FUZZY SOFT $\theta$- CLOSURE OPERATOR IN FUZZY SOFT TOPOLOGICAL SPACES

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Abstract. In this paper, the concept of fuzzy soft $\theta$- closure operation is introduced and studied. We observe that fuzzy soft $\theta$- closure operator on a fuzzy soft topological space $(X, E, \tau)$ does not satisfy the Kuratowski closure axioms. Fuzzy soft $\theta$- continuous mapping is introduced and characterized in different ways. Finally, an attempt has made to introduce and characterize fuzzy soft $\theta$- connectedness.

1. Introduction

Uncertainty is the faith of this world and all its creatures. Traditional mathematical tools are not sufficient to handle all the practical problems in many disciplines such as medical science, social science, economics, engineering, environment etc involving uncertainty of various types. In 1965, Zadeh [16] was the first to come up with his remarkable theory of fuzzy set for dealing these types of uncertainties where conventional tools fail. His theory brought a grand paradigmatic change in mathematics but this theory has its inherent difficulties [10]. The reason for these difficulties is possibly the inadequacy of parameterization tool of the theories as point out by Molodtsov in [10].

In 1999, Molodtsov [10] introduced the concept of soft sets as a mathematical tool for dealing with uncertainties which is free from the above mentioned difficulties. Shabir and Naz [13] studied the topological structures of soft sets.

In recent times, fuzzification of soft set theory is progress rapidly. Combining fuzzy sets and soft sets, Maji et al. [9] put forward a new model known as fuzzy soft set. Kharal and Ahmad [8] defined the concept of mapping on fuzzy soft classes. Topological structure of fuzzy soft sets was started by Tanay and Burc Kandemir [14]. The study was pursued by some others [1, 2, 3, 4, 5, 6, 7, 11, 12, 15].

In the present paper, we introduce fuzzy soft $\theta$- closure, fuzzy soft $\theta$- open set, fuzzy soft $\theta$- closed set and fuzzy soft $\theta$- interior and some of their basic properties are studied. Fuzzy soft $\theta$- continuous mapping, fuzzy soft $\theta$- open mapping, fuzzy

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2. Preliminaries

Throughout this paper $X$ denotes initial universe, $E$ denotes the set of all possible parameters for $X$, $P(X)$ denotes the power set of $X$, $I^X$ denotes the set of all fuzzy sets on $X$ and $I$ stands for $[0,1]$.

**Definition 2.1** [16] A fuzzy set $A$ in $X$ is defined by a membership function $\mu_A : X \rightarrow [0,1]$ whose value $\mu_A(x)$ represents the "grade of membership" of $x$ in $A$ for $x \in X$.

If $A, B \in I^X$ then from [16] we have the following:

(i) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \ \forall \ x \in X$
(ii) $A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \ \forall \ x \in X$
(iii) $C = A \lor B \Leftrightarrow \mu_C(x) = \max(\mu_A(x), \mu_B(x)), \ \forall \ x \in X$
(iv) $D = A \land B \Leftrightarrow \mu_D(x) = \min(\mu_A(x), \mu_B(x)), \ \forall \ x \in X$
(v) $E = A^c \Leftrightarrow \mu_E(x) = 1 - \mu_A(x), \ \forall \ x \in X$

**Definition 2.2** [10] Let $A \subseteq E$. A pair $(F, A)$ is called a soft set over $X$ if $F$ is a mapping from $A$ into $P(X)$, i.e. $F : A \rightarrow P(X)$.

In other words, a soft set is a parameterized family of subsets of the set $X$. For $e \in A$, $F(e)$ may be considered as the set of $e$–approximate elements of the soft set $(F, A)$.

**Definition 2.3** [9] Let $A \subseteq E$. A pair $(f, A)$ is called a fuzzy soft set over $X$ if $f : A \rightarrow I^X$ is a function, i.e., for each $a \in A$, $f(a) = f_a : X \rightarrow [0,1]$ is a fuzzy set on $X$.

We will use $FS(X, E)$ to denote the family of all fuzzy soft sets over $X$.

Modification in the above definition was done by Roy and Samanta [12], which goes as follows.  

**Definition 2.4** [12] Let $A \subseteq E$. A fuzzy soft set $f_A$ over $X$ is a mapping from the parameter set $E$ to $I^X$, i.e. $f_A : E \rightarrow I^X$, where $f_A(e) \neq 0_X$ if $e \in A \subseteq E$ and $f_A(e) = 0_X$ if $e \notin A$, where $0_X$ denotes the empty fuzzy set on $X$.

**Definition 2.5** [12] The fuzzy soft set $f_0 \in FS(X, E)$ is called null fuzzy soft set, denoted by $0_E$, if for all $e \in E$, $f_A(e) = 0_X$.

**Definition 2.6** [12] Let $f_E \in FS(X, E)$. The fuzzy soft set $f_E$ is called universal fuzzy soft set, denoted by $1_E$, if for all $e \in E$, $f_E(e) = 1_X$ where $1_X(x) = 1$ for all $x \in X$.

