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MULTIDUAL NUMBERS AND THEIR MULTIDUAL FUNCTIONS

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Abstract. The purpose of this paper is to develop a general theory of multidual numbers. We start by defining the notion of multidual numbers and their algebraic properties. In addition, we develop a simple mathematical method based on matrices, simplifying manipulation of multidual numbers. Inspired from multicomplex analysis, we define the multidual functions and we generalize the concept of hyperholomorphicity. Moreover, we obtain a general representation of hyperholomorphic multidual functions using the notion of generator polynomials. As concrete examples, some usual real functions have been generalized to the case of multidual numbers, such that the exponential and logarithmic multidual function. Finally, we extend, using the multidual functions, the Galilean Trigonometric functions and their inverses functions to the multi-dimensional case as well as some of their algebraic and analytic properties.

1. Introduction

The theory of algebra of dual numbers has been originally introduced by W. K. Clifford [2] in 1873, and he showed that they form an algebra but not a field because only dual numbers with real part not zero possess an inverse element. In 1891 E. Study [14] realized that this associative algebra was ideal for describing the group of motions of three-dimensional space. At the turn of the 20th century, A. Kotelnikov [8] developed dual vectors and dual quaternions. Algebraic study of dual numbers is the topic of numerous papers, e.g. [2, 7]. This nice concept has lots of applications in many fields of fundamental sciences; as, algebraic geometry, Riemannian geometry, quantum mechanics and astronomy. It also arises in various contexts of engineering: aerospace, robotic and computer science. For more details about the applications of dual numbers, we refer the reader to [3, 5, 14, 16, 17].

However, up to now there are only a few attempts in the mathematical study of dual functions (functions of dual variable). An early attempt is due to E. E. Kramer [9] in 1930. Later, in 2011, Z. Ercan and S. Yüce [4] obtained generalized Euler's and De Moivre's formulas for functions with dual Quaternion variable. Recently, F. Messelmi [11] develops a theory, inspired from complex analysis, of dual function

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and he generalizes the notion of holomorphic dual functions and obtained some interesting properties.

Furthermore, the concept of multicomplex numbers has been defined and introduced by many mathematicians and Physicists. The starting point is the introduction of a generator i, such that $i^n = -1$ and create the space of multicomplex numbers of order n, $\mathbb{MC}_n$. In keeping with the case $n = 2$ of usual complex numbers and their trigonometric functions, an associated extended trigonometry follows. It is characterized by specific "angular" functions dubbed multisine (mus). A collection of useful relations exists between the mus-functions: additions, derivatives, etc, see for more details about multicomplex numbers the references [**?**, 6, 10, 12, 13]

More recently, the theory becomes one of the important impulses for developing some new concept of quantum mechanics and cosmology.

The purpose of this paper is to contribute to the development of multidual numbers, by generalizing the dual numbers in higher dimensions, as well as their functions. Moreover, in the study of multidual functions (functions of multidual variable) some natural question raise:

- When and under what conditions a multidual function is differentiable ?.
- How can one extend regularly real functions to multidual variable ?.

Throughout the paper, we will try to answer some of these questions.

In details, we start by generalizing the notion of hyperholomorphicity to multidual functions. To this end, as in multicomplex analysis, we study the Differentiability of multidual dual functions. The notion of hyperholomorphicity has been introduced and a general representation of hyperholomorphic functions was shown, using the new concept of generator polynomial. Moreover, we provide the basic statements that allow us to extend holomorphically real functions to the wider multidual generalized Clifford Algebra and we ensure that such an extension is meaningful. As an application, we generalize some usual real functions to multidual Algebra.

In this work we have not shown physical applications of all concepts presented here. However, we will try to find future applications.

The outline of the paper is as follows. In Section 2 we focus on the development of multidual numbers and their algebraic properties.

Section 3 is devoted to the study of multidual functions. To this aim, we generalize the concept of hyperholomorphy to multidual numbers and a few properties have been established. As example, we generalize in the last chapter some usual real function to multidual variables and we introduce the Galilean multi-Trigonometric functions and their inverses.

2. Multidual Numbers

We introduce the concept of Multidual numbers as follows.

A multidual number z is an ordered $(n+1)$ −tuple of real numbers $(x_0, x_1, ..., x_n)$ associated with the real unit 1 and the powers of the multidual unit ε, where ε is an $(n+1)$ −nilpotent number i.e. $\varepsilon^{n+1} = 0$ and $\varepsilon^i \neq 0$ for $i = 1, ..., n$. A multidual number is usually denoted in the form

$$z = \sum_{i=0}^{n} x_i \varepsilon^i. \tag{1}$$

for which, we admit that $\varepsilon^0 = 1$.

We denote by $\mathbb{D}_n$ the set of multidual numbers defined as

$$\mathbb{D}_n = \left\{ z = \sum_{i=0}^{n} x_i \varepsilon^i \mid x_i \in \mathbb{R} \text{ where } \varepsilon^{n+1} = 0 \text{ and } \varepsilon^i \neq 0 \text{ for } i = 1, ..., n \right\} \tag{2}$$

Furthermore, every element $z = \sum\limits_{i=0}^{n} x_i \varepsilon^i$ of $\mathbb{D}_n$ can be also written

$$z = \mathcal{X}(z)^T \mathcal{E}, \tag{3}$$

where $\mathcal{X}(z)$ is the real vector associated to the multidual number z given by

$$\mathcal{X}(z) = \begin{bmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ x_n \end{bmatrix}, \tag{4}$$

and $\mathcal{E}$ represents the following vector, called multidual vector,

$$\mathcal{E} = \begin{bmatrix} 1 \\ \varepsilon \\ \varepsilon^2 \\ . \\ . \\ \varepsilon^n \end{bmatrix}. \tag{5}$$

There are many ways to choose the multidual unit number ε. As simple example, we can take the real matrix

$$\varepsilon = \begin{bmatrix} 0 & 0 & . & . & . & 0 \\ 1 & 0 & . & . & . & 0 \\ 0 & 1 & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & 0 \\ 0 & 0 & . & 0 & 1 & 0 \end{bmatrix}. \tag{6}$$

Addition and multiplication of the multidual numbers are defined by

$$\sum_{i=0}^{n} x_i \varepsilon^i + \sum_{i=0}^{n} y_i \varepsilon^i = \sum_{i=0}^{n} (x_i + y_i) \varepsilon^i, \tag{7}$$

$$\left\{ \begin{aligned} \left(\sum_{i=0}^{n} x_i \varepsilon^i \right) \cdot \left(\sum_{i=0}^{n} y_i \varepsilon^i \right) &= \sum_{i=0}^{n} \left(\sum_{j=0}^{i} x_j y_{i-j} \right) \varepsilon^i \\ &= \sum_{i=0}^{n} \left(\sum_{j=0}^{i} y_j x_{i-j} \right) \varepsilon^i. \end{aligned} \right. \tag{8}$$

If $z = \sum\limits_{i=0}^{n} x_i \varepsilon^i$ is a multidual number, we will denote by $\operatorname{real}(z)$ the real part of z given by

$$\operatorname{real}(z) = x_0. \tag{9}$$

Thus, the multidual numbers form a commutative ring with characteristic 0. Moreover the inherited multiplication gives the multidual numbers the structure of $(n+1)$ −dimensional generalized Clifford Algebra.

