Electronic Journal of Mathematical Analysis and Applications
Vol. 3(2) July 2015, pp. 31-45.
ISSN: 2090-792X (online)
http://ejmaa.6te.net/

DOMAIN OF THE DOUBLE BAND MATRIX DEFINED BY FIBONACCI NUMBERS IN THE MADDOX'S SPACE $\ell(p)$*

HÜSAMETTİN ÇAPAN AND FEYZİ BAŞAR**

Abstract. In the present paper, some algebraic and topological properties of the domain $\ell(F,p)$ of the double band matrix F defined by a sequence of Fibonacci numbers in the sequence space $\ell(p)$ are studied, where $\ell(p)$ denotes the space of all sequences $x = (x_k)$ such that $\sum_k |x_k|^{p_k} < \infty$ and was defined by Maddox in [*Spaces of strongly summable sequences*, Quart. J. Math. Oxford (2) **18** (1967), 345–355]. Furthermore, the alpha-, beta- and gamma-duals of the space $\ell(F,p)$ are determined, and the Schauder basis is given. The classes of matrix transformations from the space $\ell(F,p)$ to the spaces ℓ_∞, c and c_0 are characterized. Additionally, the characterizations of some other matrix transformations from the space $\ell(F,p)$ to the Euler, Riesz, difference, etc., sequence spaces are obtained from the main results of the paper.

1. Preliminaries, Background and Notation

By ω, we denote the space of all sequences with complex terms which contains ϕ, the set of all finitely non-zero sequences, that is,

$$\omega : \quad = \quad \{x = (x_k) : x_k \in \mathbb{C} \ \text{ for all } \ k \in \mathbb{N}\},$$

where $\mathbb{C}$ denotes the complex field and $\mathbb{N} = \{0, 1, 2, \ldots\}$. By a sequence space, we understand a linear subspace of the space ω. We write ℓ_∞, c, c_0 and ℓ_p for the classical sequence spaces of all bounded, convergent, null and absolutely p-summable sequences which are the Banach spaces with the norms $\|x\|_\infty = \sup_{k\in\mathbb{N}} |x_k|$ and $\|x\|_p = (\sum_k |x_k|^p)^{1/p}$; respectively, where $1 \le p < \infty$. For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞. Also by bs and cs, we denote the spaces of all bounded and convergent series, respectively. bv is the space consisting of all sequences (x_k) such that $(x_k - x_{k+1})$ in ℓ_1 and bv_0 is the intersection of the spaces bv and c_0.

2010 *Mathematics Subject Classification.* Primary: 46A45, Secondary: 46B45, 46A35.
Key words and phrases. Paranormed sequence space, double sequential band matrix, alpha-, beta- and gamma-duals, matrix transformations in sequence spaces.
*The main results of this paper were presented in part at the conference *Algerian-Turkish International Days on Mathematics (ATIM 2013) to be held September 12–14, 2013 in İstanbul, Turkey, at the Fatih University.*
**Corresponding author.

A linear topological space X over the real field $\mathbb{R}$ is said to be a *paranormed space* if there is a function $g : X \to \mathbb{R}$ satisfying the following conditions for all $x, y \in X$:

(i) $g(x) = 0$ if $x = \theta$, (ii) $g(x) = g(-x)$, (iii) $g(x + y) \leq g(x) + g(y)$, (iv) Scalar multiplication is continuous, i.e., $|\alpha_n - \alpha| \to 0$ and $g(x_n - x) \to 0$ imply $g(\alpha_n x_n - \alpha x) \to 0$ for all α's in $\mathbb{R}$ and all x's in X, where θ is the zero vector in the linear space X.

Assume here and after that (p_k) be a bounded sequence of strictly positive real numbers with $\sup p_k = H$ and $M = \max\{1, H\}$. Then, the linear space $\ell(p)$ was defined by Maddox [1] (see also Simons [2] and Nakano [3]) as follows:

$$\ell(p) = \left\{ x = (x_k) \in w : \sum_k |x_k|^{p_k} < \infty \right\}, \ (0 < p_k \leq H < \infty)$$

which is the complete space paranormed by $g(x) = \left(\sum_k |x_k|^{p_k}\right)^{1/M}$. We assume throughout that $p_k^{-1} + (p'_k)^{-1} = 1$ and denote the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{F}$ and use the convention that any term with negative subscript is equal to naught.

The alpha-, beta- and gamma-duals of a sequence space λ, which are respectively denoted by λ^α, λ^β and λ^γ, are defined by

$$\begin{aligned} \lambda^\alpha &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in \ell_1 \text{ for all } y = (y_k) \in \lambda\}, \\ \lambda^\beta &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in cs \text{ for all } y = (y_k) \in \lambda\}, \\ \lambda^\gamma &:= \{x = (x_k) \in \omega : xy = (x_k y_k) \in bs \text{ for all } y = (y_k) \in \lambda\}. \end{aligned}$$

Let λ, μ be any two sequence spaces and $A = (a_{nk})$ be an infinite matrix of complex numbers a_{nk}, where $k, n \in \mathbb{N}$. Then, we say that A defines a *matrix transformation* from λ into μ and we denote it by writing $A : \lambda \to \mu$, if for every sequence $x = (x_k) \in \lambda$ the sequence $Ax = \{(Ax)_n\}$, the A-transform of x, is in μ; where

$$(Ax)_n = \sum_k a_{nk} x_k \tag{1.1}$$

provided the series on the right side of (1.1) converges for each $n \in \mathbb{N}$. By $(\lambda : \mu)$, we denote the class of all matrices A such that $A : \lambda \to \mu$. Thus, $A \in (\lambda : \mu)$ if and only if Ax exists, i.e. $A_n \in \lambda^\beta$ for all $n \in \mathbb{N}$ and is in μ for all $x \in \lambda$, where A_n denotes the sequence in the n-th row of A. This shows the importance of the beta-dual for the existence of matrix transformations on any given sequence space.

The matrix domain λ_A of an infinite matrix A in a sequence space λ is defined as the set of all sequences $x = (x_k) \in \omega$ such that Ax exists and is in the space λ, that is $\lambda_A := \{x = (x_k) \in w : Ax \in \lambda\}$. It is immediate that λ_A is a sequence space whenever λ is a sequence space and the spaces λ_A and λ are linearly isomorphic if A is triangle.

