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EVEN-TUPLED COINCIDENCE THEOREMS IN GENERALIZED COMPLETE METRIC SPACES

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Abstract. In this paper, we use complete asymptotically regular metric spaces due to Zeyada and Ahmed Soliman to prove a generalization of some recent even-tupled coincidence point theorems due to Imdad *et al.* [J. Oper. Volume 2013, Article ID 532867, 9 pages and J. Adv. Math. 9 (1) (2014) 1787-1805].

1. Introduction

The Banach Contraction Principle is one of the most celebrated fixed point theorems ever proved in Mathematics and still remains an inspiration to the workers of metric fixed point theory. The significance of this principle lies in its wide range of applicability within and beyond mathematics. To prove all kind of generalizations of the aforesaid principle continues to be a heavily investigated topic of metric fixed point theory and henceforth there already exists an extensive literature on this topic (e.g.([4], [8]-[12], [14], [17], [21]-[27]).
One noted variant of Banach contraction principle is essentially due to Bhaskar and Lakshmikantham [13], wherein the idea of coupled fixed point was initiated in partially ordered metric spaces besides proving some interesting coupled fixed point theorems for mappings satisfying a mixed monotone property. Recently, many authors obtained important coupled fixed point theorems (e.g. [1], [2], [5]-[7], [28]). In the same continuation, Lakshmikantham and Ćirić [19] proved coupled coincidence point theorems for nonlinear contractions in partially ordered complete metric spaces which indeed generalize the corresponding fixed point theorems contained in Bhaskar and Lakshmikantham [13].
Section 2 of the paper, as usual, is devoted to preliminaries which include basic definitions and results related to n-tupled coincidence point. Some definitions and concepts on complete asymptotically regular metric spaces are also included. In Section 3, we prove an even-tupled coincidence point theorem for ϕ-contraction mappings in partially ordered complete asymptotically regular metric spaces.
Thus, the main aim of this paper is to prove even-tupled coincidence and common

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fixed point theorems in partially ordered complete asymptotically regular metric spaces.

2. Preliminaries

Throughout this paper, n stands for a general even natural number, while $i \in \{1, 2, ..., n\}$ and $m \in \mathbb{N} \cup \{0\}$. For the sake of simplicity, we write only " for all/each/some i ", " for all/each m " and " for all/each $m \geq 1$ " instead of " for all/each/some $i \in \{1, 2, ..., n\}$", " for all/each $m \in \mathbb{N} \cup \{0\}$" and " for all/each $m \in \mathbb{N}$" respectively.

Definition 1[16] Let $(X, \preceq)$ be a partially ordered set and $F : X^n \to X$ be a mapping. The mapping F is said to have the mixed monotone property if F is non-decreasing in its odd position arguments and is non-increasing in its even position arguments, *i.e.*, for any $x^1, x^2, x^3, ..., x^n \in X$,

$$\begin{array}{l}
\text{for all } x_1^1, x_2^1 \in X,\, x_1^1 \preceq x_2^1 \text{ implies } F(x_1^1, x^2, x^3, ..., x^n) \preceq F(x_2^1, x^2, x^3, ..., x^n),\\
\text{for all } x_1^2, x_2^2 \in X,\, x_1^2 \preceq x_2^2 \text{ implies } F(x^1, x_1^2, x^3, ..., x^n) \succeq F(x^1, x_2^2, x^3, ..., x^n),\\
\text{for all } x_1^3, x_2^3 \in X,\, x_1^3 \preceq x_2^3 \text{ implies } F(x^1, x^2, x_1^3, ..., x^n) \preceq F(x^1, x^2, x_2^3, ..., x^n),\\
\qquad\qquad\qquad\qquad \vdots\\
\text{for all } x_1^n, x_2^n \in X,\, x_1^n \preceq x_2^n \text{ implies } F(x^1, x^2, x^3, ..., x_1^n) \succeq F(x^1, x^2, x^3, ..., x_2^n).
\end{array}$$

Definition 2 [16] Let $(X, \preceq)$ be a partially ordered set. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. The mappings F and g are said to have the mixed g-monotone property if F is g-non-decreasing in its odd position arguments and is g-non-increasing in its even position arguments, *i.e.*, for any $x^1, x^2, x^3, ..., x^n \in X$,

$$\begin{array}{l}
\text{for all } x_1^1, x_2^1 \in X,\, gx_1^1 \preceq gx_2^1 \text{ implies } F(x_1^1, x^2, x^3, ..., x^n) \preceq F(x_2^1, x^2, x^3, ..., x^n),\\
\text{for all } x_1^2, x_2^2 \in X,\, gx_1^2 \preceq gx_2^2 \text{ implies } F(x^1, x_1^2, x^3, ..., x^n) \succeq F(x^1, x_2^2, x^3, ..., x^n),\\
\text{for all } x_1^3, x_2^3 \in X,\, gx_1^3 \preceq gx_2^3 \text{ implies } F(x^1, x^2, x_1^3, ..., x^n) \preceq F(x^1, x^2, x_2^3, ..., x^n),\\
\qquad\qquad\qquad\qquad \vdots\\
\text{for all } x_1^n, x_2^n \in X,\, gx_1^n \preceq gx_2^n \text{ implies } F(x^1, x^2, x^3, ..., x_1^n) \succeq F(x^1, x^2, x^3, ..., x_2^n).
\end{array}$$

Definition 3 [16] Let X be a nonempty set. An element $(x^1, x^2, x^3, ..., x^n) \in X^n$ is called an n-tupled fixed point of the mapping $F : X^n \to X$ if

$$\begin{array}{l}
x^1 = F(x^1, x^2, x^3, ..., x^n)\\
x^2 = F(x^2, x^3, ..., x^n, x^1)\\
x^3 = F(x^3, ..., x^n, x^1, x^2)\\
\qquad\vdots\\
x^n = F(x^n, x^1, x^2, ..., x^{n-1}).
\end{array}$$