For $n = 1$, $\mathbb{D}_1$ is the Clifford algebra of dual numbers, see for more details regarding dual numbers the references [2, 7, 11].

In abstract algebra terms, the multidual numbers can be described as the quotient of the polynomial ring $\mathbb{R}[X]$ by the ideal generated by the polynomial X^{n+1}, i.e.

$$\mathbb{D}_n \approx \mathbb{R}[X]/X^{n+1}. \tag{10}$$

If z is a multidual number, the conjugate of z denoted by $\bar{z}$ is the multidual number described by

$$\begin{cases} \operatorname{real}(\bar{z}) = \operatorname{real}(z), \\ z\bar{z} \in \mathbb{R}. \end{cases} \tag{11}$$

Suppose now that $z = \sum_{i=0}^{n} x_i\varepsilon^i$ and $\bar{z} = \sum_{i=0}^{n} \overline{x_i}\varepsilon^i$. Then, using relation (11) we get

$$z\bar{z} = x_0^2 + \sum_{1=1}^{n} \left(\sum_{j=0}^{i} x_j \overline{x_{i-j}} \right) \varepsilon^i \in \mathbb{R}.$$

Which implies that

$$\sum_{j=0}^{i} x_j \overline{x_{i-j}} = 0 \quad \forall i = 1, ..., n.$$

This can be written in matrix form

$$\begin{bmatrix} x_0 & 0 & . & . & 0 & 0 \\ x_1 & x_0 & . & . & . & 0 \\ x_2 & x_1 & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ x_{n-1} & . & . & x_2 & x_1 & x_0 \end{bmatrix} \begin{bmatrix} \overline{x_1} \\ \overline{x_2} \\ . \\ . \\ . \\ \overline{x_n} \end{bmatrix} = -x_0 \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}. \tag{12}$$

Hence $z = \sum_{i=0}^{n} x_i\varepsilon^i$ has a unique conjugate, solution of the system (12), if and only if $\operatorname{real}(z) = x_0 \neq 0$. On the other hand if $x_0 = 0$, we remark that the number $\sum_{i=1}^{n} x_i\varepsilon^i$ is a divisor of zero in $\mathbb{D}_n$.

Proposition 1 Let $z = \sum_{i=0}^{n} x_i\varepsilon^i$ and $t = \sum_{i=0}^{n} y_i\varepsilon^i$ be two multidual numbers, such that $x_0, y_0 \neq 0$. Then,

$$\overline{zt} = \bar{z}\bar{t}. \tag{13}$$

Proof. We get from the relation (11)

$$z\bar{z} = x_0^2 \quad \forall z \in \mathbb{D}_n.$$

This allows us to write

$$(zt)\left(\overline{zt}\right) = (x_0y_0)^2.$$

Thus, under the hypothesis $x_0, y_0 \neq 0$, we find

$$\begin{aligned} \overline{zt} &= \frac{x_0^2y_0^2}{zt} \\ &= \frac{x_0^2}{z}\frac{y_0^2}{t} = \bar{z}\bar{t}. \end{aligned}$$

Particularly, the map $z \in \mathbb{D}_n - \{0\} \longmapsto \overline{z} \in \mathbb{D}_n - \{0\}$ is an automorphism of groups.

Remark 1 If $n \geq 2$, $\overline{z+t} \neq \overline{z} + \overline{t}$.

It is also important to know that every multidual number has another representation, using matrices.

To this aim, let us denote by $\mathcal{G}_{n+1}(\mathbb{R})$ the subset of $\mathcal{M}_{n+1}(\mathbb{R})$ given by

$$\begin{aligned}\mathcal{G}_{n+1}(\mathbb{R}) &= \{A = (a_{ij}) \in \mathcal{M}_{n+1}(\mathbb{R}) \mid a_{ij} = 0 \text{ if } i > j \\ &\quad \text{and } a_{i+1,j+1} = a_{ij} \text{ if } 0 \leq i \leq j \leq n\}. \end{aligned} \tag{14}$$

So, every matrix A of $\mathcal{G}_{n+1}(\mathbb{R})$ is such that

$$A = \begin{bmatrix} a_0 & 0 & . & . & 0 & 0 \\ a_1 & a_0 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ a_{n-1} & . & . & . & a_0 & 0 \\ a_n & a_{n-1} & . & . & a_1 & a_0 \end{bmatrix}. \tag{15}$$

One can easily verify that $\mathcal{G}_{n+1}(\mathbb{R})$ is a subring of $\mathcal{M}_{n+1}(\mathbb{R})$ which forms a $(n+1)-$dimensional associative and commutative Algebra.

If $a_0 \neq 0$, $\mathcal{G}_{n+1}$ becomes a field. It is also a subgroup of $GL(n+1)$.

Introducing now the map

$$\begin{cases} \mathcal{N} : \mathbb{D}_n \longrightarrow \mathcal{G}_{n+1}(\mathbb{R}), \\ \mathcal{N}\left(\sum\limits_{i=0}^{n} x_i \varepsilon^i\right) = \begin{bmatrix} x_0 & 0 & . & . & . & 0 \\ x_1 & x_0 & 0 & . & . & 0 \\ . & x_1 & . & . & . & . \\ . & . & . & . & . & . \\ x_{n-1} & . & . & . & x_0 & 0 \\ x_n & x_{n-1} & . & . & x_1 & x_0 \end{bmatrix} \end{cases} \tag{16}$$

The following results are immediate consequences of the definitions $\mathcal{G}_{n+1}(\mathbb{R})$ of and $\mathcal{N}$.

Theorem 2 $\mathcal{N}$ is an isomorphism of rings.

Corollary 3 Let $z = \sum\limits_{i=0}^{n} a_i \varepsilon^i \in \mathbb{D}_n$. Then,

(1) $z = e_1^T \mathcal{N}(z)^T \mathcal{E}$, where e_1 is the first element in the canonical base of $\mathbb{R}^{n+1}$.

(2) $z^n = e_1^T (\mathcal{N}(z)^n)^T \mathcal{E}$.

(3) In addition, if $z \neq 0$, then $\frac{1}{z} = e_1 \left(\mathcal{N}(z)^{-1}\right)^T \mathcal{E}$.

3. Multidual Functions

We start by giving some topological definitions and properties of $\mathbb{D}_n$.

Introducing the mapping

$$\begin{cases} \mathcal{P} : \mathbb{D}_n \longrightarrow \mathbb{R}_+, \\ \mathcal{P}(z) = |\mathrm{real}(z)|. \end{cases} \tag{17}$$

One can easily to verify that

$$\begin{cases} z\bar{z} = \mathcal{P}(z)^2 \quad \forall z \in \mathbb{D}_n. \\ \mathcal{P}(z_1 + z_2) \leq \mathcal{P}(z_1) + \mathcal{P}(z_2) \quad \forall z_1, z_2 \in \mathbb{D}_n, \\ \mathcal{P}(z_1 z_2) = \mathcal{P}(z_1)\mathcal{P}(z_2) \quad \forall z_1, z_2 \in \mathbb{D}_n, \\ \mathcal{P}(\lambda z) = |\lambda| \mathcal{P}(z) \quad \forall z \in \mathbb{D}_n, \forall \lambda \in \mathbb{R}, \\ \mathcal{P}(0) = 0. \end{cases} \tag{18}$$

In particularl, $\mathcal{P}$ defines a semi-norm in $\mathbb{D}_n$. It inducess a structure of pseudo-topology over $\mathbb{D}_n$.

