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APPROXIMATE SOLUTIONS OF FRACTIONAL WAVE EQUATIONS USING VARIATIONAL ITERATION METHOD AND LAPLACE TRANSFORM

YANQIN LIU, XIULING YIN, LINLIN ZHAO

Abstract. A relatively novel modification of the variational iteration method, by means of the Laplace transform, is applied to obtain an approximate solution of fractional wake-like equations with variable coefficients. The fractional derivatives described in this paper are in the Caputo sense. It is observed that the approach is a reliable tool to analytically investigating wave models with fractional derivatives and can be implemented to other fractional models.

1. Introduction

The application of fractional calculus is a hot topic in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science[1, 2, 3]. With the development of nonlinear sciences, many analytical and numerical techniques[4, 5, 6, 7, 8, 9, 10, 11] have been developed by various scientists. But these fractional differential equations are difficult to get their exact solutions[12, 13, 14]. So, the numerical methods have largely been used to solve these equations. Most of these methods have their inbuilt deficiencies like the calculation of Adomian's polynomials, the Lagrange multiplier, divergent results, and huge computational work. Recently, some improved homotopy perturbation methods[15, 16]and improved variational iteration method[17, 18] have been used by many researches.

The variational iteration method(VIM)[5, 6, 7] was extended to initial value problems of differential equations and become a widely used method. Yulita[19] applied this method to obtain analytical solutions of fractional heat- and wave-like equations and the chose Lagrange multiplier as -1. The key problem of the VIM is the correct determination of the Lagrange multiplier when the method is applied to fractional equations, combined with the Laplace transform, the crucial point of this method is solved efficiently by Wu[20, 21]. Laplace transform overcoms principle drawbacks in application of the VIM to fractional equations.

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The main objective of this paper is to extend this new modified method to solve variable coefficient and inhomogeneous space and time fractional wake equations. And the fractional derivatives are described in the Caputo sense.

2. Preliminaries

The Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$ {}_0^C D_t^{\alpha} u(x,t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} \mathrm{d}\tau, \qquad m = [\alpha]+1, m \in N. \tag{1} $$

where $\Gamma(\cdot)$ denotes the Gamma function.

The Caputo space-fractional derivative operator of order $\beta > 0$ is defined as

$$ {}_0^C D_x^{\beta} u(x,t) = \frac{1}{\Gamma(m-\beta)} \int_0^x (x-\xi)^{m-\beta-1} \frac{\partial^m u(\xi,t)}{\partial \xi^m} \mathrm{d}\xi, \qquad m = [\beta]+1, m \in N. \tag{2} $$

Laplace transform of ${}_0^C D_t^{\alpha} u$ is given as

$$ L[{}_0^C D_t^{\alpha} u(x,t)] = s^{\alpha} U(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k}, \qquad m-1 < \alpha \le m, \tag{3} $$

where $U(x,s) = L[u(x,t)] = \int_0^{\infty} e^{-st} u(x,t) dt$. And further information about fractional derivatives and its properties can be found in [1, 2].

3. Description of the method

Let us consider the time fractional equation as follows:

$$ {}_0^C D_t^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{4} $$

$$ u^{(k)}(x,0^+) = a_k, \qquad t > 0, \alpha > 0, m = [\alpha]+1, k = 0, \cdots, m-1 \tag{5} $$

where $g(x,t)$ is the source term, N represents the general nonlinear differential operator and R is the linear differential operator. Now, we consider the application of the modified VIM[20, 21]. Taking the above Laplace transform to both sides of Equation (4) and (5), the iteration formula of Eq.(4) can be constructed as

$$ \begin{aligned} U_{n+1}(x,s) = U_n(x,s) + \lambda(s)[s^{\alpha} U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k} \\ + L[R[u_n(x,t)] + N[u_n(x,t)] - g(x,t)]], \end{aligned} \tag{6} $$

Considering $L[R[u_n(x,t)] + N[u_n(x,t)]]$ as restricted terms, one can derive a Lagrange multiplier as

$$ \lambda = -1/s^{\alpha}, \tag{7} $$

With Equation (7) and the inverse-laplace transform L^{-1}, the iteration formula (6) can be explicitly given as

$$ \begin{aligned} u_{n+1}(x,t) &= u_n(x,t) - L^{-1}[\frac{1}{s^{\alpha}}[s^{\alpha} U_n(x,s) - \sum_{k=0}^{m-1} u^k(x,0^+) s^{\alpha-1-k} \\ &\quad + L[R[u_n(x,t)] + N[u_n(x,t)] - g(x,t)]]] \\ &= u_0(x,t) - L^{-1}[\frac{1}{s^{\alpha}}[L[R[u_n(x,t)] + N[u_n(x,t)]]]], \end{aligned} \tag{8} $$