**Definition 2.7** [9] Let $A, B \subseteq E$ and $f_A, g_B \in FS(X, E)$. We say that $f_A$ is a fuzzy soft subset of $g_B$ and write $f_A \subseteq g_B$ if and only if

(1) $A \subseteq B$,
(2) For every $e \in E$, $f_A(e) \leq g_B(e)$.

**Definition 2.8** [9] Let $f_A, g_B \in FS(X, E)$. Then the union of $f_A$ and $g_B$, denoted by $h_C = f_A \cup g_B$, where $C = A \cup B$ and $h_C$ is defined by
Definition 2.9 [9] Let $f_A, g_B \in FS(X, E)$. Then the intersection of $f_A$ and $g_B$ is also a fuzzy soft set $h_C$, defined by $h_C(e) = f_A(e) \wedge g_B(e)$ for all $e \in E$, where $C = A \cap B$. Here we write $h_C = f_A \cap g_B$.

Definition 2.10 [9] Let $f_A \in FS(X, E)$. Then the complement of $f_A$, denoted by $f_A^c$, is a fuzzy soft set defined by $f_A^c(e) = 1_X \setminus f_A(e)$, for all $e \in E$.

Let us call $f_A^c$ to be the fuzzy soft complement of $f_A$ in $FS(X, E)$.

Clearly $(f_A^c)^c = f_A$, $(\bar{1}_E)^c = \bar{0}_E$ and $(\bar{0}_E)^c = \bar{1}_E$.

Definition 2.11 [2] A fuzzy soft set $f_A \in FS(X, E)$ is called fuzzy soft point if $A = \{e\} \subseteq E$ and $f_A(e)$ is a fuzzy point in $X$, i.e., there exists $x \in X$ such that $f_A(e)(x) = \alpha (0 < \alpha \leq 1)$ and $f_A(e)(y) = 0$ for all $y \in X - \{x\}$. We denoted by $e_f$.

Definition 2.12 [2] Let $f_A, g_B \in FS(X, E)$. Then $f_A$ is said to be soft quasi-coincident with $g_B$, denoted by $f_A \approx g_B$, if there exist $e \in E$ and $x \in X$ such that $f_A(e)(x) + g_B(e)(x) > 1$.

If $f_A$ is not soft quasi-coincident with $g_B$, then we write $f_A \approx \neg g_B$.

Definition 2.13 [8] Let $FS(X, E)$ and $FS(Y, K)$ be families of all fuzzy soft sets over $X$ and $Y$, respectively. Let $u : X \to Y$ and $p : E \to K$ be two functions. Then $f_{up}$ is called a fuzzy soft mapping from $FS(X, E)$ to $FS(Y, K)$ and denoted by $f_{up} : FS(X, E) \to FS(Y, K)$.

(1) Let $f_A \in FS(X, E)$. Then the image of $f_A$ under the fuzzy soft mapping $f_{up}$ is the fuzzy soft set over $Y$ defined by $f_{up}(f_A)$, where

$$f_{up}(f_A)(k)(y) = \left\{ \begin{array}{ll} \bigvee_{x \in u^{-1}(y)} \bigvee_{e \in p^{-1}(k) \cap A} f_A(e)(x) & \text{if } u^{-1}(y) \neq \emptyset \text{ and if } p^{-1}(k) \cap A \neq \emptyset \\ 0_Y & \text{otherwise.} \end{array} \right.$$ 

(2) Let $g_B \in FS(X, E)$. Then the pre-image of $g_B$ under the fuzzy soft mapping $f_{up}$ is the fuzzy soft set over $X$ defined by $f_{up}^{-1}(g_B)$, where

$$f_{up}^{-1}(g_B)(e)(x) = \left\{ \begin{array}{ll} g_B(p(e)(u(x))) & \text{for } p(e) \in B \\ 0_X & \text{otherwise.} \end{array} \right.$$ 

If $u$ and $p$ are injective then the fuzzy soft mapping $f_{up}$ is said to be injective. If $u$ and $p$ are surjective then the fuzzy soft mapping $f_{up}$ is said to be surjective. The fuzzy soft mapping $f_{up}$ is called constant if $u$ and $p$ are constant.

Definition 2.14 [14] A fuzzy soft topological space is a triple $(X, E, \tau)$ where $X$ is a non-empty set and $\tau$ is a family of fuzzy soft sets over $(X, E)$ satisfying the following properties:

1. $0_E, 1_E \in \tau$
2. If $f_A, g_B \in \tau$, then $f_A \cap g_B \in \tau$
3. If $f_{A_\alpha} \in \tau$ for all $\alpha \in \Lambda$, an index set, then $\bigcup_{\alpha \in \Lambda} f_{A_\alpha} \in \tau$
Then \( \tau \) is called a fuzzy soft topology over \((X, E)\). Also each member of \( \tau \) is called a fuzzy soft open set in \((X, E, \tau)\).

\( g_B \) is called fuzzy soft closed in \((X, E, \tau)\) if \((g_B)^c \in \tau \).

**Definition 2.15** [15] Let \((X, E, \tau)\) be a fuzzy soft topological space and \( f_A \in FS(X, E) \). The fuzzy soft closure of \( f_A \) denoted by \( \overline{f_A} \) is the intersection of all fuzzy soft closed supersets of \( f_A \).