Thus, we can define the multidual disk and multidual sphere of centre $t = \sum_{i=0}^{n} y_i \varepsilon^i \in \mathbb{D}_n$ and radius $r > 0$, respectively, by

$$\begin{aligned} D(t,r) &= \left\{ z = \sum_{i=0}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid p(z-t) < r \right\} \\ &= \left\{ z = \sum_{i=0}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid |x_i - y_i| < r,\ x_i \in \mathbb{R},\ i = 1, ..., n \right\}, \end{aligned} \tag{19}$$

$$\begin{aligned} S(t,r) &= \left\{ z = \sum_{i=0}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid p(z-t) = r \right\} \\ &= \left\{ z = \sum_{i=0}^{n} x_i \varepsilon^i \in \mathbb{D}_n \mid |x_i - y_i| = r,\ x_i \in \mathbb{R},\ i = 1, ..., n \right\}. \end{aligned} \tag{20}$$

$S(t,r)$ can be also called the generalized Galilean sphere.

Definition 1

(1) We say that Ω is a multidual subset of the multidual algebra $\mathbb{D}_n$ if there exists a subset $O \subset \mathbb{R}$ such that

$$\Omega = O \times \mathbb{R}^n. \tag{21}$$

O is called the generator of Ω.

(2) We say that Ω is an open multidual subset of the multidual algebra $\mathbb{D}_n$ if the generator of Ω is an open subset of $\mathbb{R}$.

(3) Ω is said to be a closed multidual subset of $\mathbb{D}_n$ if the complemente is an open subset of $\mathbb{D}$.

(4) Ω is said to be a connected multidual subset of $\mathbb{D}_n$ if the generator is a connected subset of $\mathbb{R}$.

(5) Ω is said to be a compact multidual subset of $\mathbb{D}_n$ if the generator is a compact subset of $\mathbb{R}$.

We discuss now some properties of multidual functions. We investigate the continuity of multidual functions and the differentiability in the multidual sense, which can be also called hyperholomorphicity, as in multicomplexe case.

In the following definitions, we suppose that $\mathbb{D}_n$ is equipped with the usual topology of $\mathbb{R}^{n+1}$.

Definition 2 A multidual function is a mapping from a subset $\Omega \subset \mathbb{D}_n$ to $\mathbb{D}_n$.

Let Ω be an open subset of $\mathbb{D}_n$, $t = \sum_{i=0}^{n} y_i \varepsilon^i \in \Omega$ and $f : \Omega \longrightarrow \mathbb{D}_n$ a multidual function.

Definition 3 We say that the function f is continuous at t if

$$\lim_{z\longrightarrow t} f(z) = f(t). \tag{22}$$

where the limit is calculated coordinate by coordinate, this means that

$$\lim_{z\longrightarrow t} f(z) = \lim_{x_i\longrightarrow y_i \ i=0,\dots,n} f(z) = f(t). \tag{23}$$

Definition 4 The function is continuous in $\Omega \subset \mathbb{D}_n$ if it is continuous at every point of Ω.

Definition 5 The multidual function f is said to be differentiable in the multidual sense at $t = \sum_{i=0}^{n} y_i \varepsilon^i$ if the following limit exists

$$\frac{df}{dz}(t) = \lim_{z\longrightarrow t} \frac{f(z) - f(t)}{z - t}, \tag{24}$$

$\frac{df}{dz}(t)$ is called the derivative of f at the point t.
If f is differentiable for all points in a neighbourhood of the point t then f is called hyperholomorphic at t.

Definition 6 The function f is hyperholomorphic in $\Omega \subset \mathbb{D}_n$ if it is hyperholomorphic at every point of Ω.

The definition of derivative in the multidual sense has to be treated with a little more care than its real companion; this is illustrated by the following example.

Example 1 The function $f : \mathbb{D}_1 \longrightarrow \mathbb{D}_1$ such that $f(z) = \overline{z}$ is nowhere differentiable. To this aim, a simple calculation gives

$$\begin{aligned}
\lim_{z\rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} &= \lim_{z\rightarrow z_0} \frac{\overline{z} - \overline{z_0}}{z - z_0} \\
&= \lim_{z\rightarrow z_0} \frac{\overline{z - z_0}^2}{(x - x_0)^2} \\
&= 1 - 2\varepsilon \lim_{x\rightarrow x_0, y\rightarrow y_0} \frac{y - y_0}{x - x_0}.
\end{aligned}$$

But this limit does not exist.

The basic properties for derivatives are similar to those we know from real calculus. In fact, one should convince oneself that the following rules follow mostly from properties of the limit.

Proposition 4 Suppose f and g are differentiable at $z \in \mathbb{D}_n$, and that $c \in \mathbb{D}_n$, $n \in \mathbb{Z}$, and h is differentiable at $g(z)$. Then, we have

$$\begin{cases}
\dfrac{d(f + cg)}{dz} = \dfrac{df}{dz} + c\dfrac{dg}{dz}, \\
\dfrac{d(f.g)}{dz} = \dfrac{df}{dz}g + f\dfrac{dg}{dz}. \\
\dfrac{d\left(\frac{f}{g}\right)}{dz} = \dfrac{\frac{df}{dz}g - f\frac{dg}{dz}}{g^2} \text{ (we have to be aware of division by zero).} \\
\dfrac{d(h \circ g)}{dz} = \dfrac{dh}{dz}(g)\dfrac{dg}{dz}.
\end{cases}$$

In the following results we generalize the Cauchy-Riemann formulas to multidual functions.

Theorem 5 Let f be a multidual function in $\Omega \subset \mathbb{D}_n$, which can be written in terms of its real and multidual parts as

$$f(z) = \sum_{i=0}^{n} \varphi_i(x_0, x_1, ..., x_n)\, \varepsilon^i. \tag{25}$$

f is hyperholomorphic in $\Omega \subset \mathbb{D}_n$ if and only if the derivative of f satisfies

$$\frac{df}{dz} = \frac{\partial f}{\partial x_0} = \sum_{i=0}^{n} \frac{\partial \varphi_i}{\partial x_0} \varepsilon^i. \tag{26}$$

Proof. The proof will be done by recurrence on n.
For $n = 1$ the result was already proved by F. Messelmi in the reference [11].
Suppose that $n = 2$. We can compute the derivative of f at $t = \sum\limits_{i=0}^{2} y_i \varepsilon^i \in \Omega$ as follows