Definition 4 [16] Let X be a nonempty set. An element $(x^1, x^2, x^3, ..., x^n) \in X^n$ is called an n-tupled coincidence point of the mappings $F : X^n \to X$ and $g : X \to X$ if

$$\begin{array}{l}
g(x^1) = F(x^1, x^2, x^3, ..., x^n)\\
g(x^2) = F(x^2, x^3, ..., x^n, x^1)\\
g(x^3) = F(x^3, ..., x^n, x^1, x^2)\\
\qquad\vdots\\
g(x^n) = F(x^n, x^1, x^2, ..., x^{n-1}).
\end{array}$$

Definition 5 [33] Let X be a nonempty set. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. An element $(x^1, x^2, x^3, ..., x^n) \in X^n$ is called an n-tupled common fixed point of the mappings F and g if

$$\begin{aligned} x^1 = \quad & g(x^1) & = F(x^1, x^2, ..., x^n), \\ x^2 = \quad & g(x^2) & = F(x^2, x^1, ..., x^n), \\ & \vdots & \\ x^n = \quad & g(x^n) & = F(x^n, x^2, ..., x^1). \end{aligned}$$

Definition 6 [16] Let X be a nonempty set. The mappings $F : X^n \to X$ and $g : X \to X$ are said to be commutating if for all $x^1, x^2, ..., x^n \in X$

$$g(F(x^1, x^2, ..., x^n)) = F(gx^1, gx^2, ..., gx^n).$$

Definition 7 [32] Let $(X, \preceq)$ be a partially ordered set equipped with a metric d. We say that X has *MCB* (Monotone-Convergence-Boundedness) property if it satisfies the following conditions:

(a) Every non-decreasing convergent sequence $\{x_m\}$ in X is bounded above by its limit (as an upper bound), *i.e.*,

$$x_m \succeq x_{m+1} \text{ and } x_m \xrightarrow{d} x \text{ implies } g(x_m) \succeq g(x), \text{ for all } m.$$

(b) Every non-increasing convergent sequence $\{x_m\}$ in X is bounded below by its limit (as a lower bound), *i.e.*,

$$x_m \succeq x_{m+1} \text{ and } x_m \xrightarrow{d} x \text{ implies } g(x_m) \succeq g(x), \text{ for all } m.$$

Definition 8 [32] Let $(X, \preceq)$ be a partially ordered set equipped with a metric d We say that X has *g-MCB* property if it satisfies the following conditions:

(a) g-image of every non-decreasing convergent sequence $\{x_m\}$ in X is bounded above by g-image of its limit (as an upper bound), *i.e.*,

$$x_m \preceq x_{m+1} \text{ and } x_m \xrightarrow{d} x \text{ implies } g(x_m) \preceq g(x), \text{ for all } m.$$

(b) g-image of every non-increasing convergent sequence $\{x_m\}$ in X is bounded below by g-image of its limit (as a lower bound), *i.e.*,

$$x_m \succeq x_{m+1} \text{ and } x_m \xrightarrow{d} x \text{ implies } g(x_m) \succeq g(x), \text{ for all } m.$$

If g is set to be the identity mapping on X, then Definition 8 reduces to Definition 7.

Definition 9 [19] We denote by Φ the family of functions $\phi : [0, \infty) \to [0, \infty)$ satisfying

(a) $\phi(t) < t$ for each $t > 0$,

(b) $\lim\limits_{r \to t^+} \phi(r) < t$ for each $t > 0$.

Definition 10 [30, 31] A sequence $\{x_n\}$ in a metric space (X, d) is said to be asymptotically regular if

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0.$$

Definition 11 [30, 31] Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is called Cauchy if $\lim\limits_{n,m \to \infty} d(x_n, x_m) = 0$ for all $n, m \in \mathbb{N}$ such that $m > n$.

Definition 12 [30, 31] A metric space (X, d) is called complete if every Cauchy sequence in X converges.

Lemma 1 Every Cauchy sequence is asymptotically regular but converse need not be true in general.

Example 1 Let $X = R$ and $d(x, y) = |x - y| \ \forall \ x, y \in X$. Then $\{x_n\}$ in X defined by $x_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ is asymptotically regular but not Cauchy.

Definition 13 [30, 31] A metric space is called complete asymptotically regular if every asymptotically regular sequence $\{x_n\}$ in X, converges to some point in X.

3. Main Results

Before proving our main result, we prove the following lemma:

Lemma 2 Let $(X, \preceq)$ be a partially ordered set. Let $F : X^n \to X$ and $g : X \to X$ be two mappings. If F has mixed $g-$monotone property and $x^1, x^2, ..., x^n, y^1, y^2, ..., y^n \in X$ such that $g(x^1) = g(y^1), g(x^2) = g(y^2), ..., g(x^n) = g(y^n)$, then

$$\begin{array}{rcl} F(x^1, x^2, ..., x^n) & = & F(y^1, y^2, ..., y^n) \\ F(x^2, ..., x^n, x^1) & = & F(y^2, ..., y^n, y^1) \\ & \vdots & \\ F(x^n, x^1, ..., x^{n-1}) & = & F(y^n, y^1, ..., y^{n-1}). \end{array}$$

Proof Given that $g(x^1) = g(y^1), g(x^2) = g(y^2), ..., g(x^n) = g(y^n)$, by using reflexivity of $\preceq$, we have

$$\begin{bmatrix} g(x^1) \preceq g(y^1) \text{ and } & g(x^1) \succeq g(y^1) \\ g(x^2) \preceq g(y^2) \text{ and } & g(x^2) \succeq g(y^2) \\ \vdots & \\ g(x^n) \preceq g(y^n) \text{ and } & g(x^n) \succeq g(y^n) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} g(x^1) \preceq g(y^1), & g(x^2) \succeq g(y^2), & \cdots, & g(x^n) \succeq g(y^n) \\ & \text{and} & & \\ g(x^1) \succeq g(y^1), & g(x^2) \preceq g(y^2), & \cdots, & g(x^n) \preceq g(y^n) \end{bmatrix}$$