Clearly, \( \overline{f_A} \) is the smallest fuzzy soft closed set over \((X, E)\) which contains \( f_A \).

**Definition 2.16** [14] Let \((X, E, \tau)\) be a fuzzy soft topological space and \( f_A \in FS(X, E) \). The fuzzy soft interior of \( f_A \) denoted by \( f_A^0 \) is the union of all fuzzy soft open subsets of \( f_A \).

Clearly, \( f_A^0 \) is the largest fuzzy soft open set over \((X, E)\) which contained in \( f_A \).

**Theorem 2.17** [6] Let \((X, E, \tau)\) be a fuzzy soft topological space and \( f_A \in FS(X, E) \). Then

\[
\begin{align*}
(1) \quad & (f_A^0)^c = (\overline{f_A})^0, \\
(2) \quad & (f_A)^c = (f_A^0)^0.
\end{align*}
\]

**Definition 2.18** [2] Let \((X, E, \tau)\) be a fuzzy soft topological space and \( f_A \in FS(X, E) \) is called fuzzy soft \( q \)-neighbourhood (briefly fuzzy soft \( q \)-nbd) of \( g_B \) only if there exists a fuzzy soft open set \( h_C \) in \( \tau \) such that \( g_B \sqsubseteq h_C \subseteq f_A \).

If, in addition, \( f_A \) is fuzzy soft open then \( f_A \) is called a fuzzy soft open \( q \)-nbd of \( g_B \).

**Definition 2.19** [3] Let \((X, E, \tau_1)\) and \((X, K, \tau_2)\) be two fuzzy soft topological spaces. A fuzzy soft function \( f_{up} : (X, E, \tau_1) \rightarrow (X, K, \tau_2) \) is called a fuzzy soft continuous if \( f_{up}^{-1}(g_B) \in \tau_1 \) for all \( g_B \in \tau_2 \).

### 3. Fuzzy soft \( \theta \)-open and fuzzy soft \( \theta \)-closed sets

In this section, we define fuzzy soft \( \theta \)- closure, fuzzy soft \( \theta \)-open and fuzzy soft \( \theta \)-closed sets and its related properties.

**Definition 3.1** A fuzzy soft point \( e_\alpha^\theta \) is called a fuzzy soft \( \theta \)-cluster point of a fuzzy soft set \( f_A \) in an fuzzy soft topological space \((X, E, \tau)\) if and only if fuzzy soft closure of every fuzzy soft open \( q \)-nbd of \( e_\alpha^\theta \) is soft quasi-coincident with \( f_A \).

The union of all fuzzy soft \( \theta \)-cluster points of \( f_A \) is called the fuzzy soft \( \theta \)-closure of \( f_A \) and is denoted by \( fs\theta cl(f_A) \).

**Remark 3.2** For a fuzzy soft set \( f_A \) in a fuzzy soft topological space \((X, E, \tau)\), \( f_A \subseteq \overline{f_A} \subseteq fs\theta cl(f_A) \).

**Theorem 3.3** For a fuzzy soft open set \( f_A \) in a fuzzy soft topological space \((X, E, \tau)\), \( \overline{f_A} = fs\theta cl(f_A) \).

**Proof:** From Remark 3.2 it is sufficient to show that \( fs\theta cl(f_A) \subseteq \overline{f_A} \). Let \( e_\alpha^\theta \) be a fuzzy soft point in \((X, E, \tau)\) such that \( e_\alpha^\theta \notin \overline{f_A} \). Then there exists a fuzzy soft open \( q \)-nbd \( g_B \) of \( e_\alpha^\theta \) such that \( g_B \sqsubseteq f_A \). Then \( g_B \subseteq \overline{(1_E - f_A)} = 1_E - f_A \). Thus \( g_B \sqsubseteq f_A \) and consequently \( e_\alpha^\theta \notin fs\theta cl(f_A) \).

**Theorem 3.4** Let \((X, E, \tau)\) be a fuzzy soft topological space and \( f_A \in FS(X, E) \). Then \( fs\theta cl(f_A) = \cap \{g_B : f_A \subseteq g_B \in \tau\} \).
Proof: Obviously, \( L.H.S \subseteq R.H.S. \) Now if possible, let \( e_x^o \in R.H.S \) but \( e_x^o \notin \text{fscl}(f_A) \). Then there exists a fuzzy soft open \( q-nbd \) \( g_B \) of \( e_x^o \) such that \( g_B \notin \text{fscl}(f_A) \) and hence \( f_A \subseteq (1_E - g_B) \). Then \( e_x^o \notin \text{fscl}(1_E - g_B) \) and consequently, \( g_B \notin (1_E - g_B) \) which is impossible and this completes the proof.

**Definition 3.5** Let \((X, E, \tau)\) be a fuzzy soft topological space and \(f_A \in FS(X, E)\). Then \(f_A\) is called a fuzzy soft \(\theta\)-closed if and only if \(f_A = \text{fscl}(f_A)\).

The complement of a fuzzy soft \(\theta\)-closed set is called a fuzzy soft \(\theta\)-open set.