2. The Sequence Space $\ell(F, p)$

Consider the sequence (f_n) of Fibonacci numbers defined by the linear recurrence relations

$$f_n := \begin{cases} 1 & , \quad n = 0, 1, \\ f_{n-1} + f_{n-2} & , \quad n \geq 2. \end{cases}$$

Let us define the double band matrix $F = (f_{nk})$ by the sequence (f_n), as follows:

$$f_{nk} := \begin{cases} -\frac{f_{n+1}}{f_n} & , \quad k = n-1, \\ \frac{f_n}{f_{n+1}} & , \quad k = n, \\ 0 & , \quad 0 \le k < n-1 \text{ or } k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. The usual inverse $F^{-1} = (c_{nk})$ of the matrix F is calculated as

$$c_{nk} := \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}} & , \quad 0 \le k \le n, \\ 0 & , \quad k > n \end{cases}$$

for all $k, n \in \mathbb{N}$. It is easy to show that the matrix F is neither regular nor coercive while it is conservative.

The domain $\ell(F, p)$ of the double band matrix F in the sequence space $\ell(p)$ is introduced, that is to say that

$$\ell(F,p) := \left\{ x = (x_k) \in \omega : \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^{p_k} < \infty \right\},$$

where $0 < p_k \le H < \infty$. In the case $p_k = p$ for all $k \in \mathbb{N}$, the space $\ell(F,p)$ is reduced to the space $\ell_p(F)$, i.e.,

$$\ell_p(F) := \left\{ x = (x_k) \in \omega : \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^{p} < \infty \right\}, \ (p \ge 1).$$

Furthermore, the alpha-, beta- and gamma-duals of the space $\ell(F,p)$ are determined, and the Schauder basis is constructed. The classes of matrix transformations from the space $\ell(F,p)$ to the spaces ℓ_∞, c and c_0 are characterized.

Now, we define the sequence $y = (y_k)$ by the F-transform of a sequence $x = (x_k)$, i.e.,

$$y_k = (Fx)_k = -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \tag{2.1}$$

for all $k \in \mathbb{N}$. At this situation we can express x in terms of y that

$$x_k = \left(F^{-1} y\right)_k = \sum_{j=0}^{k} \frac{f_{k+1}^2}{f_j f_{j+1}} y_j \tag{2.2}$$

for all $k \in \mathbb{N}$.

Theorem 2.1. *$\ell(F,p)$ is a linear, complete metric space paranormed by h defined by*

$$h(x) = \left(\sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^{p_k} \right)^{1/M}, \tag{2.3}$$

where $0 < p_k \le H < \infty$ for all $k \in \mathbb{N}$.

Proof. To show the linearity of the space $\ell(F,p)$ with respect to the coordinatewise addition and scalar multiplication is trivial. Firstly, we show that $\ell(F,p)$ is a paranormed space with the paranorm h defined by (2.3).

It is clear that $h(\theta) = 0$, where $\theta = (0, 0, \ldots)$ and $h(x) = h(-x)$ for all $x \in \ell(F,p)$.

Let $x = (x_k), y = (y_k) \in \ell(F,p)$. Then, by Minkowski's inequality and the inequality $|a+b|^p \leq |a|^p + |b|^p$; where $0 < p \leq 1$ and $a, b \in \mathbb{C}$, we have

$$\begin{aligned} h(x+y) &= \left[\sum_k \left|-\frac{f_{k+1}}{f_k}(x_{k-1}+y_{k-1}) + \frac{f_k}{f_{k+1}}(x_k+y_k)\right|^{p_k}\right]^{1/M} \\ &= \left[\sum_k \left(\left|-\frac{f_{k+1}}{f_k}x_{k-1} + \frac{f_k}{f_{k+1}}x_k - \frac{f_{k+1}}{f_k}y_{k-1} + \frac{f_k}{f_{k+1}}y_k\right|^{p_k/M}\right)^M\right]^{1/M} \\ &\leq \left[\sum_k \left(\left|-\frac{f_{k+1}}{f_k}x_{k-1} + \frac{f_k}{f_{k+1}}x_k\right|^{p_k/M} + \left|-\frac{f_{k+1}}{f_k}y_{k-1} + \frac{f_k}{f_{k+1}}y_k\right|^{p_k/M}\right)^M\right]^{1/M} \\ &\leq \left(\sum_k \left|-\frac{f_{k+1}}{f_k}x_{k-1} + \frac{f_k}{f_{k+1}}x_k\right|^{p_k}\right)^{1/M} + \left(\sum_k \left|-\frac{f_{k+1}}{f_k}y_{k-1} + \frac{f_k}{f_{k+1}}y_k\right|^{p_k}\right)^{1/M} \\ &= h(x) + h(y). \end{aligned}$$

Also, since the inequality $|\alpha|^{p_k} \leq \max\{1, |\alpha|^M\}$ holds for $\alpha \in \mathbb{R}$, we get

$$\begin{aligned} h(\alpha x) &= \left[\sum_k \left|-\frac{f_{k+1}}{f_k}(\alpha x_{k-1}) + \frac{f_k}{f_{k+1}}(\alpha x_k)\right|^{p_k}\right]^{1/M} \\ &= \left(\sum_k |\alpha|^{p_k} \left|-\frac{f_{k+1}}{f_k}x_{k-1} + \frac{f_k}{f_{k+1}}x_k\right|^{p_k}\right)^{1/M} \\ &\leq \max\{1, |\alpha|\} h(x). \end{aligned}$$

Let (α_n) be a sequence of scalars with $\alpha_n \to \alpha$, as $n \to \infty$, and $\left\{x^{(n)}\right\}_{n=0}^{\infty}$ be a sequence of elements $x^{(n)} \in \ell(F,p)$ with $h\left[x^{(n)} - x\right] \to 0$, as $n \to \infty$. Then, we observe that

$$\begin{aligned} 0 \leq h\left[\alpha_n x^{(n)} - \alpha x\right] &= h\left[\alpha_n x^{(n)} - \alpha x^{(n)} + \alpha x^{(n)} - \alpha x\right] \\ &= h\left[(\alpha_n - \alpha)x^{(n)} + \alpha\left(x^{(n)} - x\right)\right] \\ &\leq h\left[(\alpha_n - \alpha)x^{(n)}\right] + h\left[\alpha\left(x^{(n)} - x\right)\right] \\ &= |\alpha_n - \alpha| h\left[x^{(n)}\right] + \max\{1, |\alpha|\} h\left[x^{(n)} - x\right]. \end{aligned} \tag{2.4}$$

If we combine the facts $\alpha_n - \alpha \to 0$, as $n \to \infty$, and $h\left[x^{(n)} - x\right] \to 0$, as $n \to \infty$, with (2.4) we obtain that $h\left[\alpha_n x^{(n)} - \alpha x\right] \to 0$, as $n \to \infty$. That is to say that the scalar multiplication is continuous. This shows that h is a paranorm on $\ell(F,p)$.