As F has mixed $g-$monotone property,we have

$$\begin{bmatrix} F(x^1, x^2, ..., x^n) \preceq F(y^1, y^2, ..., y^n) \\ \text{and} \\ F(x^1, x^2, ..., x^n) \succeq F(y^1, y^2, ..., y^n) \end{bmatrix}$$

By using anti-symmetry of $\preceq$, implies that

$$F(x^1, x^2, ..., x^n) = F(y^1, y^2, ..., y^n).$$

Similarly, one can show that

$$\begin{array}{rcl} F(x^2, ..., x^n, x^1) & = & F(y^2, ..., y^n, y^1) \\ & \vdots & \\ F(x^n, x^1, ..., x^{n-1}) & = & F(y^n, y^1, ..., y^{n-1}). \end{array}$$

Now, we prove our main result as follows:

Theorem 1 Let $(X, \preceq)$ be a partially ordered set equipped with a metric d such that (X, d) is a complete asymptotically regular metric space. Let $F : X^n \to X$ and $g : X \to X$ be two mappings, where n stands for fixed even natural number. Suppose that the following conditions hold:

(i) $F(X^n) \subseteq g(X)$,

(ii) F has the mixed $g-$monotone property,

(iii) (g, F) is a commutating pair,
(iv) g is continuous,
(v) either F is continuous or $(X, d, \preceq)$ has g-MCB property,
(vi) $\exists\, x_0^1, x_0^2, x_0^3, ..., x_0^n \in X$ such that

$$\begin{array}{l} g(x_0^1) \preceq F(x_0^1, x_0^2, ..., x_0^n), \\ g(x_0^2) \succeq F(x_0^2, ..., x_0^n, x_0^1), \\ g(x_0^3) \preceq F(x_0^3, ..., x_0^n, x_0^1, x_0^2), \\ \quad\vdots \\ g(x_0^n) \succeq F(x_0^n, x_0^1, ..., x_0^{n-1}), \end{array} \tag{1}$$

(vii) $\exists\, \phi \in \Phi$ such that

$$d(F(U), F(V)) \leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx^i, gy^i)\Big), \tag{2}$$

for all $U = (x^1, x^2, ..., x^n), V = (y^1, y^2, ..., y^n) \in X^n$, with

$$g(x^1) \preceq g(y^1), g(x^2) \succeq g(y^2), ..., g(x^n) \succeq g(y^n).$$

Then F and g have an n-tupled coincidence point.
Proof In view of the hypothesis $F(X^n) \subseteq g(X)$, we construct the sequences $\{x_m^1\}$, $\{x_m^2\}$, $\{x_m^3\}$,...,$\{x_m^n\}$ in X as follows:

$$\begin{array}{c} g(x_{m+1}^1) = F(x_m^1, x_m^2, ..., x_m^n), \\ g(x_{m+1}^2) = F(x_m^2, ..., x_m^n, x_m^1), \\ \vdots \\ g(x_{m+1}^n) = F(x_m^n, x_m^1, ..., x_m^{n-1}). \end{array} \tag{3}$$

Now, we prove that for all m,

$$g(x_m^1) \preceq g(x_{m+1}^1), g(x_m^2) \succeq g(x_{m+1}^2), \cdots, g(x_m^n) \succeq g(x_{m+1}^n). \tag{4}$$

By using (1) and (3), we obtain

$$\begin{array}{c} g(x_0^1) \preceq F(x_0^1, x_0^2, ..., x_0^n) = g(x_1^1), \\ g(x_0^2) \succeq F(x_0^2, ..., x_0^n, x_0^1) = g(x_1^2), \\ \vdots \\ g(x_0^n) \succeq F(x_0^n, x_0^1, ..., x_0^{n-1}) = g(x_1^n). \end{array}$$

So (4) holds for $m = 0$. Suppose (4) holds for some $m > 0$. By using mixed g-monotone property of F, we have

$$\begin{array}{rcl} g(x_{m+1}^1) & = & F(x_m^1, x_m^2, ..., x_m^{n-1}, x_m^n) \\ & \preceq & F(x_{m+1}^1, x_m^2, ..., x_m^{n-1}, x_m^n) \\ & \preceq & F(x_{m+1}^1, x_{m+1}^2, ..., x_m^{n-1}, x_m^n) \\ & \vdots & \\ & \preceq & F(x_{m+1}^1, x_{m+1}^2, ..., x_{m+1}^{n-1}, x_m^n) \\ & \preceq & F(x_{m+1}^1, x_{m+1}^2, ..., x_{m+1}^{n-1}, x_{m+1}^n) \\ & = & g(x_{m+2}^1), \end{array}$$

$$\begin{aligned}
g(x^2_{m+1}) &= F(x^2_m, x^3_m, ..., x^n_m, x^1_m)\\
&\succeq F(x^2_{m+1}, x^3_m, ..., x^n_m, x^1_m)\\
&\succeq F(x^2_{m+1}, x^3_{m+1}, ..., x^n_m, x^1_m)\\
&\vdots\\
&\succeq F(x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}, x^1_m)\\
&\succeq F(x^2_{m+1}, x^3_{m+1}, ..., x^n_{m+1}, x^1_{m+1})\\
&= g(x^2_{m+2}),\\
&\vdots
\end{aligned}$$

$$\begin{aligned}
g(x^n_{m+1}) &= F(x^n_m, x^1_m, ..., x^{n-2}_m, x^{n-1}_m)\\
&\succeq F(x^n_{m+1}, x^1_m, ..., x^{n-2}_m, x^{n-1}_m)\\
&\succeq F(x^n_{m+1}, x^1_{m+1}, ..., x^{n-2}_m, x^{n-1}_m)\\
&\vdots\\
&\succeq F(x^n_{m+1}, x^1_{m+1}, ..., x^{n-2}_{m+1}, x^{n-1}_m)\\
&\succeq F(x^n_{m+1}, x^1_{m+1}, ..., x^{n-2}_{m+1}, x^{n-1}_{m+1})\\
&= g(x^n_{m+2}).
\end{aligned}$$