**Theorem 3.6** Let \((X, E, \tau)\) be a fuzzy soft topological space. Then

1. the fuzzy soft sets \(0_E\) and \(1_E\) are fuzzy soft \(\theta\)-closed;
2. for any two fuzzy soft sets \(f_A\) and \(g_B\), if \(f_A \subseteq g_B\) then \(\text{fscl}(f_A) \subseteq \text{fscl}(g_B)\);
3. the union of any two fuzzy soft \(\theta\)-closed sets is a fuzzy soft \(\theta\)-closed set in \((X, E, \tau)\);
4. the intersection of any collection of fuzzy soft \(\theta\)-closed sets is a fuzzy soft \(\theta\)-closed set in \((X, E, \tau)\).

Proof: (1) and (2) are obviously true.

(3) Let \(f_A\) and \(g_B\) be any two fuzzy soft \(\theta\)-closed sets in \((X, E, \tau)\). Let \(e_x^o\) be a fuzzy soft point in \((X, E, \tau)\) such that \(e_x^o \notin \text{fscl}(f_A \cup g_B)\). Then there exists a fuzzy soft open \(q-nbd\) \(h_C\) of \(e_x^o\) such that \(h_C \cap f_A \subseteq g_B\). Now \(h_C \cap g_B\) is a fuzzy soft open \(q-nbd\) of \(e_x^o\), and for any \(e \in E\) and \(x \in X\) we have

\[
[h_C \cap g_B](e)(x) + (f_A \cup g_B)(e)(x) \leq (h_C \cap f_A)(e)(x) + \max(f_A(e)(x), g_B(e)(x))
\]

leq \[
\leq \min(h_C(e)(x), j_D(e)(x)) + \max(f_A(e)(x), g_B(e)(x))
\]

Thus \(h_C \cap g_B \notin \text{fscl}(f_A \cup g_B)\), which shows that \(e_x^o \notin \text{fscl}(f_A \cup g_B)\). Hence \((f_A \cup g_B) = \text{fscl}(f_A \cup g_B)\) and \((f_A \cup g_B)\) is fuzzy soft \(\theta\)-closed.

(4) The proof is straightforward.

**Remark 3.7** The fuzzy soft \(\theta\)-closure operation on a fuzzy soft topological space \((X, E, \tau)\) does not satisfy the Kuratowski closure axioms, which follows from the following example.

**Example 3.8** Let \(X = \{a, b\}, E = \{e_1, e_2, e_3\}, A = \{e_1, e_2\} \subseteq E, B = \{e_1, e_3\} \subseteq E, C = \{e_2, e_3\} \subseteq E, D = \{e_1\} \subseteq E, F = \{e_2\} \subseteq E, G = \{e_3\} \subseteq E\). Let us consider the following fuzzy soft sets over \((X, E)\).

\[
\begin{align*}
\text{fs}(e_1) &= \{a/0.5, b/0.6\}, f(e_2) = \{a/0.7, b/0.4\}, f(e_3) = \{a/0, b/0\} \\
\text{fs}(e_1) &= \{a/0.5, b/0.6\}, g(e_2) = \{a/0, b/0\}, g(e_3) = \{a/0.4, b/0.7\} \\
h_C &= \{b(e_1) = \{a/0, b/0\}, b(e_2) = \{a/0.7, b/0.4\}, b(e_3) = \{a/0.4, b/0.7\} \\
i_E &= \{i(e_1) = \{a/0.5, b/0.6\}, i(e_2) = \{a/0.7, b/0.4\}, i(e_3) = \{a/0.4, b/0.7\} \\
j_D &= \{j(e_1) = \{a/0.5, b/0.6\}, j(e_2) = \{a/0, b/0\}, j(e_3) = \{a/0, b/0\} \\
k_F &= \{k(e_1) = \{a/0, b/0\}, k(e_2) = \{a/0.7, b/0.4\}, k(e_3) = \{a/0, b/0\} \\
u_G &= \{u(e_1) = \{a/0, b/0\}, u(e_2) = \{a/0, b/0\}, u(e_3) = \{a/0.4, b/0.7\}
\end{align*}
\]

Let us consider the fuzzy soft topology \(\tau = \{0_E, 1_E, f_A, g_B, h_C, i_E, j_D, k_F, u_G\}\) over \((X, E)\). Now \(f_A e = \{f_A(e_1) = \{a/0.5, b/0.4\}, f_A(e_2) = \{a/0.3, b/0.6\}, f_A(e_3) = \{a/1, b/1\}\)
$g_B = \{g^c(e_1) = \{a/0.5, b/0.4\}, g^c(e_2) = \{a/1, b/1\}, g^c(e_3) = \{a/0.6, b/0.3\}\}$

$h_C = \{h^c(e_1) = \{a/1, b/1\}, h^c(e_2) = \{a/0.3, b/0.6\}, h^c(e_3) = \{a/0.6, b/0.3\}\}$

$i_E = \{i^c(e_1) = \{a/0.5, b/0.4\}, i^c(e_2) = \{a/0.3, b/0.6\}, i^c(e_3) = \{a/0.6, b/0.3\}\}$

$j_D = \{j^c(e_1) = \{a/0.5, b/0.4\}, j^c(e_2) = \{a/1, b/1\}, j^c(e_3) = \{a/1, b/1\}\}$

$k_F = \{k^c(e_1) = \{a/0.5, b/0.4\}, k^c(e_2) = \{a/1, b/1\}, k^c(e_3) = \{a/1, b/1\}\}$

Clearly, $f_A = u_G^c, \overline{g_B} = k_F, \overline{h_C} = j_D, \overline{1_E} = 1_E, \overline{f_A} = h_C, \overline{k_F} = g_B, \overline{w} = f_A^c$

Let us consider the following fuzzy soft set over $(X, E)$.