$$\lim_{z \to t} \frac{f(z) - f(t)}{z - t} = \lim_{x_i \to y_i \ i=0,1,2} \frac{(f(z) - f(t))\left(\overline{z - t}\right)}{(x_0 - y_0)^2}$$
$$= \lim_{x_i \to y_i \ i=0,1,2} \frac{1}{(x_0 - y_0)^2} \left\{ (\varphi_0(x_0, x_1, x_2) - \varphi_0(y_0, y_1, y_2)) + \right.$$
$$\left. (\varphi_1(x_0, x_1, x_2) - \varphi_1(y_0, y_1, y_2))\, \varepsilon + (\varphi_2(x_0, x_1, x_2) - \varphi_2(y_0, y_1, y_2))\, \varepsilon^2 \right\} \times$$
$$\left\{ (x_0 - y_0) - (x_1 - y_1)\, \varepsilon + \left(\frac{(x_1 - y_1)^2}{x_0 - y_0} - (x_2 - y_2) \right) \varepsilon^2 \right\}$$
$$= \lim_{x_i \to y_i \ i=0,1,2} \left\{ \frac{\varphi_0(x_0, x_1, x_2) - \varphi_0(y_0, y_1, y_2)}{x_0 - y_0} - \right.$$
$$\frac{x_1 - y_1}{(x_0 - y_0)^2} (\varphi_0(x_0, x_1, x_2) - \varphi_0(y_0, y_1, y_2))\, \varepsilon +$$
$$\left(\frac{(x_1 - y_1)^2}{(x_0 - y_0)^3} - \frac{x_2 - y_2}{(x_0 - y_0)^2} \right) (\varphi_0(x_0, x_1, x_2) - \varphi_0(y_0, y_1, y_2))\, \varepsilon^2 +$$
$$\frac{\varphi_1(x_0, x_1, x_2) - \varphi_1(y_0, y_1, y_2)}{x_0 - y_0} \varepsilon - \frac{x_1 - y_1}{(x_0 - y_0)^2} (\varphi_1(x_0, x_1, x_2) - \varphi_1(y_0, y_1, y_2))\, \varepsilon^2$$
$$\left. + \frac{\varphi_2(x_0, x_1, x_2) - \varphi_2(y_0, y_1, y_2)}{x_0 - y_0} \varepsilon^2 \right\}$$

Hence, this limit becomes

$$\lim_{z\to t}\frac{f(z)-f(t)}{z-t}=\frac{\partial\varphi_0}{\partial x_0}(y_0,y_1,y_2)+\frac{\partial\varphi_1}{\partial x_0}(y_0,y_1,y_2)\,\varepsilon+\frac{\partial\varphi_2}{\partial x_0}(y_0,y_1,y_2)\,\varepsilon^2-$$
$$\lim_{x_i\to y_i\ i=0,1,2}\left(\frac{x_1-y_1}{x_0-y_0}\right)^2\frac{\varphi_0(x_0,x_1,x_2)-\varphi_0(y_0,y_1,y_2)}{x_1-y_1}\varepsilon-$$
$$\lim_{x_i\to y_i\ i=0,1,2}\left(\frac{x_2-y_2}{x_0-y_0}\right)^2\frac{\varphi_0(x_0,x_1,x_2)-\varphi_0(y_0,y_1,y_2)}{x_2-y_2}\varepsilon^2+$$
$$\lim_{x_i\to y_i\ i=0,1,2}\left(\frac{x_1-y_1}{x_0-y_0}\right)^2\left\{\frac{\varphi_0(x_0,x_1,x_2)-\varphi_0(y_0,y_1,y_2)}{x_0-y_0}-\right.$$
$$\left.\frac{\varphi_1(x_0,x_1,x_2)-\varphi_1(y_0,y_1,y_2)}{x_1-y_1}\right\}\varepsilon^2.$$

However, it is well-known that the limit exists if and only if it is independent of limit of the bounded ratios $\left(\frac{x_1-y_1}{x_0-y_0}\right)^2$ and $\left(\frac{x_2-y_2}{x_0-y_0}\right)^2$. Hence, we should impose the following conditions

$$\begin{aligned}\frac{\partial\varphi_0}{\partial x_0}(y_0,y_1,y_2)&=\frac{\partial\varphi_1}{\partial x_1}(y_0,y_1,y_2),\\ \frac{\partial\varphi_0}{\partial x_1}(y_0,y_1,y_2)&=0,\\ \frac{\partial\varphi_0}{\partial x_2}(y_0,y_1,y_2)&=0.\end{aligned}$$

So, the formula (26) follows.

Suppose now that the formula is true for $n\geq 2$ and let us prove that it remains true for $n+1$.

Denoting by ε_n and ε_{n+1} the unit multidual numbers of the algebras $\mathbb{D}_n$ and $\mathbb{D}_{n+1}$, respectively.

Considering a multidual function in $\Omega\subset\mathbb{D}_{n+1}$ and denoting by

$$z_{n+1}=\sum_{i=0}^{n+1}x_{n+1,i}\varepsilon_{n+1}^i \text{ and } t_{n+1}=\sum_{i=0}^{n+1}y_{n+1,i}\varepsilon_{n+1}^i$$

two elements of $\mathbb{D}_{n+1}$.

Clearly, we have

$$\begin{aligned}\frac{f(z_{n+1})-f(t_{n+1})}{z_{n+1}-t_{n+1}}&=\frac{f(z_{n+1})-f(t_{n+1})\left(\overline{z_{n+1}-t_{n+1}}\right)}{(x_{n+1,0}-y_{n+1,0})^2}\\ &=\frac{1}{(x_{n+1,0}-y_{n+1,0})^2}\left(\sum_{i=0}^{n+1}\left(\overline{x_{n+1,i}-y_{n+1,i}}\right)\varepsilon_{n+1}^i\right)\end{aligned}$$
$$\times\left\{\sum_{i=0}^{n+1}\left(\varphi_i(x_{n+1,0},...,x_{n+1,n+1})-\varphi_i(x_{n+1,0},...,x_{n+1,n+1})\right)\varepsilon_{n+1}^i\right\}.$$

So, we obtain after some algebraic calculations

$$\frac{f(z_{n+1})-f(t_{n+1})}{z_{n+1}-t_{n+1}}=\frac{f(z_{n+1})-f(t_{n+1})\left(\overline{z_{n+1}-t_{n+1}}\right)}{\left(x_{n+1,0}-y_{n+1,0}\right)^2}$$
$$=\frac{1}{\left(x_{n+1,0}-y_{n+1,0}\right)^2}\left(\sum_{i=0}^{n}\left(\overline{x_{n+1,i}-y_{n+1,i}}\right)\varepsilon_{n+1}^{i}\right)\times$$
$$\left\{\sum_{i=0}^{n}\left(\varphi_i\left(x_{n+1,0},...,x_{n+1,n+1}\right)-\varphi_i\left(x_{n+1,0},...,x_{n+1,n+1}\right)\right)\varepsilon_{n+1}^{i}\right\}+$$
$$\frac{\overline{x_{n+1,n+1}-y_{n+1,n+1}}}{\left(x_{n+1,0}-y_{n+1,0}\right)^2}\left(\varphi_0\left(x_{n+1,0},...,x_{n+1,n+1}\right)-\varphi_0\left(x_{n+1,0},...,x_{n+1,n+1}\right)\right)\varepsilon_{n+1}^{i}+$$
$$\frac{1}{x_{n+1,0}-y_{n+1,0}}\left(\varphi_{n+1}\left(x_{n+1,0},...,x_{n+1,n+1}\right)-\varphi_{n+1}\left(x_{n+1,0},...,x_{n+1,n+1}\right)\right)\varepsilon_{n+1}^{i}.$$

Since in the proof of the case $n=2$ there is no influence of the unit multidual number, we can infer for every n

$$\lim_{\substack{x_i\to y_i\\ i=0,...,n+1}}\frac{1}{\left(x_{n+1,0}-y_{n+1,0}\right)^2}\left(\sum_{i=0}^{n}\left(\overline{x_{n+1,i}-y_{n+1,i}}\right)\varepsilon_{n+1}^{i}\right)$$
$$\times\left\{\sum_{i=0}^{n}\left(\varphi_i\left(x_{n+1,0},...,x_{n+1,n+1}\right)-\varphi_i\left(x_{n+1,0},...,x_{n+1,n+1}\right)\right)\varepsilon_{n+1}^{i}\right\}$$
$$=\sum_{i=0}^{n}\frac{\partial\varphi_i\left(x_{n+1,0},...,x_{n+1,n+1}\right)}{\partial x_0}\varepsilon_{n+1}^{i}.$$

Furthermore, since the formula is supposed to be true for n, we find get, as in the proof of the case $n=2$,

$$\frac{\partial\varphi_0}{\partial x_i}\left(y_{n+1,0},...,y_{n+1,n+1}\right)=0\quad\forall i=1,...,n+1.$$

By the use of this relation we reach the expression

$$\lim_{z_{n+1}\to t_{n+1}}\frac{f(z_{n+1})-f(t_{n+1})}{z_{n+1}-t_{n+1}}=\sum_{i=0}^{n+1}\frac{\partial\varphi_i\left(x_{n+1,0},...,x_{n+1,n+1}\right)}{\partial x_0}\varepsilon_{n+1}^{i}.$$

This achieves the proof.