Then, by induction, (4) holds for all m. We define a sequence $\delta_m \geq 0$ as follows:

$$\delta_m = d(gx^1_m, gx^1_{m+1}) + d(gx^2_m, gx^2_{m+1}) + ... + d(gx^n_m, gx^n_{m+1}).$$

In case, $\delta_{m_0} = 0$ for some $m_0 \in \mathbb{N} \cup \{0\}$, we have

$$d(gx^1_{m_0}, gx^1_{m_0+1})) = d(gx^2_{m_0}, gx^2_{m_0+1}) = ... = d(gx^n_{m_0}, gx^n_{m_0+1}) = 0.$$

Consequently by using (3), we get

$$\begin{aligned}
g(x^1_{m_0}) &= g(x^1_{m_0+1}) = F(x^1_{m_0}, x^2_{m_0}, x^3_{m_0}, ..., x^n_{m_0}),\\
g(x^2_{m_0}) &= g(x^2_{m_0+1}) = F(x^2_{m_0}, x^3_{m_0}, ..., x^n_{m_0}, x^1_{m_0}),\\
&\vdots\\
g(x^n_{m_0}) &= g(x^n_{m_0+1}) = F(x^n_{m_0}, x^1_{m_0}, x^2_{m_0}, ..., x^{r-1}_{m_0})
\end{aligned}$$

so that $(x^1_{m_0}, x^2_{m_0}, ..., x^n_{m_0})$ is an n-tupled coincidence point of F and g and hence we are through. Otherwise, if $\delta_m > 0$ for all m, then we have to show that for each i and all m,

$$d(gx^i_{m+1}, gx^i_{m+2}) \leq \phi\Big(\frac{\delta_m}{n}\Big). \tag{5}$$

On using (2), (3) and (4), we have

$$\begin{aligned}
d(gx^1_{m+1}, gx^1_{m+2}) &= d(F(x^1_m, x^2_m, ..., x^n_m, x^1_m), F(x^1_{m+1}, x^2_{m+1}, ..., x^n_{m+1}))\\
&\leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx^i_m, gx^i_{m+1})\Big)\\
&= \phi\Big(\frac{\delta_m}{n}\Big).
\end{aligned}$$

Similarly, we can inductively write

$$\begin{aligned} d(gx^2_{m+1}, gx^2_{m+2}) &\leq \phi\Big(\frac{\delta_m}{n}\Big), \\ &\vdots \\ d(gx^i_{m+1}, gx^i_{m+2}) &\leq \phi\Big(\frac{\delta_m}{n}\Big). \end{aligned}$$

Hence (5) holds for each i and for all m. By taking summation of (5) over i, we get

$$\sum_{i=1}^{n} d(gx^i_{m+1}, gx^i_{m+2}) \leq n\phi\Big(\frac{\delta_m}{n}\Big),$$

so that

$$\delta_{m+1} \leq n\phi\Big(\frac{\delta_m}{n}\Big). \tag{6}$$

Since $\phi(t) < t$ for all $t > 0$, therefore $\delta_{m+1} < \delta_m$ for all m so that $\{\delta_m\}$ is a decreasing sequence. Since it is bounded below (as $\delta_m > 0$), there is some $\delta \geq 0$ such that

$$\lim_{m\to\infty} \delta_m = \delta^+.$$

Now, we show that $\delta = 0$. Suppose, on contrary that $\delta > 0$. Taking limit of both the sides of (6) as $m \to \infty$ and keeping in mind our supposition that $\lim_{r\to t^+} \phi(r) < t$ for all $t > 0$, we have

$$\delta = \lim_{m\to\infty} \delta_{m+1} \leq n \lim_{m\to\infty} \phi\Big(\frac{\delta_m}{n}\Big) = n \lim_{\delta_m \to\ \delta^+} \phi\Big(\frac{\delta_m}{n}\Big) < n\Big(\frac{\delta}{n}\Big) = \delta,$$

which is a contradiction so that $\delta = 0$ yielding thereby

$$\lim_{m\to\infty} d(gx^1_m, gx^1_{m+1}) + d(gx^2_m, gx^2_{m+1}) + ... + d(gx^n_m, gx^n_{m+1}) = 0. \tag{7}$$

which implies that $\{gx^1_m\}, \{gx^2_m\}, ..., \{gx^n_m\}$ are asymptotically regular sequences in (X, d). Since the metric space (X, d) is complete asymptotically regular, so there exist $x^1, x^2, ..., x^n \in X$ such that

$$\begin{aligned} \lim_{m\to\infty} g(x^1_m) &= x^1, \\ \lim_{m\to\infty} g(x^2_m) &= x^2, \\ &\vdots \\ \lim_{m\to\infty} g(x^n_m) &= x^n. \end{aligned} \tag{8}$$

By the continuity of g and (8), we can have

$$\begin{aligned} \lim_{m\to\infty} g(gx^1_m) &= g(x^1), \\ \lim_{m\to\infty} g(gx^2_m) &= g(x^2), \\ &\vdots \\ \lim_{m\to\infty} g(gx^n_m) &= g(x^n). \end{aligned} \tag{9}$$

Using (3) and the commutativity of F with g, we get

$$\begin{aligned} g(gx^1_{m+1}) &= g(F(x^1_m, x^2_m, ..., x^n_m)) = F(gx^1_m, gx^2_m, ..., gx^n_m), \\ g(gx^2_{m+1}) &= g(F(x^2_m, x^3_m, ..., x^n_m)) = F(gx^2_m, gx^3_m, ..., gx^1_m), \\ &\vdots \\ g(gx^n_{m+1}) &= g(F(x^n_m, x^1_m, ..., x^{n-1}_m)) = F(gx^n_m, gx^1_m, ..., gx^{n-1}). \end{aligned} \tag{10}$$