Let $w_A = \{w(e_1) = \{a/0.4, b/0.3\}, w(e_2) = \{a/0.5, b/0.2\}, w(e_3) = \{a/0, b/0\}\}$

Now, $f_A^c \cap w_A = u_G^c$

Again, $f_A^c \cap f_A = f_A^c \neq f_A^c$

**Definition 3.9** A fuzzy soft set $f_A$ in a fuzzy soft topological space $(X, E, \tau)$, we define a fuzzy soft $\theta$- interior of $f_A$ (denoted by $\theta f_A$), which is defined by $\theta f_A = 1_E - \theta f_A = f_A^0$

**Remark 3.10** A fuzzy soft set $f_A$ in a fuzzy soft topological space $(X, E, \tau)$ is fuzzy soft $\theta$- open if and only if $f_A = f_A^0$

**Remark 3.11** For any fuzzy soft set $f_A$ in a fuzzy soft topological space $(X, E, \tau)$, $f_A^0 \subseteq f_A$

**Remark 3.12** The set of all fuzzy soft $\theta$- open sets in a fuzzy soft topological space $(X, E, \tau)$ forms a topology, called the fuzzy soft $\theta$- topology and denoted by $\tau_\theta$

**Theorem 3.13** For any fuzzy soft closed set $f_A$ in a fuzzy soft topological space $(X, E, \tau)$, $f_A^0 = f_A^c$

**Proof:** If $f_A$ is a fuzzy soft closed set. Then $f_A^0$ is fuzzy soft open, then from Theorem 3.3 $f_A^0 = f_A^c$, and now $f_A^c = 1_E - f_A^0 = f_A^c$, by Theorem 2.17

**Theorem 3.14** For any fuzzy soft set $f_A$ in a fuzzy soft topological space $(X, E, \tau)$, $f_A^c = (f_A^c)^c$

**Proof:** The proof is obvious.

**Definition 3.15** A fuzzy soft set $f_A$ in a fuzzy soft topological space $(X, E, \tau)$ is said to be a fuzzy soft $\theta$- nb of $f_A$ of a fuzzy soft point $e_x^a$ if and only if there exists a fuzzy soft open $q$-nb of $f_A$ $g_B$ of $e_x^a$ such that $g_B f_A^c$

4. **Fuzzy soft $\theta$- continuous mapping**

In this section, we introduce fuzzy soft $\theta$- continuous mapping, fuzzy soft $\theta$- open mapping, fuzzy soft $\theta$- closed mapping and investigate their related properties.

**Definition 4.1** A function $f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2)$ is said to be a fuzzy soft $\theta$- continuous if and only if for each fuzzy soft point $e_x^a$ in $FS(X, E)$ and each fuzzy soft open $q$-nb of $f_{up}(e_x^a)$ in $FS(Y, K)$, there exists a fuzzy soft open $q$-nb of $f_A$ of $e_x^a$ such that $f_{up}(f_A) \subseteq g_B$.
**Definition 4.2** Let $f_{up} : (X, E, \tau_1) \rightarrow (Y, K, \tau_2)$ be a fuzzy soft mapping. Then

(i) $f_{up}$ is said to be a fuzzy soft $\theta$- open mapping if for each fuzzy soft $\theta$- open set $f_A$ in $(X, E, \tau_1)$, $f_{up}(f_A)$ is fuzzy soft $\theta$- open in $(Y, K, \tau_2)$;

(ii) $f_{up}$ is said to be a fuzzy soft $\theta$- closed mapping if for each fuzzy soft $\theta$- closed set $g_B$ in $(X, E, \tau_1)$, $f_{up}(g_B)$ is fuzzy soft $\theta$- closed in $(Y, K, \tau_2)$.

**Theorem 4.3** A function $f_{up} : (X, E, \tau_1) \rightarrow (Y, K, \tau_2)$ is fuzzy soft $\theta$- continuous if and only if for each fuzzy soft point $e^\alpha_x$ in $FS(X, E)$ and each fuzzy soft $\theta$- nbd $g_B$ of $f_{up}(e^\alpha_x)$, $f_{up}^{-1}(g_B)$ is a fuzzy soft $\theta$- nbd of $e^\alpha_x$.