Moreover, if we assume $h(x) = 0$, then we get

$$\left|-\frac{f_{k+1}}{f_k}x_{k-1} + \frac{f_k}{f_{k+1}}x_k\right| = 0$$

for each $k \in \mathbb{N}$. If we put $k = 0$, since $x_{-1} = 0$ and $f_0/f_1 \neq 0$, we have $x_0 = 0$. For $k = 1$, since $x_0 = 0$ and $f_1/f_2 \neq 0$, we have $x_1 = 0$. Continuing in this way, we obtain $x_k = 0$ for all $k \in \mathbb{N}$. Namely, we obtain $x = \theta = (0, 0, \ldots)$. This shows that h is a total paranorm.

Now, we show that $\ell(F,p)$ is complete. Let (x^n) be any Cauchy sequence in $\ell(F,p)$; where $x^n = \left\{x_0^{(n)}, x_1^{(n)}, \ x_2^{(n)}, \ldots\right\}$. Then, for a given $\varepsilon > 0$, there exists a positive integer $n_0(\varepsilon)$ such that $[h(x^n - x^m)]^M < \varepsilon^M$ for all $n, m > n_0(\varepsilon)$. Since for each fixed $k \in \mathbb{N}$

$$\begin{aligned}
|(Fx^n)_k - (Fx^m)_k|^{p_k} &\leq \sum_k |(Fx^n)_k - (Fx^m)_k|^{p_k} \\
&= \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1}^{(n)} + \frac{f_k}{f_{k+1}} x_k^{(n)} - \left[-\frac{f_{k+1}}{f_k} x_{k-1}^{(m)} + \frac{f_k}{f_{k+1}} x_k^{(m)} \right] \right|^{p_k} \\
&= \sum_k \left| -\frac{f_{k+1}}{f_k} \left[x_{k-1}^{(n)} - x_{k-1}^{(m)} \right] + \frac{f_k}{f_{k+1}} \left[x_k^{(n)} - x_k^{(m)} \right] \right|^{p_k} \\
&= [h(x^n - x^m)]^M < \varepsilon^M
\end{aligned}$$

for every $n, m > n_0(\varepsilon)$, $\{(Fx^0)_k, (Fx^1)_k, (Fx^2)_k, \ldots\}$ is a Cauchy sequence of real numbers for every fixed $k \in \mathbb{N}$. Since $\mathbb{R}$ is complete, it converges, say $(Fx^n)_k \to (Fx)_k$ as $n \to \infty$. Using these infinitely many limits $(Fx)_0, (Fx)_1, (Fx)_2, \ldots$ we define the sequence $\{(Fx)_0, (Fx)_1, (Fx)_2, \ldots\}$. For each $k \in \mathbb{N}$ and $n > n_0(\varepsilon)$

$$\begin{aligned}
[h(x^n - x)]^M &= \sum_k \left| -\frac{f_{k+1}}{f_k} \left[x_{k-1}^{(n)} - x_{k-1} \right] + \frac{f_k}{f_{k+1}} \left[x_k^{(n)} - x_k \right] \right|^{p_k} \\
&= \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1}^{(n)} + \frac{f_k}{f_{k+1}} x_k^{(n)} - \left[-\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right] \right|^{p_k} \\
&= \sum_k |(Fx^n)_k - (Fx)_k|^{p_k} < \varepsilon^M.
\end{aligned}$$

This shows that $x^n - x \in \ell(F,p)$. Since $\ell(F,p)$ is a linear space, we conclude that $x \in \ell(F,p)$. It follows that $x^n \to x$, as $n \to \infty$, in $\ell(F,p)$ which means that $\ell(F,p)$ is complete.

Now, one can easily check that the absolute property does not hold on the space $\ell(F,p)$, that is

$$h(x) = \left(\sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^{p_k} \right)^{1/M} \neq \left(\sum_k \left| -\frac{f_{k+1}}{f_k} |x_{k-1}| + \frac{f_k}{f_{k+1}} |x_k| \right|^{p_k} \right)^{1/M} = h(|x|),$$

where $|x| = (|x_k|)$. This says that $\ell(F,p)$ is the sequence space of non-absolute type. □

Theorem 2.2. *Convergence in $\ell(F,p)$ is strictly stronger than coordinatewise convergence, but the converse is not true, in general.*

Proof. First we show that $h(x^n - x) \to 0$, as $n \to \infty$ implies $x_k^{(n)} \to x_k$, as $n \to \infty$ for all $k \in \mathbb{N}$. If we fix k, then we have

$$\begin{aligned}
0 &\leq \left| -\frac{f_{k+1}}{f_k} x_{k-1}^{(n)} + \frac{f_k}{f_{k+1}} x_k^{(n)} - \left(-\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right) \right|^{p_k} \\
&\leq \sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1}^{(n)} + \frac{f_k}{f_{k+1}} x_k^{(n)} - \left(-\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right) \right|^{p_k} \\
&= \sum_k \left| -\frac{f_{k+1}}{f_k} \left(x_{k-1}^{(n)} - x_{k-1} \right) + \frac{f_k}{f_{k+1}} \left(x_k^{(n)} - x_k \right) \right|^{p_k} \\
&= [h(x^n - x)]^M.
\end{aligned}$$

Hence, we have for $k=0$

$$\lim_{n\to\infty}\left|-\frac{f_1}{f_0}x_{-1}^{(n)}+\frac{f_0}{f_1}x_0^{(n)}-\left(-\frac{f_1}{f_0}x_{-1}+\frac{f_0}{f_1}x_0\right)\right|=0,$$

that is, $\left|\frac{f_0}{f_1}\left[x_0^{(n)}-x_0\right]\right|\to 0$, as $n\to\infty$, and $f_0/f_1=1\neq 0$, then $\left|x_0^{(n)}-x_0\right|\to 0$, as $n\to\infty$. Likewise, for each $k\in\mathbb{N}$, we have $\left|x_k^{(n)}-x_k\right|\to 0$, as $n\to\infty$.