Now, we show that F and g have an $n-$tupled coincidence point. To accomplish this, we use assumption (v)$'$. Suppose that F is continuous, then on using (8), (9), (10) and continuity of F, we obtain

$$\begin{aligned} g(x^1) &= \lim_{m\to\infty} g(gx^1_{m+1}) \\ &= \lim_{m\to\infty} F(gx^1_m, gx^2_m, ..., gx^n_m) \\ &= F\Big(\lim_{m\to\infty} gx^1_m, \lim_{m\to\infty} gx^2_m, ..., \lim_{m\to\infty} gx^n_m\Big) \\ &= F(x^1, x^2, ..., x^n). \end{aligned}$$

Similarly, we can also show that

$$\begin{aligned} g(x^2) &= F(x^2, x^3, ..., x^n, x^1), \\ g(x^3) &= F(x^3, ..., x^n, x^1, x^2), \\ &\vdots \\ g(x^n) &= F(x^n, x^1, ..., x^{n-1}). \end{aligned}$$

Thus, the element $(x^1, x^2, ..., x^n) \in X^n$ is an n-tupled coincidence point of the mappings F and g. Otherwise, assume that X has *g-MCB* property. Since $\{gx^i_m\}$ is non-decreasing or non-increasing according as i is odd or even respectively and for each i, $g(x^i_m) \to x^i$ as $m \to \infty$, we have

$$g(gx^1_m) \preceq g(x^1),\ g(gx^2_m) \succeq g(x^2),\ \cdots, g(gx^n_m) \succeq g(x^n). \tag{11}$$

We claim that for all m

$$d(F(gx^1_m, gx^2_m, ..., gx^n_m), F(x^1, ..., x^n)) \leq \frac{1}{n}\sum_{i=1}^{n} d(ggx^i_m, gx^i). \tag{12}$$

On account of two different possibilities arising here, we consider a partition $\{\mathbb{N}^0, \mathbb{N}^+\}$ of $\mathbb{N}$, *i.e.*, $\mathbb{N}^0 \cup \mathbb{N}^+ = \mathbb{N}$ and $\mathbb{N}^0 \cap \mathbb{N}^+ = \emptyset$ verifying that

(i)$d(ggx^i_m, gx^i) = 0$ for each i and for all $m \in \mathbb{N}^0$.

(ii)$d(ggx^i_m, gx^i) > 0$ for some i and for all $m \in \mathbb{N}^+$.

In case (i), by using Lemma 2, we get

$$\begin{aligned} F(gx^1_{m_0}, gx^2_{m_0}, ..., gx^n_{m_0}) &= F(x^1, x^2, ..., x^n) \\ F(gx^2_{m_0}, ..., gx^n_{m_0}, gx^1_{m_0}) &= F(x^2, ..., x^n, x^1) \\ &\vdots \\ F(gx^n_{m_0}, gx^1_{m_0}, ..., gx^{n-1}_{m_0}) &= F(x^n, x^1, ..., x^{n-1}) \end{aligned}$$

which implies that $d(F(gx^1_{m_0}, gx^2_{m_0}, ..., gx^n_{m_0}), F(x^1, ..., x^n)) = 0$ and hence in this case (12) trivially holds.

In case (ii), we have $\frac{1}{n}\sum\limits_{i=1}^{n} d(ggx^i_m, gx^i) > 0$. By using (2) and (11), we get

$$d(F(gx^1_m, gx^2_m, ..., gx^n_m), F(x^1, ..., x^n)) \leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} d(ggx^i_m, gx^i)\Big) < \frac{1}{n}\sum_{i=1}^{n} d(ggx^i_m, gx^i)$$

i. e. (12) is proved. Hence (12) holds for all m. By using triangular inequality, (10) and (12), we get

$$\begin{aligned} d(gx^1, F(x^1, ..., x^n)) &\leq d(gx^1, ggx^1_{m+1}) + d(ggx^1_{m+1}, F(x^1, ..., x^n)) \\ &= d(gx^1, ggx^1_{m+1}) + d(F(gx^1_m, gx^2_m, ..., gx^n_m), F(x^1, ..., x^n)) \\ &\leq d(gx^1, ggx^1_{m+1}) + \frac{1}{n}\sum_{i=1}^{n} d(ggx^i_m, gx^i). \end{aligned}$$

Letting $m \to \infty$ in the preceding inequality and using (9), we have

$$g(x^1) = F(x^1, x^2..., x^n).$$

Similarly, we can also show that

$$\begin{aligned} g(x^2) &= F(x^2, x^3, ..., x^n, x^1), \\ g(x^3) &= F(x^3, ..., x^n, x^1, x^2), \\ &\vdots \\ g(x^n) &= F(x^n, x^1, ..., x^{n-1}) \end{aligned}$$

which show that $(x^1, x^2, ..., x^n)$ is an n-tupled coincidence point of F and g. This completes the proof.

On setting $g = I$, the identity mapping on X, in Theorem 1, we get the corresponding $n-$tupled fixed point theorem.

Corollary 1 Let $(X, \preceq)$ be a partially ordered set equipped with a metric d such that (X, d) is a complete asymptotically regular metric space. Suppose that the following conditions hold:

(i) F has the mixed monotone property,

(ii) either F is continuous or $(X, d, \preceq)$ has *MCB* property,

(iii) $\exists\, x^1_0, x^2_0, x^3_0, ..., x^n_0 \in X$ such that

$$\begin{aligned} x^1_0 &\preceq F(x^1_0, x^2_0, ..., x^n_0), \\ x^2_0 &\succeq F(x^2_0, ..., x^n_0, x^1_0), \\ &\vdots \\ x^n_0 &\succeq F(x^n_0, x^1_0, ..., x^{n-1}_0), \end{aligned}$$

(iv) $\exists\, \phi \in \Phi$ such that

$$d(F(U), F(V)) \leq \phi\Big(\tfrac{1}{n}\sum_{i=1}^{n} d(x^i, y^i)\Big),$$

for all $U = (x^1, x^2, ..., x^n), V = (y^1, y^2, ..., y^n) \in X^n,$ with

$$x^1 \preceq y^1, x^2 \succeq y^2, ..., x^n \succeq y^n.$$

Then F has an n-tupled fixed point.