**Proof:** Let $e^\alpha_x \in FS(X, E)$ and $g_B$ be a fuzzy soft $\theta$- nbd of $f_{up}(e^\alpha_x)$. Then there exists a fuzzy soft open $q$-nbd $h_C$ of $f_{up}(e^\alpha_x)$ such that $h_C \subseteq g_B$. Since $f_{up}$ is fuzzy soft $\theta$- continuous, there exists a fuzzy soft open $q$-nbd $f_A$ of $e^\alpha_x$ such that $f_{up}(f_A) \subseteq f_{up}(e^\alpha_x) \subseteq h_C$. Therefore, $f_{up}^{-1}(h_C) \subseteq f_{up}^{-1}(g_B)$.

Conversely, let $e^\alpha_x \in FS(X, E)$ and $g_B$ be a fuzzy soft open $q$-nbd of $f_{up}(e^\alpha_x)$. Then $\overline{g_B}$ is a fuzzy soft $\theta$- nbd of $f_{up}(e^\alpha_x)$. By hypothesis, $f_{up}^{-1}(\overline{g_B})$ is a fuzzy soft $\theta$- nbd of $e^\alpha_x$. Therefore, there exists a fuzzy soft open $q$-nbd $f_A$ of $e^\alpha_x$ such that $f_{up}(f_A) \subseteq f_{up}(e^\alpha_x) \subseteq \overline{g_B}$. Hence $f_{up}$ is fuzzy soft $\theta$- continuous.

**Theorem 4.4** Let $f_{up} : (X, E, \tau_1) \rightarrow (Y, K, \tau_2)$ be a fuzzy soft $\theta$- continuous mapping. Then the following hold:

i) $f_{up}(f_s\theta cl(f_A)) \subseteq f_s\theta cl(f_{up}(f_A))$ for each fuzzy soft set $f_A$ in $FS(X, E)$.

ii) $f_s\theta cl(f_{up}^{-1}(f_A)) \subseteq f_{up}^{-1}[f_s\theta cl(f_A)]$ for each fuzzy soft set $f_A$ in $FS(Y, K)$.

iii) For each fuzzy soft $\theta$- closed set $f_A$ in $(Y, K, \tau_2)$, $f_{up}^{-1}(f_A)$ is a fuzzy soft $\theta$- closed set in $(X, E, \tau_1)$.

iv) For each fuzzy soft $\theta$- open set $f_A$ in $(Y, K, \tau_2)$, $f_{up}^{-1}(f_A)$ is a fuzzy soft $\theta$- open set in $(X, E, \tau_1)$.

v) For each fuzzy soft open set $f_A$ in $(Y, K, \tau_2)$, $f_s\theta cl(f_{up}^{-1}(f_A)) \subseteq f_{up}^{-1}[f_s\theta cl(f_A)]$.

**Proof:** (i) Let $e^\alpha_x \in f_s\theta cl(f_A)$ and $g_B$ be any fuzzy soft open $q$-nbd of $f_{up}(e^\alpha_x)$. Then there exists a fuzzy soft open $q$-nbd $h_C$ of $e^\alpha_x$ such that $f_{up}(h_C) \subseteq g_B$. Since $e^\alpha_x \in f_s\theta cl(f_A)$, we have $h_C \subseteq f_A$. Thus $f_{up}(h_C) \subseteq f_{up}(f_A)$ and hence $f_{up}(e^\alpha_x) \subseteq f_s\theta cl(f_A)$. So $f_{up}(f_s\theta cl(f_A)) \subseteq f_s\theta cl(f_{up}(f_A))$.

(ii) By (i), $f_{up}(f_s\theta cl(f_{up}^{-1}(f_A))) \subseteq f_s\theta cl(f_{up}(f_{up}^{-1}(f_A))) \subseteq f_s\theta cl(f_A)$. Hence $f_s\theta cl(f_{up}^{-1}(f_A)) \subseteq f_{up}^{-1}[f_s\theta cl(f_A)]$.

(iii) Let $f_A$ be a fuzzy soft $\theta$- closed set in $(Y, K, \tau_2)$. Then $f_s\theta cl(f_A) = f_A$. By (ii), $f_s\theta cl(f_{up}^{-1}(f_A)) \subseteq f_{up}^{-1}[f_s\theta cl(f_A)] = f_{up}^{-1}(f_A)$. Hence $f_{up}^{-1}(f_A)$ is fuzzy soft $\theta$- closed in $(X, E, \tau_1)$.

(iv) Let $f_A$ be a fuzzy soft $\theta$- open set in $(Y, K, \tau_2)$. Then $f_A \subseteq f_{up}^{-1}(f_A) \subseteq f_{up}^{-1}(f_A)$ is a fuzzy soft $\theta$- open set in $(X, E, \tau_1)$. Since $f_{up}^{-1}(f_A) = 1_E - f_{up}^{-1}(f_A)$, $f_{up}^{-1}(f_A)$ is a fuzzy soft $\theta$- open set in $(X, E, \tau_1)$.
Let us consider the following fuzzy soft topology since $g$ is continuous, since $f$ is fuzzy soft continuous, and $\theta$ is soft connected.

Example 4.5 Let $X = \{a, b\}$, $E = \{e_1, e_2\}$

Let us consider the following fuzzy soft sets over $(X, E)$.

