\sum_{u=0}^{T-1} |x(u+\alpha)|^{\theta+1} \leq M_1$. Thus

$$
\max\left\{|x(u+\alpha)|^{\theta+1} :\ u = 0, \cdots, T-1\right\} \leq M_1.
$$

Hence $|x(n+\alpha)| \leq M_1^{1/(\theta+1)}$ for all $n \in \{0, \cdots, T-1\}$. Then $||x|| = \max_{n\in Z} |x(n)| \leq M_1^{1/(\theta+1)}$.

Now, we consider $\max_{n\in Z}|y(n)|$. Since (16) implies that

$$\begin{aligned}
& \lambda\phi(\lambda)\sum_{s=n}^{n+T-1} y(s+\alpha)\phi^{-1}\left(\frac{y(s+\alpha)}{p(s+\alpha)}\right) \\
= & \lambda\phi(\lambda)\sum_{s=n}^{n+T-1} p(s+\alpha)\frac{y(s+\alpha)}{p(s+\alpha)}\phi^{-1}\left(\frac{y(s+\alpha)}{p(s+\alpha)}\right) \\
= & \sum_{s=n}^{n+T-1} p(s+\alpha)\phi\left(\lambda\phi^{-1}\left(\frac{y(s+\alpha)}{p(s+\alpha)}\right)\right)\lambda\phi^{-1}\left(\frac{y(s+\alpha)}{p(s+\alpha)}\right) \\
= & \sum_{s=n}^{n+T-1} p(s+\alpha)\phi\Big(\Delta^{\alpha}x(s+\alpha)\Big)\Delta^{\alpha}x(s+\alpha) \\
= & (-1)^{\alpha}\sum_{s=n}^{n+T-1}\Delta^{\alpha}\Big[p(s)\phi\Big(\Delta^{\alpha}x(s)\Big)\Big]x(s+\alpha) \\
= & \lambda\phi(\lambda)\sum_{s=n}^{n+T-1} f(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))x(s+\alpha).
\end{aligned}$$

We get

$$\sum_{s=n}^{n+T-1} y(s+\alpha)\phi^{-1}\left(\frac{y(s+\alpha)}{p(s+\alpha)}\right)=\sum_{s=n}^{n+T-1} f(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))x(s+\alpha).$$

Then $\phi^{-1}(x)=|x|^{t-2}x$ implies $\phi^{-1}(ab)\phi^{-1}(a)\phi^{-1}(b)$, and $x\phi^{-1}(x)\geq 0$. (9),(10) and (11) imply that

$$\begin{aligned}
& \phi^{-1}\left(\frac{1}{||p||}\right)\sum_{s=n}^{n+T-1} y(s+\alpha)\phi^{-1}(y(s+\alpha)) \\
\leq & \sum_{s=n}^{n+T-1}\phi^{-1}\left(\frac{1}{p(s+\alpha)}\right)y(s+\alpha)\phi^{-1}(y(s+\alpha)) \\
= & \sum_{s=n}^{n+T-1}\Big[g(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))x(s+\alpha) \\
& \quad +h(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))x(s+\alpha)\Big] \\
\leq & \sum_{s=n}^{n+T-1}\Big[-\beta|x(s+\alpha)|^{\theta+1}+h(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))x(s+\alpha))\Big] \\
\leq & \sum_{s=n}^{n+T-1}\Big|h(s,x(s+\alpha),x(\tau_1(s)),\cdots,x(\tau_m(s))x(s+\alpha)\Big| \\
\leq & \sum_{s=n}^{n+T-1}p_0(s)|x(s+\alpha)|^{\theta+1}+\sum_{i=1}^{m}\sum_{s=n}^{n+T-1}p_i(s)|x(\tau_i(s))|^{\theta}|x(s+\alpha)|+\sum_{s=n}^{n+T-1}r(s)|x(s+\alpha)|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & ||p_0||\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}+\sum_{i=1}^{m}||p_i||\sum_{n=0}^{T-1}|x(\tau_i(n))|^{\theta}|x(n+\alpha)| \\
& +||r||\sum_{n=0}^{T-1}|x(n+\alpha)| \\
\leq\ & ||p_0||\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}+||r||T^{\theta/(\theta+1)}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
& +\sum_{i=1}^{m}||p_i||\left[\sum_{n=0}^{T-1}|x(\tau_i(n))|^{\theta+1}\right]^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
=\ & ||p_0||\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
& +\sum_{i=1}^{m}||p_i||\left[\sum_{u\in\{\tau_i(n)-\alpha:\ n=0,\cdots,T-1\}}|x(u+\alpha)|^{\theta+1}\right]^{\theta/(\theta+1)}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
\leq\ & ||p_0||\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
& +\sum_{i=1}^{m}||p_i||\left[T\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{\theta/(\theta+1)}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
\leq\ & ||p_0||\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
& +T^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{\theta/(\theta+1)}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
=\ & ||p_0||\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}+||r||T^{\frac{\theta}{\theta+1}}\left[\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1}\right]^{1/(\theta+1)} \\
& +T^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||\sum_{n=0}^{T-1}|x(n+\alpha)|^{\theta+1} \\
\leq\ & ||p_0||M_1+||r||T^{\frac{\theta}{\theta+1}}M_1^{\frac{1}{\theta+1}}+T^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||M_1 \\
=:\ & M_2.
\end{aligned}
$$