On setting $\phi(t) = k.t$ with $k \in [0, 1)$ in Theorem 1, we get the following result.

Corollary 2 Theorem 1 remains true if the contractive condition (vii) is replaced by the following contractive condition (vii)$'$ besides retaining the rest of the hypotheses.

(vii)$'$ $\quad d(F(U), F(V)) \leq \frac{k}{n}\sum_{i=1}^{n} d(gx^i, gy^i),\ k \in [0, 1)$

for all $U=(x^1,x^2,...,x^n), V=(y^1,y^2,...,y^n)\in X^n,$ with

$$g(x^1)\preceq g(y^1), g(x^2)\succeq g(y^2),...,g(x^n)\succeq g(y^n).$$

On setting $\phi(t)=k.t$ with $k\in[0,1)$, in Corollary 1, we get the following result.

Corollary 3 Corollary 1 remains true if the contractive condition (iv) is replaced by the following contractive condition (iv)$'$ besides retaining the rest of the hypotheses.

(iv)$'$ $d(F(U),F(V))\leq \frac{k}{n}\sum\limits_{i=1}^{n} d(x^n,y^n),\ k\in[0,1)$

for all $U=(x^1,x^2,...,x^n), V=(y^1,y^2,...,y^n)\in X^n,$ with

$$x^1\preceq y^1, x^2\succeq y^2,...,x^n\succeq y^n.$$

Now, we are equipped to prove the corresponding uniqueness theorem. Let $(X,\preceq)$ be a partially ordered set. Equip the product X^n with the following partial order:

For all $(x^1,x^2,...,x^n),(y^1,y^2,...,y^n)\in X^n$

$(x^1,x^2,...,x^n)\preceq_r (y^1,y^2,...,y^n)$ if and only if $x^1\preceq y^1, x^2\succeq y^2, x^3\preceq y^3,...,x^n\succeq y^n$. We say that two elements $(x^1,x^2,...,x^n)$ and $(y^1,y^2,...,y^n)$ of X^n are comparable if

either $(x^1,x^2,...,x^n)\preceq_r (y^1,y^2,...,y^n)$ or $(x^1,x^2,...,x^n)\succeq_r (y^1,y^2,...,y^n)$.

Theorem 2 In addition to the hypotheses of Theorem 1, suppose that for all even n-tupled coincidence points $(x^1,x^2,...,x^n)$, $(y^1,y^2,...,y^n)\in X^n$ of F and g, there exists a $(u^1,u^2,...,u^n)\in X^n$ such that

$$(F(u^1,u^2,...,u^n),F(u^2,...,u^n,u^1),...,F(u^n,u^1,...,u^{n-1}))$$

is comparable to

$$(F(x^1,x^2,...,x^n),F(x^2,...,x^n,x^1),...,F(x^n,x^1,...,x^{n-1}))$$

and

$$(F(y^1,y^2,...,y^n),F(y^2,...,y^n,y^1),...,F(y^n,y^1,...,y^{n-1})).$$

Then F and g have a unique n-tupled common fixed point.

Proof In view of Theorem 1, the set of even n-tupled coincidence points is nonempty. If $(x^1,x^2,...,x^n)$, and $(y^1,y^2,...,y^n)$ are two n-tupled coincidence points, then

$$\begin{aligned} g(x^1)&=F(x^1,x^2,...,x^n),\\ g(x^2)&=F(x^2,x^3,...,x^n,x^1),\\ &\vdots\\ g(x^n)&=F(x^n,x^1,x^2,...,x^{n-1}). \end{aligned} \tag{13}$$

and

$$\begin{aligned} g(y^1)&=F(y^1,y^2,...,y^n),\\ g(y^2)&=F(y^2,y^3,...,y^n,y^1),\\ &\vdots\\ g(y^n)&=F(y^n,y^1,...,y^{n-1}). \end{aligned} \tag{14}$$

Now, we have to show that

$$g(x^1)=g(y^1), g(x^2)=g(y^2),...,g(x^n)=g(y^n). \tag{15}$$

By one of the assumptions, there exists $(u^1,u^2,...,u^n)\in X^n$ such that

$$(F(u^1,u^2,...,u^n),F(u^2,...,u^n,u^1),...,F(u^n,u^1,...,u^{n-1}))$$

is comparable with

$$(F(x^1,x^2,...,x^n),F(x^2,...,x^n,x^1),...,F(x^n,x^1,...,x^{n-1}))$$

and

$$(F(y^1, y^2, ..., y^n), F(y^2, ..., y^n, y^1), ..., F(y^n, y^1, ..., y^{n-1})).$$

Put $u_0^1 = u^1$, $u_0^2 = u^2, ..., u_0^n = u^n$. Since $F(X^n) \subseteq g(X)$, on the lines similar to the proof of Theorem 1, we can inductively define sequences $\{u_m^1\}$, $\{u_m^2\}$, $\{u_m^3\}$,..., $\{u_m^n\}$ such that

$$\begin{aligned} g(u_{m+1}^1) &= F(u_m^1, u_m^2, ..., u_m^n), \\ g(u_{m+1}^2) &= F(u_m^2, u_m^1, ..., u_m^n), \\ &\vdots \\ g(u_{m+1}^n) &= F(u_m^n, u_m^2, ..., u_m^1). \end{aligned} \tag{16}$$

Since
$(F(x^1, x^2, ..., x^n), F(x^2, ..., x^n, x^1), ..., F(x^n, x^1, ..., x^{n-1})) = (gx^1, gx^2, ..., gx^n)$
and
$(F(u^1, u^2, ..., u^n), F(u^2, ..., u^n, u^1), ..., F(u^n, u^1, ..., u^{n-1})) = (gu_1^1, gu_1^2, ..., gu_1^n)$
are comparable, therefore

$$g(x^1) \preceq g(u_1^1), g(x^2) \succeq g(u_1^2), \cdots, g(x^n) \succeq g(u_1^n).$$