Now, we show that the converse is not true in general. We assume $x_k^{(n)}\to x_k$, as $n\to\infty$. Then, there exists an $N\in\mathbb{N}$ such that $\left|x_k^{(n)}-x_k\right|<1$ for each fixed k and for all $n\geq N$. Therefore, we see that

$$\begin{aligned}
0 \leq h(x^n-x) &= \left[\sum_k\left|-\frac{f_{k+1}}{f_k}\left(x_{k-1}^{(n)}-x_{k-1}\right)+\frac{f_k}{f_{k+1}}\left(x_k^{(n)}-x_k\right)\right|^{p_k}\right]^{1/M} \qquad (2.5)\\
&= \left\{\sum_k\left[\left|-\frac{f_{k+1}}{f_k}\left(x_{k-1}^{(n)}-x_{k-1}\right)+\frac{f_k}{f_{k+1}}\left(x_k^{(n)}-x_k\right)\right|^{p_k/M}\right]^M\right\}^{1/M}\\
&\leq \left\{\sum_k\left[\left|-\frac{f_{k+1}}{f_k}\left(x_{k-1}^{(n)}-x_{k-1}\right)\right|^{p_k/M}+\left|\frac{f_k}{f_{k+1}}\left(x_k^{(n)}-x_k\right)\right|^{p_k/M}\right]^M\right\}^{1/M}\\
&\leq \left[\sum_k\left|-\frac{f_{k+1}}{f_k}\left(x_{k-1}^{(n)}-x_{k-1}\right)\right|^{p_k}\right]^{1/M}+\left[\sum_k\left|\frac{f_k}{f_{k+1}}\left(x_k^{(n)}-x_k\right)\right|^{p_k}\right]^{1/M}\\
&\leq \left(\sum_k\left|-\frac{f_{k+1}}{f_k}\right|^{p_k}\left|x_{k-1}^{(n)}-x_{k-1}\right|^{p_k}\right)^{1/M}+\left(\sum_k\left|\frac{f_k}{f_{k+1}}\right|^{p_k}\left|x_k^{(n)}-x_k\right|^{p_k}\right)^{1/M}\\
&\leq \left(\sum_k\left|-\frac{f_{k+1}}{f_k}\right|^{p_k}\right)^{1/M}+\left(\sum_k\left|\frac{f_k}{f_{k+1}}\right|^{p_k}\right)^{1/M}
\end{aligned}$$

for all k and $n\geq N$. Since $|-f_{k+1}/f_k|\to 1.6$ and $|f_k/f_{k+1}|\to 0.6$, as $k\to\infty$, $h(x^n-x)$ in (2.5) does not converge for each fixed $k\in\mathbb{N}$ and for all $n\geq N$. This implies that the converse is not true. Let us consider the elements of the sequence x^n be equal, then we observe $h(x^n-x)=0$, that is to say that coordinatewise convergence requires convergence. Hence, we can say that the converse is not true in general. □

Definition 2.3. *A sequence space λ with a linear topology is called a K-space, provided each of the maps $q_i:\lambda\to\mathbb{C}$ defined by $q_i(x)=x_i$ is continuous for all $i\in\mathbb{N}$. If a sequence space λ is complete and convergence in λ requires coordinatewise convergence, then λ is called FK-space. An FK-space whose topology is normable is called a BK-space.*

Now, we give the followings:

Theorem 2.4. *$\ell(F,p)$ is a K-space.*

Proof. Firstly, we show that $q_i(x)=x_i$ is linear for all $i\in\mathbb{N}$. Let $x=(x_i), y=(y_i)\in\ell(F,p)$ and $\alpha\in\mathbb{C}$. Then, we get

$$q_i(x+y)=(x+y)_i=x_i+y_i=q_i(x)+q_i(y) \quad\text{and}\quad q_i(\alpha x)=(\alpha x)_i=\alpha x_i=\alpha q_i(x)$$

for all $i \in \mathbb{N}$. Hence, q_i is linear.

Now, we prove that q_i is continuous. For this, it is sufficient to show that q_i is bounded.

Let $x = (x_i) \in \ell(F, p)$ be any vector. Then, since $|q_i(x)| = |x_i|$ for all $i \in \mathbb{N}$, one can see that

$$\|q_i\| = \sup_{x \neq \theta} \frac{|q_i(x)|}{\|x\|_{\ell(F,p)}} = \sup_{x \neq \theta} \frac{|x_i|}{\|x\|_{\ell(F,p)}} \leq \sup_{x \neq \theta} \frac{\|x\|_{\ell(F,p)}}{\|x\|_{\ell(F,p)}} = 1 < \infty,$$

i.e. q_i is bounded. Hence, q_i is a linear and continuous operator. That is to say that $\ell(F, p)$ is a K-space. □

Theorem 2.5. *$\ell(F, p)$ is an FK-space.*

Proof. It is easy to see by Theorems 2.1 and 2.2 that $\ell(F, p)$ is complete sequence space and convergence requires coordinatewise convergence. Hence, $\ell(F, p)$ is an FK-space. □

Theorem 2.6. *$\ell_p(F)$ is the linear space under the coordinatewise addition and scalar multiplication which is a BK-space with the norm*

$$\|x\| \quad = \quad \left(\sum_k \left| -\frac{f_{k+1}}{f_k} x_{k-1} + \frac{f_k}{f_{k+1}} x_k \right|^p \right)^{1/p},$$

where $x = (x_k) \in \ell_p(F)$ and $1 \leq p < \infty$.

Proof. Since the first part of the theorem is a routine verification, we omit the detail. Since ℓ_p is a BK-space with respect to its usual norm and F is a triangle matrix, Theorem 4.3.2 of Wilansky [4, p. 61] gives the fact that $\ell_p(F)$ is a BK-space, where $1 \leq p < \infty$. This completes the proof. □

Definition 2.7. *Let d be a metric on a linear space X. If algebraic operations are continuous, namely (x_n) and (y_n) are two sequences in X, and (α_n) is a sequence of scalars such that*

$$\begin{array}{lllll} \lim_{n\to\infty} d(x_n, x) = 0 & \textit{and} & \lim_{n\to\infty} d(y_n, y) = 0 & \textit{implies} & \lim_{n\to\infty} d(x_n + y_n, x + y) = 0, \\ \lim_{n\to\infty} \alpha_n = \alpha & \textit{and} & \lim_{n\to\infty} d(x_n, x) = 0 & \textit{implies} & \lim_{n\to\infty} d(\alpha_n x_n, \alpha x) = 0 \end{array}$$

then, (X, d) is called linear metric space; (see Malkowsky and Rakočević [5]). If X is a complete linear metric space then it is called Frechet sequence space (see Wilansky [6]). Now, we may give the following:

Theorem 2.8. *$\ell_p(F)$ is a Frechet space.*

Proof. To avoid the repetition of the similar statements, we only show that the algebraic operations are continuous on the space $\ell_p(F)$. Let (x_n) and (y_n) be two sequences in $\ell_p(F)$, and (α_n) be a sequence of scalars such that $d(x_n, x) \to 0$,

$d(y_n, y) \to 0$ and $\alpha_n \to \alpha$, as $n \to \infty$. Then, we get that