$u_0(x,t)$ is an initial approximation of Eq.(4), and

$$u_0(x,t) = L^{-1}(\sum_{k=0}^{m-1} u^k(x,0^+)s^{\alpha-1-k}) + L^{-1}[\frac{1}{s^\alpha}L[g(x,t)]]$$

$$= u(x,0) + u'(x,0)t + \cdots + \frac{u^{m-1}(x,0)t^{m-1}}{(m-1)!} + L^{-1}[\frac{1}{s^\alpha}L[g(x,t)]], \tag{9}$$

then the approximations $u_n(x,t)$ can be completely determined. Finally, the approximate solution is

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{10}$$

4. Applications

We will apply the new modified VIM to the following fractional wave equations, wave equation is a kind of important evolution equation, it plays an important role in describe the vibration of thin film, the spread of electromagnetic waves or sound waves in space. All the results in this paper are calculated by using the Mathematica symbolic computation software.

Example 1: Consider the following one-dimensional linear inhomogeneous fractional wave equation[22]

$${}_0^C D_t^\alpha u(x,t) + \frac{\partial u}{\partial x} = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) + t\cos(x), \qquad t>0, x\in R, 0<\alpha\le 1. \tag{11}$$

$$u(x,0) = 0. \tag{12}$$

After taking the Laplace transform to both sides of Equation (11) and (12), we get the following iteration formula:

$$U_{n+1}(x,s) = U_n(x,s) + \lambda(s)[s^\alpha U_n(x,s) - s^{\alpha-1}u(x,0)$$

$$+ L[\frac{\partial u_n(x,t)}{\partial x} - \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) - t\cos(x)]], \tag{13}$$

Considering $L[\frac{\partial u_n(x,t)}{\partial x}]$ as restricted terms, Lagrange multiplier can be defined as $\lambda(s) = -1/s^\alpha$, with the inverse-Laplace transform, the approximate solution of Equation (11) can be given as

$$u_{n+1}(x,t) = L^{-1}[\frac{1}{s^\alpha}[L[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) - t\cos(x)]]] - L^{-1}[\frac{1}{s^\alpha}[L[\frac{\partial u_n(x,t)}{\partial x}]]], \tag{14}$$

which reads

$$u_0(x,t) = L^{-1}[\frac{1}{s^\alpha}[L[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin(x) - t\cos(x)]]] = t\sin(t) + \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\cos(x), \tag{15}$$

$$u_1 = -L^{-1}[\frac{1}{s^\alpha}[L[\frac{\partial u_0(x,t)}{\partial x}]]]$$

$$= t\sin(t) + \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\cos(x) - \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x), \tag{16}$$

$$u_2 = -L^{-1}[\frac{1}{s^\alpha}[L[\frac{\partial u_1(x,t)}{\partial x}]]]$$

$$= t\sin(t) + \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\cos(x) - \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}\cos(x) + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x)$$

$$-\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\sin(x)-\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\cos(x), \tag{17}$$
$$\vdots$$

Canceling the noise terms and keeping the non-noise terms yield the exact solution $u(x,t)=t\sin(x)$ of Equation (11).

Example 2: Consider the following two-dimensional space-time fractional initial boundary value problem which describes the wave-like models

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha}=\frac{1}{12}(x^2\frac{\partial^\beta u(x,y,t)}{\partial x^\beta}+y^2\frac{\partial^\gamma u(x,y,t)}{\partial y^\gamma}),\qquad 0<x,y<1,t>0 \tag{18}$$

subject to the Neumann boundary conditions

$$u_x(0,y,t)=0,\qquad u_x(1,y,t)=4\cosh(t), \tag{19}$$
$$u_y(x,0,t)=0,\qquad u_y(x,1,t)=4\sinh(t), \tag{20}$$

and initial conditions

$$u(x,y,0)=x^4,\qquad u_t(x,y,0)=y^4. \tag{21}$$

the time-space fractional derivative defined here is in the Caputo sense, and $1<\alpha,\beta,\gamma\le 2$.