Theorem 6 Let f be a multidual function in $\Omega\subset\mathbb{D}_n$, which can be written in terms of its real and multidual parts as in the relation (25) and suppose that the partial derivatives of f exist. Then,

(1) f is hyperholomorphic in Ω if and only if following formulas hold

$$\begin{cases}\frac{\partial\varphi_i}{\partial x_j}=\frac{\partial\varphi_{i-j}}{\partial x_0}\text{ if }j\le i,\\ \frac{\partial\varphi_i}{\partial x_j}=0\text{ if }j>i.\end{cases}\tag{27}$$

(2) f is hyperholomorphic in Ω if and only if its partial derivatives satisfy

$$\frac{\partial f}{\partial x_j}=\varepsilon^j\frac{\partial f}{\partial x_0}\quad\forall j=0,...,n.\tag{28}$$

Proof. (1) Consider f as a multi-valued real function. We can assert that her total differential can be written

$$df = \sum_{j=0}^{n} \frac{\partial f}{\partial x_j} dx_j.$$

So, making use (25), it follows

$$df = \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{\partial \varphi_j}{\partial x_i} dx_i \varepsilon^i. \tag{29}$$

On the other hand, by virtue of the relation (26) we have

$$\begin{aligned} df &= \frac{\partial f}{\partial x_0} dz \\ &= \frac{\partial}{\partial x_0} \left(\sum_{i=0}^{n} \varphi_i \varepsilon^i \right) \left(\sum_{i=0}^{n} dx_i \varepsilon^i \right) \\ &= \sum_{i=0}^{n} \left(\sum_{j=0}^{i} \frac{\partial \varphi_{i-j}}{\partial x_0} dx_j \right) \varepsilon^i. \end{aligned}$$

Combining this with (29), we find

$$\sum_{j=0}^{n} \frac{\partial \varphi_j}{\partial x_i} dx_i = \sum_{j=0}^{i} \frac{\partial \varphi_{i-j}}{\partial x_0} dx_j \quad \forall i = 0, ..., n. \tag{30}$$

Which eventually gives (27), after simple developments.
(2) By defintion of f, we know that

$$\frac{\partial f}{\partial x_j} = \sum_{i=0}^{n} \frac{\partial \varphi_i}{\partial x_j} \varepsilon^i.$$

So, in view of (27), we get

$$\begin{aligned} \frac{\partial f}{\partial x_j} &= \sum_{i=j}^{n} \frac{\partial \varphi_{i-j}}{\partial x_0} \varepsilon^i \\ &= \varepsilon^j \sum_{k=0}^{n} \frac{\partial \varphi_k}{\partial x_0} \varepsilon^k. \end{aligned}$$

Hence, the proof is done.
In the following statement, we give another représentation of the Cauchy-Riemann formulas as a differential matrix equations.

Corollary 7 Let f be a multidual function in $\Omega \subset \mathbb{D}_n$, which can be written in terms of its real and multidual parts as

$$f(z) = \sum_{i=0}^{n} \varphi_i (x_0, x_1, ..., x_n) \varepsilon^i,$$

and admit that the partial derivatives of f exist. Then, f is hyperholomorphic in Ω if and only if f is solution of the following differential matrix equation

$$\nabla \mathcal{X}(f) = \frac{\partial \mathcal{N}(f)}{\partial x_0}. \tag{31}$$

Proof. If we consider f as a multi-valued function, then that her gradient can be computed as

$$\nabla\mathcal{X}(f)=\begin{bmatrix} \frac{\partial\varphi_0}{\partial x_0} & \frac{\partial\varphi_0}{\partial x_1} & . & . & . & \frac{\partial\varphi_0}{\partial x_n} \\ \frac{\partial\varphi_1}{\partial x_0} & \frac{\partial\varphi_1}{\partial x_1} & . & . & . & \frac{\partial\varphi_1}{\partial x_n} \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \frac{\partial\varphi_n}{\partial x_0} & \frac{\partial\varphi_n}{\partial x_1} & . & . & . & \frac{\partial\varphi_n}{\partial x_n} \end{bmatrix}.$$

Taking into account the relation (27), we can infer

$$\begin{aligned}\nabla\mathcal{X}(f) &= \begin{bmatrix} \frac{\partial\varphi_0}{\partial x_0} & 0 & . & . & . & 0 \\ \frac{\partial\varphi_1}{\partial x_0} & \frac{\partial\varphi_0}{\partial x_0} & 0 & . & . & 0 \\ . & \frac{\partial\varphi_1}{\partial x_0} & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & 0 \\ \frac{\partial\varphi_n}{\partial x_0} & . & . & . & \frac{\partial\varphi_1}{\partial x_0} & \frac{\partial\varphi_0}{\partial x_0} \end{bmatrix} \\ &= \frac{\partial\mathcal{N}(f)}{\partial x_0}.\end{aligned}$$

Which permits us to conclude the proof.

4. Extension and Generator Polynomials

Definition 7 A multidual function defined in $\Omega\subset\mathbb{D}_n$ is said to be homogeneous if

$$f(\mathrm{real}(z))\in\mathbb{R}. \tag{32}$$

The following Theorem shows that we can extend any homogeneous hyperholomorphic function defined in a subset $\Omega\subset\mathbb{D}_n$ to the wider multidual subset generated by the first orthogonal projection of Ω.

Theorem 8 Let f be an homogeneous multidual function in $\Omega\subset\mathbb{D}_n$, which can be written in terms of its real and multidual parts as

$$f(z)=\sum_{i=0}^{n}\varphi_i(x_0,x_1,...,x_n)\,\varepsilon^i,$$

and suppose that the partial derivatives of f exist. Then, if f is hyperholomorphic in Ω, the functions φ_i verify

(1) $\varphi_0\in\mathcal{C}^n(\mathcal{P}_1(\Omega))$ and $\frac{d^{n+1}\varphi_0}{dx^{n+1}}$ exists in $\mathcal{P}_1(\Omega)$,

(2) $\varphi_i\in\mathcal{C}^{n-i}(\mathcal{P}_1(\Omega))\times\mathcal{C}^\infty(\mathbb{R}^n)$ and $\frac{\partial^{n-i+1}\varphi_i}{\partial x^{n-i+1}}$ exists in $\mathcal{P}_1(\Omega)$, $\forall i=1,...,n$,

where $\mathcal{P}_1(\Omega)$ represents the first projection of Ω on $\mathbb{R}$.