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} d(x_n + y_n, x + y) &(2.6)\\ &= \lim_{n\to\infty} [\|x_n + y_n - (x + y)\|] \\ &\leq \lim_{n\to\infty} (\|x_n - x\| + \|y_n - y\|) \\ &= \lim_{n\to\infty} d(x_n, x) + \lim_{n\to\infty} d(y_n, y) = 0, \\ 0 &\leq \lim_{n\to\infty} d(\alpha_n x_n, \alpha x) &(2.7)\\ &= \lim_{n\to\infty} \|\alpha_n x_n - \alpha x\| \\ &= \lim_{n\to\infty} \|(\alpha_n - \alpha)x_n + \alpha(x_n - x)\| \\ &\leq \lim_{n\to\infty} (|\alpha_n - \alpha|\|x_n\| + |\alpha|\|x_n - x\|) \\ &= \lim_{n\to\infty} |\alpha_n - \alpha|\|x_n\| + |\alpha| \lim_{n\to\infty} d(x_n, x) = 0. \end{aligned}$$

It is easy to see from (2.6) and (2.7) that the algebraic operations are continuous on the linear metric space $\ell_p(F)$. Hence, $\ell_p(F)$ is a Frechet space. □

With the notation of (2.1), the transformation T defined from $\ell(F, p)$ to $\ell(p)$ by $x \mapsto y = Tx$ is linear bijection, so we have the following:

Corollary 2.1. *The sequence space $\ell(F, p)$ of the non-absolute type is linearly paranorm isomorphic to the space $\ell(p)$, where $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$.*

It is known from Theorem 2.3 of Jarrah and Malkowsky [7] that the domain λ_T of an infinite matrix $T = (t_{nk})$ in a normed sequence space λ has a basis if and only if λ has a basis, if T is a triangle. As a direct consequence of this fact, we have:

Corollary 2.2. *Let $0 < p_k \leq H < \infty$ and $\lambda_k = (Fx)_k$ for all $k \in \mathbb{N}$. Define the sequence $b^{(k)} = \left\{b_n^{(k)}\right\}_{n\in\mathbb{N}}$ of the elements of the spaces $\ell(F, p)$ by*

$$b_n^{(k)} = \begin{cases} \frac{f_{k+1}^2}{f_n f_{n+1}} & , \quad 0 \leq n \leq k, \\ 0 & , \quad n > k \end{cases} \tag{2.8}$$

for every fixed $k \in \mathbb{N}$. Then, the sequence $\left\{b^{(k)}\right\}_{k\in\mathbb{N}}$ is a basis for the space $\ell(F, p)$ and any $x \in \ell(F, p)$ has a unique representation of the form $x = \sum_k \lambda_k b^{(k)}$.

3. The alpha-, beta- and gamma-duals of the space $\ell(F, p)$

Prior to giving the alpha-, beta- and gamma-duals of the space $\ell(F, p)$, we quote some required lemmas for proving our theorems.

Lemma 3.1. [8, Theorem 5.1.0] *Let $A = (a_{nk})$ be an infinite matrix over the complex field. The following statements hold:*

(i) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if $\sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left|\sum_{n\in N} a_{nk}\right|^{p_k} < \infty$.*

(ii) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_1)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{N\in\mathcal{F}} \sum_k \left|\sum_{n\in N} a_{nk} B^{-1}\right|^{p_k'} < \infty. \tag{3.1}$$

Lemma 3.2. [9, (i) and (ii) of Theorem 1] *Let $A = (a_{nk})$ be an infinite matrix over the complex field. The following statements hold:*

(i) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if*

$$\sup_{n,k\in\mathbb{N}} |a_{nk}|^{p_k} < \infty. \tag{3.2}$$

(ii) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : \ell_\infty)$ if and only if there exists an integer $B > 1$ such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left|a_{nk}B^{-1}\right|^{p_k'} < \infty. \tag{3.3}$$

Lemma 3.3. [9, Corollary for Theorem 1] *Let $A = (a_{nk})$ be an infinite matrix over the complex field and $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A \in (\ell(p) : c)$ if and only if (3.2), (3.3) hold, and*

$$\lim_{n\to\infty} a_{nk} = \beta_k \quad \textit{for each } k \in \mathbb{N} \tag{3.4}$$

also holds.

Let us define the sets $E_1(p)$, $E_2(p)$, $E_3(p)$, $E_4(p)$ and $E_5(p)$, as follows:

$$\begin{aligned}
E_1(p) &:= \left\{a = (a_k) \in \omega : \sup_{N\in\mathcal{F}} \sup_{k\in\mathbb{N}} \left|\sum_{n\in N} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n\right|^{p_k} < \infty\right\}, \\
E_2(p) &:= \bigcup_{B>1} \left\{a = (a_k) \in \omega : \sup_{N\in\mathcal{F}} \sum_k \left|\sum_{n\in N} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n B^{-1}\right|^{p_k'} < \infty\right\}, \\
E_3(p) &:= \left\{a = (a_k) \in \omega : \sup_{k,n\in\mathbb{N}} \left|\sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j\right|^{p_k} < \infty\right\}, \\
E_4(p) &:= \left\{a = (a_k) \in \omega : \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j \text{ is convergent}\right\}, \\
E_5(p) &:= \bigcup_{B>1} \left\{a = (a_k) \in \omega : \sup_{n\in\mathbb{N}} \sum_k \left|\sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j B^{-1}\right|^{p_k'} < \infty\right\}.
\end{aligned}$$

Because of Part (i) can be established in a similar way to the proof of Part (ii), we give the proof only for Part (ii) in Theorems 3.4 and 3.5, below.

Theorem 3.4. *The following statements hold:*

(i) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $\{\ell(F,p)\}^\alpha = E_1(p)$.*
(ii) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $\{\ell(F,p)\}^\alpha = E_2(p)$.*

Proof. Let us take any $a = (a_n) \in \omega$. By using (2.2), we obtain that

$$a_n x_n = \sum_{k=0}^{n} \frac{f_{n+1}^2}{f_k f_{k+1}} a_n y_k = (Ey)_n \quad \text{for all } n \in \mathbb{N}, \tag{3.5}$$

where $E=(e_{nk})$ is defined by $e_{nk}=\begin{cases} \frac{f_{n+1}^2}{f_kf_{k+1}}a_n & , \quad 0\le k\le n, \\ 0 & , \quad k>n \end{cases}$ for all $k,n\in\mathbb{N}$. Thus, we observe by combining (3.5) with the condition (3.1) of Part (ii) of Lemma 3.1 that $ax=(a_nx_n)\in\ell_1$ whenever $x=(x_k)\in\ell(F,p)$ if and only if $Ey\in\ell_1$ whenever $y=(y_k)\in\ell(p)$. This leads to the fact that $\{\ell(F,p)\}^\alpha=E_2(p)$, as asserted. □