Hence

$$
\sum_{s=0}^{T-1}|y(s+\alpha)|^t=\sum_{s=n}^{n+T-1}|y(s+\alpha)|^t=\sum_{s=n}^{n+T-1}y(s+\alpha)\phi^{-1}(y(s+\alpha))\leq M_2\phi^{-1}\left(||p||\right). \tag{17}
$$

It follows from $y\in X_1$ that $||y||=\max_{n\in Z}|y(n+\alpha)|\leq\left(M_2\phi^{-1}\left(||p||\right)\right)^{1/t}$. Hence

$$||(x,y)|| \leq \max\left\{ M_1^{1/(\theta+1)}, \ \left(M_2\phi^{-1}\left(||p||\right)\right)^{1/t}\right\} \text{ for } (x,y) \in X.$$

So Ω_1 is bounded.

Step 2. Prove that $\Omega_2 = \{(a,b) \in \text{Ker}L: \ N(a,b) \in \text{Im}L\}$ is bounded.

For $(a,b) \in \text{Ker}L$, we have $N(a,b) = (\phi^{-1}(b/p(n)), f(n,a,\cdots,a))$. $Nx \in \text{Im}L$ implies that

$$\sum_{n=0}^{T-1}\phi^{-1}(b/p(n)) = 0, \quad \sum_{n=0}^{T-1} f(n,a,\cdots,a) = 0.$$

It follows from condition (B) that $|a| \leq M$ and $b = 0$. Thus Ω_2 is bounded.

Step 3. Prove that $\Omega_3 = \{(a,b) \in \text{Ker}L: \ \lambda \wedge (a,b) + (1-\lambda)QN(a,b) = 0, \ \lambda \in [0,1]\}$ or $\Omega_3 = \{(a,b) \in \text{Ker}L: \ -\lambda \wedge (a,b) + (1-\lambda)QN(a,b) = 0, \ \lambda \in [0,1]\}$ is bounded.

If (12) holds, consider

$$\Omega_3 = \{(a,b) \in \text{Ker}L: \ \lambda \wedge (a,b) + (1-\lambda)QN(a,b) = 0, \ \lambda \in [0,1]\}.$$

We will prove that Ω_3 is bounded. For $(a,b) \in \Omega_3$, and $\lambda \in [0,1]$, we have

$$-(1-\lambda)\sum_{n=0}^{T-1}\phi^{-1}(b/p(n)) = \lambda b, \ \ -(-1)^{\alpha}(1-\lambda)\sum_{n=0}^{T-1} f(n,a,\cdots,a) = \lambda a T.$$

If $\lambda = 1$, then $a = b = 0$. If $\lambda \neq 1$, and $|a| > M$, it follows from (B) that

$$0 \geq -(-1)^{\alpha}(1-\lambda)a\sum_{n=0}^{T-1} f(n,a,\cdots,a) = \lambda a^2 T > 0,$$

a contradiction. So $|a| \leq M$. Similarly, we get $|b| \leq M$. Hence Ω_3 is bounded.

If (13) holds, consider

$$\Omega_3 = \{(a,b) \in \text{Ker}L: \ -\lambda \wedge (a,b) + (1-\lambda)QN(a,b) = 0, \ \lambda \in [0,1]\},$$

Similarly, we can get a contradiction. So Ω_3 is bounded.

Set Ω be a open bounded subset of X such that $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}$. By the definition of Ω, we have $\Omega \supset \overline{\Omega_1}$ and $\Omega \supset \overline{\Omega_2}$, thus, from Step 1, Step 2 and Step 3, that $Lx \neq \lambda Nx$ for $x \in D(L) \setminus \text{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$; $Nx \notin \text{Im}L$ for $x \in \text{Ker}L \cap \partial\Omega$.

In fact, let $H(x,\lambda) = \pm\lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $\Omega \supset \overline{\Omega_3}$, thus $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \text{Ker}L$, thus by homotopy property of degree,

$$\begin{aligned} & \deg(QN|_{D(L)}, \Omega \cap \text{Ker}L, 0) = \deg(H(\cdot,0), \Omega \cap \text{Ker}L, 0) \\ = \ & \deg(H(\cdot,1), \Omega \cap \text{Ker}L, 0) = \deg(\pm\wedge, \Omega \cap \text{Ker}L, 0) \neq 0 \text{ since } 0 \in \Omega. \end{aligned}$$

Thus by Theorem 2.1 (Theorem IV.13[5]), $L(x,y) = N(x,y)$ has at least one solution $(x,y) \in \Omega$, then x is a solution of equation (8). The proof is completed.