Particularly, f can be hyperholomorphically extended to the multidual subset $\mathcal{P}_1(\Omega)\times\mathbb{R}^n$.

Proof. Suppose that f is hyperholomorphic in Ω, then by Cauchy-Riemann formulas we find

$$\varphi_i(x_0,x_1,...,x_n)=\varphi_i(x_0,x_1,...,x_i),\ \forall i=0,...,n, \tag{33}$$

Moreover, Cauchy-Riemann formulas lead also to

$$\begin{cases} \frac{\partial \varphi_i}{\partial x_0} = \frac{\partial \varphi_i}{\partial x_0}, \\ \frac{\partial \varphi_{i-1}}{\partial x_0} = \frac{\partial \varphi_i}{\partial x_1}, \\ \vdots \\ \frac{\partial \varphi_1}{\partial x_0} = \frac{\partial \varphi_i}{\partial x_{i-1}}, \\ \frac{d\varphi_0}{dx_0} = \frac{\partial \varphi_i}{\partial x_i}. \end{cases}$$

These yield for $i = 1$

$$\varphi_1(x_0, x_1) = \frac{d\varphi_0}{dx_0} x_1.$$

For $i = 2$, we get by similar argument

$$\varphi_2(x_0, x_1, x_2) = \frac{1}{2} \frac{d^2\varphi_0}{dx_0^2} x_1^2 + \frac{d\varphi_0}{dx_0} x_2.$$

Since f is homogeneous, we can then generalize these relations by recurence on $i = 1, ..., n$, to find

$$\varphi_i(x_0, x_1, ..., x_i) = \sum_{j=1}^{i} P_{ij}(x_1, ..., x_i) \frac{d^{i+1-j}\varphi_0}{dx_0^{i+1-j}},$$

where $P_{ij} \in \mathbb{R}[x_1, ..., x_i]$, called the generator polynomials.

We deduce that the multidual function f can be written in explicit from

$$f(z) = \varphi_0(x_0) + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}(x_1, ..., x_i) \frac{d^{i+1-j}\varphi_0}{dx_0^{i+1-j}} \varepsilon^i. \quad (34)$$

Thus, we can also obtain employing (26)

$$\frac{df}{dz} = \frac{d\varphi_0}{dx_0} + \sum_{i=1}^{n} \sum_{j=1}^{i} P_{ij}(x_1, ..., x_i) \frac{d^{i+2-j}\varphi_0}{dx_0^{i+2-j}} \varepsilon^i. \quad (35)$$

In particular, the proof is an immediate consequence.

In the following proposition, we give some properties of the generator polynomials P_{ij}.

Proposition 9 The generator polynomials verify the following statements:

$$\begin{cases} P_{ij} = 0 \ \forall i = 1, ..., n \text{ and } j = i+1, ..., n, \\ \frac{\partial P_{ij}}{\partial x_k} = 0 \ \ \forall i = 1, ..., n, k = 1, ..., i \text{ and } j = 1, ..., k-1, \\ \frac{\partial P_{ij}}{\partial x_k} = P_{i-k, j-k+1} \ \ \forall i = 2, ..., n, k = 1, ..., i-1 \text{ and } j = k, ..., i-1, \\ P_{ii}(x_1, ..., x_i) = x_i \ \ \forall i = 1, ..., n. \end{cases} \quad (36)$$

The proof follows directly from the Cauchy-Riemann formulas.

The object of the following corollary is the study of multidual constant functions.

Corollary 10 (*Constant functions*) Let f be an homogeneous multidual function defined in a connected multidual subset $O \times \mathbb{R}^n \subset \mathbb{D}_n$. The following statements hold

(1) If $\frac{df}{dz} = 0$ in $O \times \mathbb{R}^n$, then $f \equiv const$.

(2) If f is bounded in $O \times \mathbb{R}^n$ (in the sense that $|\varphi_i(x_0, x_1, ..., x_i)| \leq c \ \forall i = 1, ..., n$ and $\forall (x_0, x_1, ..., x_n) \in O \times \mathbb{R}^n$), then $f \equiv const$.

The following proposition ensures us that every regular real function can be holomorphically extended to the multidual numbers.

Proposition 11 (*Extension of real functions*) Let $f : O \longrightarrow \mathbb{R}$ be a real function, where O is an open connected domain of $\mathbb{R}$.

Suppose that $f \in \mathcal{C}^n(O)$ and $\frac{d^{n+1}f}{dx^{n+1}}$ exists in O. Then, there exists a unique hyperholomorphic homogeneous multidual function $F : O \times \mathbb{R}^n \longrightarrow \mathbb{D}_n$ satisfying

$$F(x_0) = f(x_0) \quad \forall x_0 \in O. \tag{37}$$

In addition, if $f \in \mathcal{C}^q(O)$, $q \geq n+1$, then $F \in \mathcal{C}^{q-n-1}(O \times \mathbb{R}^n)$. In Particular, if $f \in \mathcal{C}^\infty(O)$, then $F \in \mathcal{C}^\infty(O \times \mathbb{R}^n)$. We say in that case that f is an analytic multidual function in the multidual subset $O \times \mathbb{R}^n$ and we write $f \in \mathcal{A}(O \times \mathbb{R}^n)$.

5. Concrete Examples

We can think of applying the statement of proposition 8 to build homogeneous multidual functions similar to the usual real functions, obtained as their extensions. In detail, we define the multidual exponential function, the multiduale Logarithmic function. Also we introduce the concept of Galilean multi-trigonometric functions and we give and discuss some of their interesting properties.

5.1. **Multidual Exponential function.** The multidual exponential function can be obtained as extension of the exponential real function to multidual numbers, but there is some technical difficulties to work with such definition. To this aim, we prefer to use the exponential of matrices.

Let $A \in \mathcal{G}_{n+1}(\mathbb{R})$ and suppose that $\|A\| < +\infty$ for some norm. It is well known that the exponential of A can be defined by the series

$$\exp(A) = e^A = \sum_{m=0}^{+\infty} \frac{1}{m!} A^m. \tag{38}$$

In addition, the series converges normally in each bounded domain of $\mathcal{G}_{n+1}(\mathbb{R})$.

Since $A \in \mathcal{G}_{n+1}(\mathbb{R})$, we can affirm that for all $m \in \mathbb{N}$ we have $A^m \in \mathcal{G}_{n+1}(\mathbb{R})$. Thus, by passage to the limit it follows that

$$\exp(A) \in \mathcal{G}_{n+1}(\mathbb{R}). \tag{39}$$

Introducing now the multidual function f defined for every $z \in \mathbb{D}_n$ by

$$f(z) = \mathcal{N}^{-1} \circ \exp \circ \mathcal{N}(z). \tag{40}$$

We find via this definition

$$\mathcal{N}(f(z)) = \exp(\mathcal{N}(z)), \tag{41}$$

and so

$$f(z) = e_1^T.\exp(\mathcal{N}(z))^T.\mathcal{E}. \tag{42}$$

We are now ready to define rigorously the multidual exponential function.

Definition 8 We define the multidual exponential function by

$$\left\{ \begin{array}{c} \exp : \mathbb{D}_n \longrightarrow \mathbb{D}_n \\ \exp(z) = e^z = e_1^T.\exp(\mathcal{N}(z))^T.\mathcal{E} \quad \forall z \in \mathbb{D}_n. \end{array} \right. \tag{43}$$

Some properties of the multidual exponential function are collected in the following.