Theorem 3.5. *The following statements hold:*

(i) *Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $\{\ell(F,p)\}^\beta=E_3(p)\cap E_4(p)$.*
(ii) *Let $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $\{\ell(F,p)\}^\beta=E_4(p)\cap E_5(p)$.*

Proof. Take any $a=(a_j)\in\omega$. Then, one can obtain by (2.2) that

$$\sum_{j=0}^n a_jx_j=\sum_{j=0}^n\left(\sum_{k=0}^j\frac{f_{j+1}^2}{f_kf_{k+1}}y_k\right)a_j=\sum_{k=0}^n\left(\sum_{j=k}^n\frac{f_{j+1}^2}{f_kf_{k+1}}a_j\right)y_k=(Dy)_n \tag{3.6}$$

for all $n\in\mathbb{N}$, where $D=(d_{nk})$ is defined by

$$d_{nk}=\begin{cases} \sum_{j=k}^n\frac{f_{j+1}^2}{f_kf_{k+1}}a_j & , \quad 0\le k\le n, \\ 0 & , \quad k>n \end{cases} \tag{3.7}$$

for all $n,k\in\mathbb{N}$. Thus, we deduce from Lemma 3.3 with (3.6) that $ax=(a_jx_j)\in cs$ whenever $x=(x_j)\in\ell(F,p)$ if and only if $Dy\in c$ whenever $y=(y_k)\in\ell(p)$. Therefore, we derive from (3.3) and (3.4) that

$$\sup_{n\in\mathbb{N}}\sum_k\left|\sum_{j=k}^n\frac{f_{j+1}^2}{f_kf_{k+1}}a_jB^{-1}\right|^{p_k'}<\infty,\quad \sum_{j=k}^\infty\frac{f_{j+1}^2}{f_kf_{k+1}}a_j<\infty.$$

This shows that $\{\ell(F,p)\}^\alpha=E_4(p)\cap E_5(p)$. □

Theorem 3.6. *The following statements hold:*

(i) *Let $0<p_k\le 1$ for all $k\in\mathbb{N}$. Then, $\{\ell(F,p)\}^\gamma=E_3(p)$.*
(ii) *Let $1<p_k\le H<\infty$ for all $k\in\mathbb{N}$. Then, $\{\ell(F,p)\}^\gamma=E_5(p)$.*

Proof. From Lemma 3.2 and (3.6), we obtain that $ax=(a_jx_j)\in bs$ whenever $x=(x_j)\in\ell(F,p)$ if and only if $Dy\in\ell_\infty$ whenever $y=(y_k)\in\ell(p)$, where $D=(d_{nk})$ is defined by (3.7). Therefore we obtain from (3.2) and (3.3) that $\{\ell(F,p)\}^\gamma=\begin{cases} E_3(p) & , \quad p_k\le 1, \\ E_5(p) & , \quad p_k>1 \end{cases}$, as desired. □

4. Matrix Transformations On The Space $\ell(F,p)$

In this section, we characterize some matrix transformations on the space $\ell(F,p)$. Since the cases $0<p_k\le 1$ and $1<p_k\le H<\infty$ are combined, Theorem 4.1 gives the exact conditions of the general case $0<p_k\le H<\infty$. We consider only the case $1<p_k\le H<\infty$ and omit the proof of the case $0<p_k\le 1$, since it can be proved in a similar way.

Theorem 4.1. *The following statements hold:*

(i) *Let* $0 < p_k \leq 1$ *for all* $k \in \mathbb{N}$*. Then,* $A = (a_{nk}) \in (\ell(F,p) : \ell_\infty)$ *if and only if*

$$\sup_{k,n\in\mathbb{N}} \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} \right|^{p_k} < \infty, \tag{4.1}$$

$$\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} < \infty. \tag{4.2}$$

(ii) *Let* $1 < p_k \leq H < \infty$ *for all* $k \in \mathbb{N}$*. Then* $A = (a_{nk}) \in (\ell(F,p) : \ell_\infty)$ *if and only if (4.2) holds and there exists an integer* $B > 1$ *such that*

$$\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} < \infty. \tag{4.3}$$

Proof. Let $A \in (\ell(F,p) : \ell_\infty)$ and $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, Ax exists for every $x \in \ell(F,p)$ and this implies that $A_n \in \{\ell(F,p)\}^\beta$ for each fixed $n \in \mathbb{N}$. Therefore, the necessities of (4.2) and (4.3) are immediate.

Conversely, suppose that the conditions (4.2) and (4.3) hold, and take any $x \in \ell(F,p)$. Since $A_n \in \{\ell(F,p)\}^\beta$ for every $n \in \mathbb{N}$, the A-transform of x exists. By using (2.2), we obtain that

$$\sum_{j=0}^{m} a_{nj} x_j = \sum_{j=0}^{m} \sum_{k=0}^{j} \frac{f_{j+1}^2}{f_k f_{k+1}} y_k a_{nj} = \sum_{k=0}^{m} \sum_{j=k}^{m} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k \tag{4.4}$$

for all $m, n \in \mathbb{N}$. Taking into account the hypothesis, we drive from (4.4), as $m \to \infty$ that

$$\sum_j a_{nj} x_j = \sum_k \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k \quad \text{for all} \quad n \in \mathbb{N}. \tag{4.5}$$

By combining (4.5) and the inequality which holds for any complex numbers a, b and any $B > 0$

$$|ab| \leq B \left(\left| aB^{-1} \right|^{p'} + |b|^p \right),$$

where $p > 1$ and $p^{-1} + p'^{-1} = 1$, we obtain that

$$\begin{aligned} \sup_{n\in\mathbb{N}} \left| \sum_j a_{nj} x_j \right| &= \sup_{n\in\mathbb{N}} \left| \sum_k \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k \right| \leq \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} y_k \right| \\ &\leq \sup_{n\in\mathbb{N}} \sum_k B \left(\left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} + |y_k|^{p_k} \right) \\ &= B \left(\sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} + \sup_{n\in\mathbb{N}} \sum_k |y_k|^{p_k} \right) < \infty. \end{aligned}$$

This shows that $Ax \in \ell_\infty$. □

Theorem 4.2. *The following statements hold:*

(i) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : c)$ if and only if (4.1) and (4.2) hold, and there is a sequence $\alpha = (\alpha_k)$ of scalars such that*

$$\lim_{n\to\infty} \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} = \alpha_k \quad \text{for all} \quad k \in \mathbb{N}. \tag{4.6}$$

(ii) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : c)$ if and only if (4.2), (4.3) and (4.6) hold.*

Proof. Let $A \in (\ell(F,p) : c)$ and $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, since the inclusion $c \subset \ell_\infty$ holds, the necessities of (4.2) and (4.3) are immediately obtained from Theorem 4.1.