Let us consider the following fuzzy soft topology $\tau_1 = \{\hat{0}_E, \hat{1}_E, f_E\}$, $\tau_2 = \{\hat{0}_E, \hat{1}_E, g_E\}$ over $(X, E)$. We define the fuzzy soft mapping $f_{up} : (X, E, \tau_1) \rightarrow (X, E, \tau_2)$ where $u : X \rightarrow X$ and $p : E \rightarrow E$ are mappings, defined by $u(a) = a, u(b) = b, p(e_1) = e_1, p(e_2) = e_2$. Obviously, the fuzzy soft $\theta$- open set in $(X, E, \tau_1)$ is $\hat{0}_E, \hat{1}_E$ $f_{up}^{-1}(\hat{0}_E) = \hat{0}_E \varepsilon (X, E, \tau_1)$ and $f_{up}^{-1}(\hat{1}_E) = \hat{1}_E \varepsilon (X, E, \tau_1)$. Thus $f_{up} : (X, E, \tau_1) \rightarrow (X, E, \tau_2)$ is fuzzy soft $\theta$- continuous, but not fuzzy soft continuous, since $f_{up}^{-1}(g_E) = g_E$ and $g_E$ is $\tau_2$- open but not $\tau_2$- open.

Theorem 4.6 Let $X$, $Y$ and $Z$ be fuzzy soft topological spaces and $f_{up} : X \rightarrow Y$ and $g_{up} : Y \rightarrow Z$ be fuzzy soft $\theta$- continuous. Then the composite mapping $g_{up} \circ f_{up} : X \rightarrow Z$ is fuzzy soft $\theta$- continuous.

Proof: Let $e_x^\alpha \varepsilon FS(X, E)$ and $g_B$ be a fuzzy soft $\theta$- nbd of $g_{up}(f_{up}(e_x^\alpha))$. Since $g_{up}$ is fuzzy soft $\theta$- continuous, $g_{up}^{-1}(g_B)$ is a fuzzy soft $\theta$- nbd of $f_{up}(e_x^\alpha)$. Also since $f_{up}$ is fuzzy soft $\theta$- continuous, $f_{up}^{-1}(g_{up}^{-1}(g_B))$ is a fuzzy soft $\theta$- nbd of $e_x^\alpha$. But $(g_{up} \circ f_{up})^{-1}(g_B) = f_{up}^{-1}(g_{up}^{-1}(g_B))$. Therefore, $g_{up} \circ f_{up}$ is fuzzy soft $\theta$- continuous.

5. Fuzzy soft $\theta$- connectedness

In this section, we introduce fuzzy soft $\theta$- connectedness, characterised in terms of fuzzy soft $\theta$- continuous mapping and investigate their related properties.

Definition 5.1 A pair $(f_A, g_B)$ of non-null fuzzy soft sets $f_A$ and $g_B$ in a fuzzy soft topological space $(X, E, \tau)$ is said to be a fuzzy soft $\theta$- separation relative to $(X, E)$ if and only if $f_{\theta} \varepsilon cl(f_A) \not\subseteq g_B$ and $f_A \varepsilon cl(g_B)$.

If, in addition, $f_A = \hat{1}_E$, then $(X, E, \tau)$ is called a $\theta$- connected space.

Remark 5.2 From $f_{\theta} \varepsilon cl(f_A) \subseteq f_A$ and $f_{\theta} \varepsilon cl(g_B) \subseteq g_B$, it follows that every fuzzy soft $\theta$- separation is a weak fuzzy soft separation.

Definition 5.3 A fuzzy soft set $f_A$ in a fuzzy soft topological space $(X, E, \tau)$ is said to be a fuzzy soft $\theta$- connected if and only if there does not exist any fuzzy soft $\theta$- separation $(g_B, h_C)$ relative to $(X, E)$ such that $f_A = g_B \sqcup h_C$.

Remark 5.4 Every fuzzy soft connected set is a fuzzy soft $\theta$- connected set but converse may not be true.

Theorem 5.5 If $(f_A, g_B)$ is a fuzzy soft $\theta$- separation relative to $(X, E, \tau)$ and $h_C, j_D$ are two non- null fuzzy soft sets such that $h_C \subseteq f_A$ and $j_D \subseteq g_B$, then $(h_C, j_D)$ is also a fuzzy soft $\theta$- separation relative to $(X, E)$.

Proof: The proof is obvious.
Theorem 5.6 Let \( f_A \) be a non-null fuzzy soft set in a fuzzy soft topological space \((X, E, \tau)\). If \( f_A \) is fuzzy soft \( \theta \)-connected, then for every fuzzy soft \( \theta \)-separation \((g_B, h_C)\) relative to \((X, E)\) with \( f_A \subseteq g_B \sqcup h_C \), exactly one of the following holds:

1. \( f_A \subseteq g_B \) and \( f_A \cap h_C = \tilde{0}_E \).
2. \( f_A \subseteq h_C \) and \( f_A \cap g_B = \tilde{0}_E \).