Proposition 12

(1) $e^{z_1+z_2} = e^{z_1}e^{z_2}$ $\forall z_1, z_2 \in \mathbb{D}_n$.

(2) $e^{-z} = \frac{1}{e^z}$ $\forall z \in \mathbb{D}_n$.

Proof. (1) For all $z_1, z_2 \in \mathbb{D}_n$ we get using formulat (35)

$$\mathcal{N}\left(e^{z_1+z_2}\right) = e^{\mathcal{N}(z_1+z_2)}.$$

Since, $\mathcal{N}(z_1).\mathcal{N}(z_2) = \mathcal{N}(z_2).\mathcal{N}(z_1)$, we find

$$\begin{aligned}\mathcal{N}\left(e^{z_1+z_2}\right) &= e^{\mathcal{N}(z_1)}e^{\mathcal{N}(z_2)} \\ &= \mathcal{N}\left(e^{z_1}\right)\mathcal{N}\left(e^{z_2}\right) \\ &= \mathcal{N}\left(e^{z_1}e^{z_2}\right).\end{aligned}$$

Hence, the first statement holds.
(2) For every $z \in \mathbb{D}_n$, it follows thinks to the previous assertion

$$\begin{aligned}\mathcal{N}\left(e^{z}\right).\mathcal{N}\left(e^{-z}\right) &= \mathcal{N}\left(e^{z}e^{-z}\right) \\ &= \mathcal{N}\left(e^{z-z}\right) \\ &= I_{n+1}.\end{aligned}$$

Which allows us to deduce that

$$\begin{aligned}\mathcal{N}\left(e^{-z}\right) &= \mathcal{N}\left(e^{z}\right)^{-1} \\ &= \mathcal{N}\left(\frac{1}{e^{z}}\right).\end{aligned}$$

So the second statement follows.

Since the extension of regular real functions to multidual numbers is unique, one can easily prove that the definition given above of multidual exponential function coincides with that obtained as extension of real exponential function. Hence we can write

$$e^{z} = e_1^{T}.\exp\left(\mathcal{N}(z)\right)^{T}.\mathcal{E} = e^{x_0}\left(1 + \sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}\left(x_1, ..., x_i\right)\varepsilon^{i}\right). \tag{44}$$

Proposition 13 The derivative of exp is given by

$$\frac{de^{z}}{dz} = e^{z} \quad \forall z \in \mathbb{D}_n. \tag{45}$$

Proof. It is enough for this to use theorem to find

$$\begin{aligned}\frac{de^{z}}{dz} &= \frac{\partial e^{z}}{\partial x_0} \\ &= \frac{\partial e^{x_0}}{\partial x_0}\left(1 + \sum_{i=1}^{n}\sum_{j=1}^{i} P_{ij}\left(x_1, ..., x_i\right)\varepsilon^{i}\right) \\ &= e^{z}\end{aligned}$$

As consequence, we have

$$\exp \in \mathcal{A}\left(\mathbb{D}_n\right) \text{ and } \frac{d^{m}e^{z}}{dz^{m}} = e^{z} \ \forall z \in \mathbb{D}_n, \ m \geq 1. \tag{46}$$

As in the proof of Corollary 7, one can easily aboutis to the following differential matrix equation verified by the multidual exponential function

$$\nabla\mathcal{X}\left(\exp\right) = \mathcal{N}\left(\exp\right). \tag{47}$$

5.2. **Galilean Multi-Trigonometric functions.** **Definition 9** For all $(x_1,...,x_n) \in \mathbb{R}^n$ and $i = 0,...,n$, we define the Galilean multi-Trigonometric functions by

$$mug_i : \mathbb{R}^i \longrightarrow \mathbb{R} \text{ such that}$$
$$\begin{cases} mug_0 = 1, \\ mug_i(x_1,...,x_i) = \sum_{j=1}^{i} P_{ij}(x_1,...,x_i). \end{cases} \tag{48}$$

these function generalizes in higher dimensions the classical Galilean Trigonometric functions, see [4].

Under the formula (44) the multidual exponential function can be rewritten

$$\exp\left(\sum_{i=1}^{n} x_i \varepsilon^i\right) = mug_0 + \sum_{i=1}^{n} mug_i(x_1,...,x_i)\,\varepsilon^i. \tag{49}$$

The expression (49) is called the generalized Euler formula for multidual numbers.

Proposition 14

(1) The Galilean multi-Trigonometric functions mug_i verify

$$\begin{cases} \frac{\partial mug_i}{\partial x_j}(x_1,...,x_i) = mug_{i-j}(x_1,...,x_{i-j}) \text{ for } i,j = 1,...,n \text{ and } j < i, \\ \frac{\partial mug_i}{\partial x_i}(x_1,...,x_i) = 1 \quad \forall i = 1,...,n, \\ mug_i(0,...,0,x_i) = x_i \quad \forall i = 1,...,n. \end{cases} \tag{50}$$

$$\begin{cases} mug_i(x_1+y_1,...,x_i+y_i) = \sum_{j=0}^{i} mug_j(x_1,...,x_j)\, mug_{i-j}(y_1,...,y_{i-j}) \\ \qquad = \sum_{j=0}^{i} mug_j(y_1,...,y_j)\, mug_{i-j}(x_1,...,x_{i-j}) \\ \forall i = 1,...,n,\ \forall (x_1,...,x_i),\ (y_1,...,y_i) \in \mathbb{R}^n. \end{cases} \tag{51}$$

(2) $\forall (x_1,...,x_n) \in \mathbb{R}^n$, $mug_i(-x_1,...,-x_i)$ is solution of the linear system

$$\begin{bmatrix} mug_1(x_1) & 0 & . & . & . & 0 \\ mug_2(x_1,x_2) & mug_1(x_1) & 0 & . & . & . \\ . & mug_2(x_1,x_2) & . & . & . & . \\ . & . & . & . & 0 & . \\ . & . & . & . & mug_1(x_1) & 0 \\ mug_i(x_1,...,x_i) & . & . & . & mug_2(x_1,x_2) & mug_1(x_1) \end{bmatrix}$$
$$\times \begin{bmatrix} mug_1(-x_1) \\ mug_2(-x_1,-x_2) \\ . \\ . \\ . \\ mug_i(-x_1,...,-x_i) \end{bmatrix} = \begin{bmatrix} mug_1(x_1) \\ mug_2(x_1,x_2) \\ . \\ . \\ . \\ mug_i(x_1,...,x_i) \end{bmatrix}. \tag{52}$$

The proof of the above proposition is simple and direct, making use the definition of mug_i and Cauchy-Riemann formulas.

5.3. **Multidual Logarithmic function.** We define the multidual Logarithmic function as the unique extension, via Corollary, of Logarithmic real function to the multidual algebra. Thus, the multidual Logarithmic function can be defined as

$$\left\{\begin{array}{c} \log:\mathbb{R}_{+}^{*}\times\mathbb{R}^{n}\subset\mathbb{D}_{n}\longrightarrow\mathbb{D}_{n}, \\ \log(z)=\log(x_{0})+\sum_{i=1}^{n}\sum_{j=1}^{i}\frac{(-1)^{i-j}}{(i-j)!}\frac{P_{ij}(x_{1},...,x_{i})}{x_{0}^{i-j+1}}\varepsilon^{i} \end{array}\right. \tag{53}$$

It is difficult to work with this definition. To this aim, we try to proceed by introducing the following homogeneous multidual function, which represents the inverse of the multidual exponential function

$$\chi:\mathbb{D}_{n}\longrightarrow\mathbb{D}_{n}\mid e^{\chi(z)}=z. \tag{54}$$

Proposition 15 The multidual function χ is hyperholomorphic in the multidual subset $\mathbb{R}_{+}^{*}\times\mathbb{R}^{n}$.