After taking the Laplace transform to both sides of Equation (18) and (21), we get the following iteration formula:

$$\begin{aligned}U_{n+1}(x,y,s)&=U_n(x,y,s)+\lambda(s)[s^\alpha U_n(x,y,s)-s^{\alpha-1}u_n(x,y,0)-s^{\alpha-2}u_{nt}(x,y,0)\\&-\frac{1}{12}L[x^2\frac{\partial^\beta u_n(x,y,t)}{\partial x^\beta}+y^2\frac{\partial^\gamma u_n(x,y,t)}{\partial y^\gamma}]],\end{aligned} \tag{22}$$

Considering $L[x^2\frac{\partial^\beta u_n(x,y,t)}{\partial x^\beta}+y^2\frac{\partial^\gamma u_n(x,y,t)}{\partial y^\gamma}]$ as restricted terms, Lagrange multiplier can be defined as $\lambda(s)=-1/s^\alpha$, with the inverse-Laplace transform, the approximate solution of Equation (18) can be given as

$$u_{n+1}(x,y,t)=u_0(x,y,t)+\frac{1}{12}L^{-1}[\frac{1}{s^\alpha}[L[x^2\frac{\partial^\beta u_n(x,y,t)}{\partial x^\beta}+y^2\frac{\partial^\gamma u_n(x,y,t)}{\partial y^\gamma}]]], \tag{23}$$

where $u_0(x,y,t)$ is an initial approximation of Equation (18), and

$$u_0(x,y,t)=u(x,y,0)+tu_t(x,y,0)=x^4+ty^4, \tag{24}$$

by the present VIM, we have the following solutions:

$$u_1(x,y,t)=x^4+ty^4+\frac{t^\alpha x^{6-\beta}\Gamma(5)}{12\Gamma(1+\alpha)\Gamma(5-\beta)}+\frac{t^{1+\alpha}y^{6-\gamma}\Gamma(5)}{12\Gamma(2+\alpha)\Gamma(5-\gamma)} \tag{25}$$

$$\begin{aligned}u_2(x,y,t)&=x^4+ty^4+\frac{t^\alpha x^{6-\beta}\Gamma(5)}{12\Gamma(1+\alpha)\Gamma(5-\beta)}+\frac{t^{1+\alpha}y^{6-\gamma}\Gamma(5)}{12\Gamma(2+\alpha)\Gamma(5-\gamma)}\\&+\frac{t^{2\alpha}x^{8-2\beta}\Gamma(5)\Gamma(7-\beta)}{144\Gamma(1+2\alpha)\Gamma(7-2\beta)}+\frac{t^{1+2\alpha}y^{8-2\gamma}\Gamma(5)\Gamma(7-\gamma)}{144\Gamma(2+2\alpha)\Gamma(7-2\gamma)},\end{aligned} \tag{26}$$
$$\vdots$$

for $n\to\infty$, the iteration solution will be given in the following formula

$$u(x,y,t)=(x^4+\frac{t^\alpha x^{6-\beta}\Gamma(5)}{12\Gamma(1+\alpha)\Gamma(5-\beta)}+\frac{t^{2\alpha}x^{8-2\beta}\Gamma(5)\Gamma(7-\beta)}{144\Gamma(1+2\alpha)\Gamma(7-2\beta)}+\cdots)$$

$$+ (ty^4 + \frac{t^{1+\alpha}y^{6-\gamma}\Gamma(5)}{12\Gamma(2+\alpha)\Gamma(5-\gamma)} + \frac{t^{1+2\alpha}y^{8-2\gamma}\Gamma(5)\Gamma(7-\gamma)}{144\Gamma(2+2\alpha)\Gamma(7-2\gamma)} + \cdots) \quad (27)$$

when the fractional derivative $\alpha = \beta = \gamma = 2$, the solution Equation (27) comes to

$$u(x,y,t) = x^4(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots) + y^4(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots) \quad (28)$$

and in a close form by

$$u(x,y,t) = x^4\cosh(t) + y^4\sinh(t), \quad (29)$$

which is in full agreement with the result given in[23].