To prove the necessity of (4.6), consider the sequence $b^{(k)}$ defined by (2.8), which belongs to the space $\ell(F,p)$ for every fixed $k \in \mathbb{N}$. Since the A-transform of every $x \in \ell(F,p)$ exists and is in c by the hypothesis, we have

$$Ab^{(k)} = \left(\sum_{j=0}^{\infty} a_{ij} b_j^{(k)} \right)_{i=0}^{\infty} = \left(\sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{ij} \right)_{i=0}^{\infty} \in c$$

for every fixed $k \in \mathbb{N}$, which shows the necessity (4.6).

Conversely, suppose that the conditions (4.2), (4.3) and (4.6) hold, and take any $x = (x_k)$ in the space $\ell(F,p)$. Then, Ax exists.

We observe for all $m, n \in \mathbb{N}$ that

$$\sum_{k=0}^{m} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} \leq \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} < \infty$$

which gives the fact by letting $m, n \to \infty$ with (4.3) and (4.6)

$$\lim_{m,n\to\infty} \sum_{k=0}^{m} \left| \sum_{j=k}^{n} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} \leq \sup_{n\in\mathbb{N}} \sum_k \left| \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj} B^{-1} \right|^{p_k'} < \infty.$$

This shows that $\sum_k \left|\alpha_k B^{-1}\right|^{p_k'} < \infty$ and $(\alpha_k) \in \{\ell(F,p)\}^\beta$ which implies that the series $\sum_k \alpha_k x_k$ converges for all $x \in \ell(F,p)$.

Now, let us consider the equality obtained from (4.5) with $a_{nj} - \alpha_j$ instead of a_{nj}

$$\sum_j (a_{nj} - \alpha_j) x_j = \sum_k \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} (a_{nj} - \alpha_j) y_k = \sum_k c_{nk} y_k, \tag{4.7}$$

where $C = (c_{nk})$ defined by $c_{nk} = \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} (a_{nj} - \alpha_j)$ for all $k, n \in \mathbb{N}$. From Lemma 3.3, $c_{nk} \to 0$, as $n \to \infty$, for all $k \in \mathbb{N}$. Therefore, we see by (4.7) that $\sum_k (a_{nk} - \alpha_k) x_k \to 0$, as $n \to \infty$. This means that $Ax \in c$ whenever $x \in \ell(F,p)$ and this step completes the proof. □

Corollary 4.3. *The following statements hold:*

(i) *Let $0 < p_k \leq 1$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : c_0)$ if and only if (4.1) and (4.2) hold, and (4.6) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

(ii) *Let $1 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Then, $A = (a_{nk}) \in (\ell(F,p) : c_0)$ if and only if (4.2) and (4.3) hold, and (4.6) also holds with $\alpha_k = 0$ for all $k \in \mathbb{N}$.*

Now, we can give the following lemma which is useful for deriving the characterization of the classes of matrix transformations from the space $\ell(F,p)$ to the space λ_A, where $\lambda \in \{\ell_\infty, c, c_0\}$ and $A \in \{\Delta, E^r, C_1, R^t, \sum, F\}$.

Lemma 4.1. [10, Lemma 5.3] *Let λ, μ be any two sequence spaces, A be an infinite matrix and B be a triangle matrix. Then, $A \in (\lambda : \mu_B)$ if and only if $BA \in (\lambda : \mu)$.*

Lemma 4.1 has several consequences depending on the choice of the space μ. Indeed, combining Lemma 4.1 with Theorems 4.1, 4.2 and Corollary 4.3, one can obtain the following results:

Corollary 4.2. *Let $A = (a_{nk})$ be an infinite matrix of complex terms. Then, the following statements hold:*

(i) *$E = (e_{nk}) \in (\ell(F,p) : bv_\infty)$ if and only if (4.1)-(4.3) hold with d_{nk} instead of a_{nk}; where $d_{nk} = e_{nk} - e_{n-1,k}$ for all $k, n \in \mathbb{N}$ and bv_∞ denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k-1}) \in \ell_\infty$, and was introduced by Başar and Altay* [10].

(ii) *$E = (e_{nk}) \in (\ell(F,p) : e^r_\infty)$ if and only if (4.1)-(4.3) hold with d_{nk} instead of a_{nk}, where $d_{nk} = \sum_{j=0}^{n} \binom{n}{j}(1-r)^{n-j} r^j e_{jk}$ for all $k, n \in \mathbb{N}$ and e^r_∞ denotes the space of all sequences $x = (x_k)$ such that $E^r x \in \ell_\infty$, and was introduced by Altay, Başar and Mursaleen* [11].

(iii) *$E = (e_{nk}) \in (\ell(F,p) : X_\infty)$ if and only if (4.1)-(4.3) hold with d_{nk} instead of a_{nk}; where $d_{nk} = \sum_{j=0}^{n} e_{jk}/(n+1)$ for all $k, n \in \mathbb{N}$ and X_∞ denotes the space of all sequences $x = (x_k)$ such that $C_1 x \in \ell_\infty$, and was introduced by Ng and Lee* [12].

(iv) *$E = (e_{nk}) \in (\ell(F,p) : r^t_\infty)$ if and only if (4.1)-(4.3) hold with d_{nk} instead of a_{nk}; where $d_{nk} = \sum_{j=0}^{n} t_j e_{jk}/T_n$ for all $k, n \in \mathbb{N}$ and r^t_∞ denotes the space of all sequences $x = (x_k)$ such that $R^t x \in \ell_\infty$, and was introduced by Altay and Başar* [13].

(v) *$E = (e_{nk}) \in (\ell(F,p) : bs)$ if and only if (4.1)-(4.3) hold with d_{nk} instead of a_{nk}, where $d_{nk} = \sum_{j=0}^{n} e_{jk}$ for all $k, n \in \mathbb{N}$.*

(vi) *$E = (e_{nk}) \in (\ell(F,p) : \ell_\infty(\widehat{F}))$ if and only if (4.1)-(4.3) hold with d_{nk} instead of a_{nk}, where $d_{nk} = -\frac{f_{n+1}}{f_n} e_{n-1,k} + \frac{f_n}{f_{n+1}} e_{nk}$ for all $k, n \in \mathbb{N}$ and $\ell_\infty(\widehat{F})$ denotes the space of all sequences $x = (x_k)$ such that $Fx \in \ell_\infty$, and was introduced by Kara* [14].