In addition, we have

$$\left\{\begin{array}{c} \nabla\mathcal{X}(\chi)=\mathcal{N}^{-1}, \\ \frac{d\chi(z)}{dz}=\frac{1}{z}\ \forall z\in\mathbb{R}_{+}^{*}\times\mathbb{R}^{n}. \end{array}\right. \tag{55}$$

Proof. First, by definition of the multidual exponential function, we know that

$$\operatorname{real}\left(e^{\chi(z)}\right)=e^{\operatorname{real}(\chi(z))}>0.$$

This means that the function χ is well-defined if and only if $\operatorname{real}(z)>0$.

Elsewhere, we get by deriving with respect to x_j, $j=1,...,n$

$$\frac{\partial\chi}{\partial x^{j}}z=\varepsilon^{j}. \tag{56}$$

It also follows

$$\frac{\partial\chi}{\partial x_{j}}=\varepsilon^{j}\frac{\partial\chi}{\partial x_{0}}.$$

Hence, theorem affirms 6 us that χ is hyperholomorphic in the multidual subset $\mathbb{R}_{+}^{*}\times\mathbb{R}^{n}$.

Now, writing χ in terms of its real and multidual parts as

$$\chi(z)=\sum_{i=0}^{n}\varphi_{i}(x_{0},x_{1},...,x_{n})\,\varepsilon^{i},$$

and deriving the two sides of the expression $e^{\chi(z)}=z$ with respect to x_k, $j=k,...,n$. The following formula holds

$$\sum_{j=0}^{i}\frac{\partial\varphi_{i-j}}{\partial x_{k}}=\delta_{ik}.$$

This formula can be rewritten explicitly

$$\mathcal{N}(z)\frac{\partial\mathcal{X}(\chi)}{\partial x_{k}}=e_{k+1}\ \forall k=0,...,n.$$

On the other hand, since χ is hyperholomorphic function, Cauchy-Riemann formulas permit us to write

$$\frac{d\chi}{dz}=\frac{\partial\chi}{\partial x_{0}}.$$

This yields, combining with equation (56)

$$\frac{d\chi}{dz}=\frac{1}{z},$$

which achieves the proof.

Proposition 16 For all $z \in \mathbb{R}_+^* \times \mathbb{R}^n$ the following relation holds

$$\chi(z) = \log z. \tag{57}$$

Proof. From relation (54), we can infer

$$e^{\mathrm{real}(\chi(z))} = \mathrm{real}(z),$$

Moreover, we find using the definition (53) of $\log z$ gives

$$\mathrm{real}(\log z) = \log(\mathrm{real}(z)).$$

By virtue of the two above relations, we obatin the identity

$$\mathrm{real}(\chi(z)) = \mathrm{real}(\log z).$$

Consequently, the result follows from the proposition 11.

The next properties follow mostly from the definition of the multidual Logarithmic function.

Proposition 17

(1) $\log\left(\frac{1}{z}\right) = -\log z$.

(2) $\log(z_1 z_2) = \log z_1 + \log(z_2)$.

(3) $\log(z^\alpha) = \alpha \log z \quad \forall z \in \mathbb{R}_+^* \times \mathbb{R}, \forall \alpha \in \mathbb{R}$.

5.4. **Galilean inverse multi-Trigonometric functions.** The fact that $e^{\log z} = z$ and $\log e^z = z$ allows us to introduce the Galilean inverse multi-trigonometric functions using the definition of the multidual Logarithmic function. To do this, putting $x_0 = 1$ in the relation, we obtain the equality

$$\log\left(1 + \sum_{i=1}^{n} x_i \varepsilon^i\right) = \sum_{i=1}^{n} \sum_{j=1}^{i} \frac{(-1)^{i-j}}{(i-j)!} P_{ij}(x_1, ..., x_i)\, \varepsilon^i. \tag{58}$$

Defining the Galilean inverse multi-trigonometric functions as follows

Definition 10 For all $(x_1, ..., x_n) \in \mathbb{R}^n$ and $i = 0, ..., n$, we define the Galilean multi-trigonometric functions by

$$mug_{-i} : \mathbb{R}^i \longrightarrow \mathbb{R} \text{ such that}$$

$$\begin{cases} mug_{-0} = mug_0 = 1, \\ mug_{-i}(x_1, ..., x_i) = \sum\limits_{j=1}^{i} \frac{(-1)^{i-j}}{(i-j)!} P_{ij}(x_1, ..., x_i). \end{cases} \tag{59}$$

Since multidual Logarithmic function represents the inverse of multidual exponential function, the following proposition holds.

Proposition 18 For all $(x_1, ..., x_n) \in \mathbb{R}^n$ and $i = 1, ..., n$, we have the inversion formulas

$$mug_{-i}(mug_1(x_1), ..., mug_i(x_1, ..., x_i)) = x_i, \tag{60}$$

$$mug_i(mug_{-1}(x_1), ..., mug_{-i}(x_1, ..., x_i)) = x_i. \tag{61}$$

The following collects some basic properties of mug_{-i}.

Proposition 19 The Galilean inverse multi-trigonometric functions mug_{-i} verify

$$\begin{cases} \frac{\partial mug_{-i}}{\partial x_j}(x_1, ..., x_i) = -\sum\limits_{k=j}^{i-1} x_{i-k} \frac{\partial mug_{-k}}{\partial x_j}(x_1, ..., x_k) \\ \text{for } i, j = 1, ..., n \text{ and } j < i, \end{cases} \tag{62}$$

$$\left\{ \begin{array}{c} \frac{\partial mug_{-i}}{\partial x_j}\left(x_1,...,x_i\right) = -\sum_{k=1}^{j-1} \frac{\partial mug_{-i}}{\partial x_k}\left(x_1,...,x_i\right) \frac{\partial mug_k}{\partial x_j}\left(x_1,...,x_k\right) \\ \text{for } i,j=1,...,n \text{ and } j<i \end{array} \right. \tag{63}$$

$$\frac{\partial mug_{-i}}{\partial x_i}\left(x_1,...,x_i\right) = 1 \quad \forall i=1,...,n, \tag{64}$$

$$mug_{-i}\left(0,...,0,x_i\right) = x_i \quad \forall i=1,...,n. \tag{65}$$

$$\left\{ \begin{array}{c} mug_{-1}\left(x_1+y_1\right) = mug_{-1}\left(x_1\right) + mug_{-1}\left(x_1\right) \\ mug_{-i}\left(x_1+y_1, x_2+y_2+x_1y_1, ..., x_i+y_i+\sum_{j=1}^{i-1} x_j y_{i-j}\right) \\ = mug_{-i}\left(x_1,...,x_i\right) + mug_{-i}\left(y_1,...,y_i\right) \\ \forall i=1,...,n,\ \forall\left(x_1,...,x_i\right),\ \ \left(y_1,...,y_i\right) \in \mathbb{R}^n. \end{array} \right. \tag{66}$$

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