Now for all $m \geq 1$, it can be easily seen that

$$\begin{aligned} g(x^1) &\preceq g(u_m^1) \preceq g(u_{m+1}^1) \\ g(x^2) &\succeq g(u_m^2) \succeq g(u_{m+1}^2) \\ &\vdots \\ g(x^n) &\succeq g(u_m^n) \succeq g(u_{m+1}^n). \end{aligned} \tag{17}$$

Denote

$$\gamma_m = d(gx^1, gu_m^1) + d(gx^2, gu_m^2) + ... + d(gx^n, gu_m^n) \qquad (18).$$

then we have to show that

$$\lim_{m\to\infty} \gamma_m = 0. \tag{19}$$

Now two cases arise. Firstly, suppose that $\gamma_{m_0} = 0$ for some $m_0 \in \mathbb{N}$ then $d(gx^1, gu_{m_0}^1) = d(gx^2, gu_{m_0}^2) = ... = d(gx^n, gu_{m_0}^n) = 0$. On using Lemma 2, we obtain

$$\begin{aligned} F(x^1, x^2, ..., x^n) &= F(u_{m_0}^1, u_{m_0}^2, ..., u_{m_0}^n) \\ F(x^2, ..., x^n, x^1) &= F(u_{m_0}^2, ..., u_{m_0}^n, u_{m_0}^1) \\ &\vdots \\ F(x^n, x^1, ..., x^{n-1}) &= F(u_{m_0}^n, u_{m_0}^1, ..., u_{m_0}^{n-1}) \end{aligned}$$

Consequently, by using (13) and (16), we get

$$\begin{aligned} d(gx^1, gu_{m_0+1}^1) &= d(F(x^1, x^2, ..., x^n), F(u_{m_0}^1, u_{m_0}^2, ..., u_{m_0}^n)) = 0 \\ d(gx^2, gu_{m_0+1}^2) &= d(F(x^2, ..., x^n, x^1), F(u_{m_0}^2, ..., u_{m_0}^n, u_{m_0}^1)) = 0 \\ &\vdots \\ d(gx^n, gu_{m_0+1}^n) &= d(F(x^n, x^1, ..., x^{n-1}), F(u_{m_0}^n, u_{m_0}^1, ..., u_{m_0}^{n-1})) = 0, \end{aligned}$$

which by using (18) implies that $\gamma_{m_0+1} = 0$. Thus by induction, we get $\gamma_m = 0 \ \forall \ m \geq m_0$, yielding thereby $\lim\limits_{n\to\infty} \gamma_m = 0$. Hence, in this case, (19) is proved.

On the other hand, suppose that $\gamma_m > 0$ for all $m \geq 1$. Then, by using (13), (16), (17), (2) and (18), we get

$$d(gx^1, gu^1_{m+1}) = d(F(x^1, x^2, ..., x^n), F(u^1_m, u^2_m, ..., u^n_m))$$
$$\leq \phi\Big(\frac{1}{n}\sum_{i=1}^{n} d(gx^i, gu^i_m)\Big) = \phi\Big(\frac{\gamma_m}{n}\Big).$$

Similarly, we can have

$$\begin{array}{rcl} d(gx^2, gu^2_{m+1}) & \leq & \phi\Big(\frac{\gamma_m}{n}\Big) \\ d(gx^3, gu^3_{m+1}) & \leq & \phi\Big(\frac{\gamma_m}{n}\Big) \\ & \vdots & \\ d(gx^n, gu^n_{m+1}) & \leq & \phi\Big(\frac{\gamma_m}{n}\Big). \end{array}$$

By adding above inequalities, we get

$$d(gx^1, gu^1_{m+1}) + d(gx^2, gu^2_{m+1}) + ... + d(gx^n, gu^n_{m+1}) \leq n\phi\Big(\frac{\gamma_m}{n}\Big),$$

so that

$$\gamma_{m+1} \leq n\phi\Big(\frac{\gamma_m}{n}\Big) \quad \text{for all } m \geq 1. \tag{20}$$

Since $\phi(t) < t$ for all $t > 0$, therefore $\gamma_{m+1} < n(\frac{\gamma_m}{n}) = \gamma_m$ for all $m \geq 1$ so that $\{\gamma_m\}$ is a decreasing sequence. Since it is bounded below (as $\gamma_m > 0$), there is some $\gamma \geq 0$ such that

$$\lim_{m\to\infty} \gamma_m = \gamma^+.$$

Now, we show that $\gamma = 0$. Suppose, on contrary that $\gamma > 0$. Taking limit on the both sides of (20) as $m \to \infty$ and keeping in mind our supposition that $\lim\limits_{r\to t^+} \phi(r) < t$ for all $t > 0$, we have

$$\gamma = \lim_{m\to\infty} \gamma_{m+1} \leq n \lim_{m\to\infty} \phi\Big(\frac{\gamma_m}{n}\Big) = n \lim_{\gamma_m \to\ \gamma^+} \phi\Big(\frac{\gamma_m}{n}\Big) < n\Big(\frac{\gamma}{n}\Big) = \gamma,$$

which is a contradiction so that $\gamma = 0$ yielding thereby $\lim\limits_{m\to\infty} \gamma_m = 0$. Hence, in both the cases, (19) hold. Consequently from (19), we have

$$\begin{array}{c} \lim\limits_{m\to\infty} d(g(x^1), g(u^1_m)) = 0, \\ \lim\limits_{m\to\infty} d(g(x^2), g(u^2_m)) = 0, \\ \vdots \\ \lim\limits_{m\to\infty} d(g(x^n), g(u^n_m)) = 0. \end{array} \tag{21}$$