Corollary 4.3. *Let $A = (a_{nk})$ be an infinite matrix of complex terms. Then, the following statements hold:*

(i) *$E = (e_{nk}) \in (\ell(F,p) : c(\Delta))$ if and only if (4.1)-(4.3) and (4.6) hold with d_{nk} instead of a_{nk}; where $d_{nk} = e_{nk} - e_{n+1,k}$ for all $k, n \in \mathbb{N}$ and $c(\Delta)$ denotes the space of all sequences $x = (x_k)$ such that $(x_k - x_{k+1}) \in c$, and was introduced by Kızmaz* [15].

(ii) *$E = (e_{nk}) \in (\ell(F,p) : e^r_c)$ if and only if (4.1)-(4.3) and (4.6) hold with d_{nk} instead of a_{nk}, where $d_{nk} = \sum_{j=0}^{n} \binom{n}{j}(1-r)^{n-j} r^j e_{jk}$ for all $k, n \in \mathbb{N}$ and e^r_c denotes the space of all sequences $x = (x_k)$ such that $E^r x \in c$, and was introduced by Altay and Başar* [16].

(iii) $E=(e_{nk})\in(\ell(F,p):\widetilde{c})$ *if and only if (4.1)-(4.3) and (4.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=\sum_{j=0}^{n}e_{jk}/(n+1)$ *for all* $k,n\in\mathbb{N}$ *and* $\widetilde{c}$ *denotes the space of all sequences* $x=(x_k)$ *such that* $C_1x\in c$*, and was introduced by Şengönül and Başar* [17].

(iv) $E=(e_{nk})\in(\ell(F,p):r_c^t)$ *if and only if (4.1)-(4.3) and (4.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=\sum_{j=0}^{n}t_je_{jk}/T_n$ *for all* $k,n\in\mathbb{N}$ *and* r_c^t *denotes the space of all sequences* $x=(x_k)$ *such that* $R^tx\in c$*, and was introduced by Altay and Başar* [18].

(v) $E=(e_{nk})\in(\ell(F,p):c(\widehat{F}))$ *if and only if (4.1)-(4.3) and (4.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=-\frac{f_{n+1}}{f_n}e_{n-1,k}+\frac{f_n}{f_{n+1}}e_{nk}$ *for all* $k,n\in\mathbb{N}$ *and* $c(\widehat{F})$ *denotes the space of all sequences* $x=(x_k)$ *such that* $Fx\in c$*, and was introduced by Başarır et al.* [19].

(vi) $E=(e_{nk})\in(\ell(F,p):cs)$ *if and only if (4.1)-(4.3) and (4.6) hold with* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=\sum_{j=0}^{n}e_{jk}$ *for all* $k,n\in\mathbb{N}$.

Corollary 4.4. *Let* $A=(a_{nk})$ *be an infinite matrix of complex terms. Then, the following statements hold:*

(i) $E=(e_{nk})\in(\ell(F,p):c_0(\Delta))$ *if and only if (4.1)-(4.3) hold and (4.6) also holds with* $\alpha_k=0$ *for all* $k\in\mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=e_{nk}-e_{n+1,k}$ *for all* $k,n\in\mathbb{N}$ *and* $c_0(\Delta)$ *denotes the space of all sequences* $x=(x_k)$ *such that* $(x_k-x_{k+1})\in c_0$*, and was introduced by Kızmaz* [15].

(ii) $E=(e_{nk})\in(\ell(F,p):e_0^r)$ *if and only if (4.1)-(4.3) hold and (4.6) also holds with* $\alpha_k=0$ *for all* $k\in\mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*, where* $d_{nk}=\sum_{j=0}^{n}\binom{n}{j}(1-r)^{n-j}r^je_{jk}$ *for all* $k,n\in\mathbb{N}$ *and* e_0^r *denotes the space of all sequences* $x=(x_k)$ *such that* $E^rx\in c_0$*, and was introduced by Altay and Başar* [16].

(iii) $E=(e_{nk})\in(\ell(F,p):\widetilde{c}_0)$ *if and only if (4.1)-(4.3) hold and (4.6) also holds with* $\alpha_k=0$ *for all* $k\in\mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=\sum_{j=0}^{n}e_{jk}/(n+1)$ *for all* $k,n\in\mathbb{N}$ *and* $\widetilde{c}_0$ *denotes the space of all sequences* $x=(x_k)$ *such that* $C_1x\in c_0$*, and was introduced by Şengönül and Başar* [17].

(iv) $E=(e_{nk})\in(\ell(F,p):r_0^t)$ *if and only if (4.1)-(4.3) hold and (4.6) also holds with* $\alpha_k=0$ *for all* $k\in\mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=\sum_{j=0}^{n}t_je_{jk}/T_n$ *for all* $k,n\in\mathbb{N}$ *and* r_0^t *denotes the space of all sequences* $x=(x_k)$ *such that* $R^tx\in c_0$*, and was introduced by Altay and Başar* [18].

(v) $E=(e_{nk})\in(\ell(F,p):c_0(\widehat{F}))$ *if and only if (4.1)-(4.3) hold and (4.6) also holds with* $\alpha_k=0$ *for all* $k\in\mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=-\frac{f_{n+1}}{f_n}e_{n-1,k}+\frac{f_n}{f_{n+1}}e_{nk}$ *for all* $k,n\in\mathbb{N}$ *and* $c_0(\widehat{F})$ *denotes the space of all sequences* $x=(x_k)$ *such that* $Fx\in c_0$*, and was introduced by Başarır et al.* [19].

(vi) $E=(e_{nk})\in(\ell(F,p):c_0s)$ *if and only if (4.1)-(4.3) hold and (4.6) also holds with* $\alpha_k=0$ *for all* $k\in\mathbb{N}$ *and* d_{nk} *instead of* a_{nk}*; where* $d_{nk}=\sum_{j=0}^{n}e_{jk}$ *for all* $k,n\in\mathbb{N}$ *and* c_0s *denotes the space of all sequences* $x=(x_k)$ *such that* $\sum_k x_k=0$.

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(H. Çapan) The Graduate School of Sciences and Engineering, Fatih University, The Hadimköy Campus, Büyükçekmece, 34500–İstanbul, Turkey

E-mail address: husamettincapan@gmail.com

(F. Başar) Department of Mathematics, Faculty of Arts and Sciences, Fatih University, The Hadimköy Campus, Büyükçekmece, 34500–İstanbul, Turkey

E-mail address: fbasar@fatih.edu.tr, feyzibasar@gmail.com