Conversely, if for every fuzzy soft \( \theta \)-separation \((g_B, h_C)\) relative to \((X, E)\) with \( f_A \subseteq g_B \sqcup h_C \) either \( f_A \cap g_B = \tilde{0}_E \) or \( f_A \cap h_C = \tilde{0}_E \) holds, then \( f_A \) is fuzzy soft \( \theta \)-connected.

**Proof:** Let \( f_A \) be fuzzy soft \( \theta \)-connected in \((X, E, \tau)\). Since \( f_A \subseteq g_B \sqcup h_C \), both of \( f_A \cap h_C = \tilde{0}_E \) and \( f_A \cap g_B = \tilde{0}_E \) do not hold simultaneously. If \( f_A \cap h_C \neq \tilde{0}_E \) and \( f_A \cap g_B \neq \tilde{0}_E \), then \( f_A \cap g_B, f_A \cap h_C \) is a fuzzy soft \( \theta \)-separation relative to \((X, E)\) such that \( f_A = (f_A \cap g_B) \cup (f_A \cap h_C) \), which contradicts the fuzzy soft \( \theta \)-connected of \( f_A \). Thus exactly one of \( f_A \cap g_B \) and \( f_A \cap h_C \) is a non-null fuzzy soft set. Now, whenever \( f_A \cap g_B = \tilde{0}_E \), we have \( f_A \subseteq h_C \), since \( f_A \subseteq g_B \sqcup h_C \). Similarly, when \( f_A \cap h_C = \tilde{0}_E \), we have \( f_A \subseteq g_B \).

Conversely, if \( f_A \) is not fuzzy soft \( \theta \)-connected, then there is a fuzzy soft \( \theta \)-separation \((g_B, h_C)\) relative to \((X, E)\) such that \( f_A = g_B \sqcup h_C \). By hypothesis, either \( f_A \cap g_B = \tilde{0}_E \), which implies that \( g_B = \tilde{0}_E \) (since \( g_B \subseteq f_A \)), or \( f_A \cap h_C = \tilde{0}_E \) implying \( h_C = \tilde{0}_E \) and so none of which is true.

Theorem 5.7 Let \( f_{up} : (X, E, \tau_1) \to (Y, K, \tau_2) \) be a fuzzy soft \( \theta \)-continuous mapping and \( f_A \) be fuzzy soft \( \theta \)-connected relative to \((X, E)\). Then \( f_{up}(f_A) \) is fuzzy soft \( \theta \)-connected relative to \((Y, K)\).

**Proof:** If possible, let \( f_{up}(f_A) \) be not fuzzy soft \( \theta \)-connected in \((Y, K, \tau_2)\), then there exists a fuzzy soft \( \theta \)-separation \((g_B, h_C)\) relative to \((Y, K)\) such that \( f_{up}(f_A) = g_B \sqcup h_C \). Put \( j_D = f_A \cap f_{up}^{-1}(g_B) \) and \( w_F = f_A \cap f_{up}^{-1}(h_C) \). Now \( f_{up}(f_A) \cap g_B \neq \tilde{0}_E \Rightarrow f_{up}^{-1}(f_{up}(f_A) \cap g_B) \neq \tilde{0}_E \Rightarrow f_A \cap f_{up}^{-1}(g_B) \neq \tilde{0}_E \Rightarrow j_D \neq \tilde{0}_E \). Similarly, \( w_F \neq \tilde{0}_E \).

If possible, let \( j_D \tilde{\varphi} w_F \), then for some \( e \in E \) and \( x \in X \), \((f_{up}^{-1}(g_B))(e)(x) + (f_{up}^{-1}(h_C))(e)(x) > 1 \) and so \( g_B(f_{up}(e)(x)) + h_C(f_{up}(e)(x)) > 1 \), a contradiction since \( g_B \tilde{\varphi} h_C \). Thus \( j_D \tilde{\varphi} w_F \). Again, \( f_{up}^{-1}(f_{up}^{-1}(h_C)) \subseteq f_{up}^{-1}(f_{up}^{-1}(f_{up}^{-1}(h_C))) \) and \( w_F \subseteq f_{up}^{-1}(h_C) \Rightarrow f_{up}^{-1}(f_{up}^{-1}(f_{up}^{-1}(h_C))) \). Then \( g_B \tilde{\varphi} f_{up}^{-1}(h_C) \Rightarrow f_{up}^{-1}(g_B) \tilde{\varphi} f_{up}^{-1}(f_{up}^{-1}(f_{up}^{-1}(h_C))) \Rightarrow f_{up}^{-1}(f_{up}^{-1}(h_C)) \). This implies that \( j_D \tilde{\varphi} w_F \). Similarly, \( w_F \tilde{\varphi} f_{up}^{-1}(f_{up}^{-1}(h_C)) \). Thus \( (j_D, w_F) \) is a fuzzy soft \( \theta \)-separation relative to \((X, E)\), and so \( f_{up}(f_A) \) is fuzzy soft \( \theta \)-connected.

6. Conclusion

Topology is an important area of mathematics with many applications in the domain of computer science and physical sciences. Fuzzy soft topology [14] is a relatively new and promising domain which can lead to the development of new mathematical models and innovative approaches that will significantly contribute to the solution of complex problems in natural sciences.
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