Similarly one can prove that

$$\begin{array}{c} \lim\limits_{m\to\infty} d(g(y^1), g(u^1_m)) = 0, \\ \lim\limits_{m\to\infty} d(g(y^2), g(u^2_m)) = 0, \\ \vdots \\ \lim\limits_{m\to\infty} d(g(y^n), g(u^n_m)) = 0. \end{array} \tag{22}$$

By the triangle inequality, (21) and (22), we have

$$\begin{array}{rcl} d(g(x^1), g(y^1)) & \leq & d(g(x^1), g(u_m^1)) + d(g(u_m^1), g(y^1)) \to 0 \;\; as \;\; m \to \infty \\ d(g(x^2), g(y^2)) & \leq & d(g(x^2), g(u_m^2)) + d(g(u_m^2), g(y^2)) \to 0 \;\; as \;\; m \to \infty \\ & \vdots & \\ d(g(x^n), g(y^n)) & \leq & d(g(x^n), g(u_m^n)) + d(g(u_m^n), g(y^n)) \to 0 \;\; as \;\; m \to \infty. \end{array}$$

Hence $g(x^1) = g(y^1), ...,$ and $g(x^n) = g(y^n)$. Thus (15) is proved.
Since

$$\begin{array}{rcl} g(x^1) & = & F(x^1, x^2, ..., x^n), \\ g(x^2) & = & F(x^2, ..., x^n, x^1), \\ & \vdots & \\ g(x^n) & = & F(x^n, x^1, ..., , x^{n-1}), \end{array}$$

owing to commutativity of F and g, we have

$$\begin{array}{c} g(g(x^1)) = g(F(x^1, x^2, ..., x^n)) = F(g(x^1), g(x^2), ..., g(x^n)), \\ g(g(x^2)) = g(F(x^2, ..., x^n, x^1)) = F(g(x^2), ..., g(x^n), g(x^1)), \\ \vdots \\ g(g(x^n)) = g(F(x^n, x^1, ..., , x^{n-1})) = F(g(x^n), g(x^1), ..., g(x^{n-1})). \end{array} \tag{23}$$

Write $g(x^1) = z^1, g(x^2) = z^2, ..., g(x^n) = z^n$. Then from (23),

$$\begin{array}{c} g(z^1) = F(z^1, z^2, ..., z^n), \\ g(z^2) = F(z^2, z^1, ..., z^n), \\ \vdots \\ g(z^n) = F(z^n, z^1, ..., z^{n-1}). \end{array} \tag{24}$$

Thus $(z^1, z^2, ..., z^n)$ is a n-tupled coincidence point. Then owing to (15) with $y^1 = z^1, y^2 = z^2, ..., y^n = z^n$, it follows that $g(z^1) = g(x^1)$, $g(z^2) = g(x^2)$,... and $g(z^n) = g(x^n)$ *i.e.*

$$g(z^1) = z^1, g(z^2) = z^2, \cdots, g(z^n) = z^n. \tag{25}$$

Using (20) and (25), we have

$$\begin{array}{rcl} z^1 = & g(z^1) & = F(z^1, z^2, ..., z^n), \\ z^2 = & g(z^2) & = F(z^2, z^1, ..., z^n), \\ & \vdots & \\ z^n = & g(z^n) & = F(z^n, z^2, ..., z^1). \end{array}$$

Hence $(z^1, z^2, ..., z^n)$ is a n-tupled common fixed point of F and g. To prove the uniqueness, assume that $(w^1, w^2, ..., w^n)$ is another n-tupled common fixed point. Then by (20), we have $w^1 = g(w^1) = g(z^1) = z^1, w^2 = g(w^2) = g(z^2) = z^2, ...w^n = g(w^n) = g(z^n) = z^n$. This completes the proof.

The following example illustrates Theorem 1.
Example 2 Let $X = [0, 1]$. Then X is a complete asymptotically metric space under natural ordering $\preceq$ of real numbers and natural metric $d(x, y) = |x - y|$ for

all $x, y \in X$. Define $g : X \to X$ as $g(x) = \frac{x}{n-1}$ wherein n is fixed and $n > 1$ for all $x \in X$. Also, define $F : X^n \to X$ by

$$F(x^1, x^2, \ldots, x^n) = \frac{x^1 - x^2 + x^3 - x^4 + \ldots + x^{n-1} - x^n}{n^2 - 1},$$

for all $x^1, x^2, \ldots, x^n \in X$.
Define $\phi : [0, \infty) \to [0, \infty)$ as $\phi(t) = \frac{n}{n+1}t$ where n is fixed as earlier. Now,

$$\begin{aligned}
& d(F(x^1, x^2, \ldots, x^n), F(y^1, y^2, \ldots, y^n)) \\
&= d\left(\tfrac{x^1-x^2+x^3-x^4+\ldots+x^{n-1}-x^n}{n^2-1}, \tfrac{y^1-y^2+y^3-y^4+\ldots+y^{n-1}-y^n}{n^2-1}\right) \\
&= \tfrac{1}{n+1}\left|\tfrac{x^1-x^2+x^3-x^4+\ldots+x^{n-1}-x^n}{n-1} - \tfrac{y^1-y^2+y^3-y^4+\ldots+y^{n-1}-y^n}{n-1}\right| \\
&\le \tfrac{n}{n+1}\tfrac{1}{n}\left\{\tfrac{|x^1-y^1|+|x^2-y^2|+\ldots+|x^n-y^n|}{n-1}\right\} \\
&= \tfrac{n}{n+1}\tfrac{1}{n}\Big(d(gx^1, gy^1) + d(gx^2, gy^2) + \ldots + d(gx^n, gy^n)\Big) \\
&= \phi\Big(\tfrac{1}{n}\sum_{i=1}^{n} d(gx^i, gy^i)\Big).
\end{aligned}$$

Thus all the conditions of Theorem 1 are satisfied and $(0, 0, \ldots, 0)$ is an n-tupled coincidence point of F and g.

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