3. Examples

In this section, we present two examples, which have applications see the text book [7], to illustrate the main results in section 2. These examples can not be solved by applying theorems in the papers [1-4,8-10]

Example 3.1. Consider the problem

$$\Delta^2(p(n)\phi(\Delta^2 x(n))) = -\beta[x(n+1)]^{2k+1} + \sum_{i=0}^{m} p_i(n)[x(n-i)]^{2k+1} + \sum_{j=0}^{l} q_j(n)[x(n+j)]^{2k+1} + r(n), \tag{18}$$

where ϕ is defined in Section 1, $\beta > 0$, l, m, k are positive integers, r, p_i, q_j are $T-$periodic sequences. Corresponding to equation (1), let

$$g(n, x_0, \cdots, x_{m+l}) = \beta x_0^{2k+1},$$

and

$$g(n, x_0, \cdots, x_{m+l}) = \sum_{i=0}^{m} p_i(n) x_i^{2k+1} + \sum_{j=0}^{l} q_j(n) x_{m+j}^{2k+1} + r(n).$$

Choose $\theta = 2k+1$. One sees that (A) holds. On the other hand, it is easy to show that there exists a constant $M > 0$ such that

$$c\sum_{n=0}^{T-1}\left[\left(-\beta + \sum_{i=0}^{m} p_i(n) + \sum_{j=0}^{l} q_j(n)\right) c^{2k+1} + r(n)\right] > 0$$

for all $|c| > M$ if $-\beta + \sum_{i=0}^{m} p_i(n) + \sum_{j=0}^{l} q_j(n) > 0$ for all $n \in [0, T-1]$ or

$$c\sum_{n=0}^{T-1}\left[\left(-\beta + \sum_{i=0}^{m} p_i(n) + \sum_{j=0}^{l} q_j(n)\right) c^{2k+1} + r(n)\right] < 0$$

for all $|c| > M$ if $-\beta + \sum_{i=0}^{m} p_i(n) + \sum_{j=0}^{l} q_j(n) < 0$ for all $n \in [0, T-1]$.

It follows from Theorem 2.3 that problem (18) has at least one solution if

$$||p_0|| + ||q_0|| + T^{\frac{2k+1}{2k+2}}\left(\sum_{i=0}^{m} ||p_i|| + \sum_{j=0}^{l} ||q_j||\right) < \beta$$

and $-\beta + \sum_{i=0}^{m} p_i(n) + \sum_{j=0}^{l} q_j(n) > 0$ for all $n \in [0, T-1]$ or $-\beta + \sum_{i=0}^{m} p_i(n) + \sum_{j=0}^{l} q_j(n) < 0$ for all $n \in [0, T-1]$.

Example 3.2. Consider the problem

$$\Delta^3(p(n)\phi(\Delta^3 x(n))) = \beta[x(n+1)]^{2k+1} - \sum_{i=0}^{m} p_i(n)[x(n-i)]^{2k+1} - \sum_{j=0}^{l} q_j(n)[x(n+j)]^{2k+1} - r(n), \tag{19}$$

where ϕ is defined in Section 1, $\beta > 0$, l, m, k are positive integers, r, p_i, q_j are $T-$periodic sequences. Corresponding to equation (2), let

$$g(n, x_0, \cdots, x_{m+l}) = \beta x_0^{2k+1},$$

and

$$g(n, x_0, \cdots, x_{m+l}) = \sum_{i=0}^{m} p_i(n) x_i^{2k+1} + \sum_{j=0}^{l} q_j(n) x_{m+j}^{2k+1} + r(n).$$

Choose $\theta = 2k+1$. One sees that (A) holds. On the other hand, it is easy to show that there exists a constant $M > 0$ such that

$$c\sum_{n=0}^{T-1}\left[\left(\beta - \sum_{i=0}^{m} p_i(n) - \sum_{j=0}^{l} q_j(n)\right) c^{2k+1} - r(n)\right] > 0$$

for all $|c| > M$ if $\beta - \sum_{i=0}^{m} p_i(n) - \sum_{j=0}^{l} q_j(n) > 0$ for all $n \in [0, T-1]$ or

$$c\sum_{n=0}^{T-1}\left[\left(\beta - \sum_{i=0}^{m} p_i(n) - \sum_{j=0}^{l} q_j(n)\right) c^{2k+1} + r(n)\right] < 0$$

for all $|c| > M$ if $\beta - \sum_{i=0}^{m} p_i(n) - \sum_{j=0}^{l} q_j(n) < 0$ for all $n \in [0, T-1]$.

It follows from Theorem 2.3 that problem (19) has at least one solution if

$$||p_0|| + ||q_0|| + T^{\frac{2k+1}{2k+2}}\left(\sum_{i=0}^{m} ||p_i|| + \sum_{j=0}^{l} ||q_j||\right) < \beta$$

and $\beta - \sum_{i=0}^{m} p_i(n) - \sum_{j=0}^{l} q_j(n) > 0$ for all $n \in [0, T-1]$ or $\beta - \sum_{i=0}^{m} p_i(n) - \sum_{j=0}^{l} q_j(n) < 0$ for all $n \in [0, T-1]$.

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