Example 3: Consider the following three-dimensional inhomogeneous time fractional initial boundary value problem which describes the wave-like models

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x^2+y^2+z^2+\frac{1}{2}(x^2u_{xx}+y^2u_{yy}+z^2u_{zz}), \qquad 0 < x,y,z < 1, t > 0, 1 < \alpha \le 2 \quad (30)$$

subject to the boundary conditions

$$u(0,y,z,t) = y^2(e^t-1)+z^2(e^{-t}-1), \quad u(1,y,z,t) = (1+y^2)(e^t-1)+z^2(e^{-t}-1), \quad (31)$$

$$u(x,0,z,t) = x^2(e^t-1)+z^2(e^{-t}-1), \quad u(x,1,z,t) = (1+x^2)(e^t-1)+z^2(e^{-t}-1), \quad (32)$$

$$u(x,y,0,t) = (x^2+y^2)(e^t-1), \quad u(x,y,1,t) = (x^2+y^2)(e^t-1)+(e^{-t}-1), \quad (33)$$

and the initial conditions

$$u(x,y,z,0) = 0, \qquad u_t(x,y,z,0) = x^2+y^2-z^2. \quad (34)$$

After taking the Laplace transform to both sides of Equation (30) and (34), we get the following iteration formula:

$$U_{n+1} = U_n(x,y,z,s) + \lambda(s)[s^\alpha U_n - s^{\alpha-1}u_n(x,y,z,0) - s^{\alpha-2}u_{nt}(x,y,z,0)$$
$$- L[x^2+y^2+z^2] - \frac{1}{2}L[x^2\frac{\partial^2 u_n(x,y,z,t)}{\partial x^2} + y^2\frac{\partial^2 u_n(x,y,z,t)}{\partial y^2} + z^2\frac{\partial^2 u_n(x,y,z,t)}{\partial z^2}]], \quad (35)$$

Considering $L[x^2\frac{\partial^2 u_n(x,y,z,t)}{\partial x^2}+y^2\frac{\partial^2 u_n(x,y,z,t)}{\partial y^2}+z^2\frac{\partial^2 u_n(x,y,z,t)}{\partial z^2}]$ as restricted terms, Lagrange multiplier can be defined as $\lambda(s) = -1/s^\alpha$, with the inverse-Laplace transform, the approximate solution of Equation (30) can be given as

$$u_{n+1} = u_0+\frac{1}{2}L^{-1}[\frac{1}{s^\alpha}[L[x^2\frac{\partial^2 u_n(x,y,z,t)}{\partial x^2}+y^2\frac{\partial^2 u_n(x,y,z,t)}{\partial y^2}+z^2\frac{\partial^2 u_n(x,y,z,t)}{\partial z^2}]]], \quad (36)$$

where u_0 is an initial approximation of Equation (30), and

$$u_0 = t(x^2+y^2-z^2) + \frac{t^\alpha x^2}{\Gamma(1+\alpha)} + \frac{t^\alpha y^2}{\Gamma(1+\alpha)} + \frac{t^\alpha z^2}{\Gamma(1+\alpha)}, \quad (37)$$

by the present VIM, we have the following solutions:

$$u_1 = (x^2+y^2)(t + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}) + z^2(-t + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}) \quad (38)$$

$$\vdots$$

for $n \to \infty$, the iteration solution will be given in the following formula

$$u(x,y,z,t) = (x^2+y^2)(t+\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}+\cdots)+z^2(-t+\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{1+\alpha}}{\Gamma(2+\alpha)}+\cdots) \tag{39}$$

when the fractional derivative $\alpha = 2$, the solution Equation (39) comes to

$$u(x,y,z,t) = (x^2+y^2)(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots)+z^2(-t+\frac{t^2}{2!}+\frac{t^3}{3!}+\cdots) \tag{40}$$

and in a close form by

$$u(x,y,z,t) = (x^2+y^2)e^t + z^2e^{-t} - (x^2+y^2+z^2), \tag{41}$$

which was given in[23].

5. Conclusion

In this letter, we implement relatively new analytical techniques, a modified variational iteration method to solve fractional equations. The key problem of the VIM is the correct determination of the Lagrange multiplier when the method is applied to fractional equations, to the best of our knowledge, there is no effective method to identify the Lagrange multipliers, by using the Laplace transform, we can easily derive Lagrange multipliers without tedious calculation and new variational iteration formulae can be derived. some inhomogeneous fractional wave equations with variable coefficients and the Caputo derivatives are illustrated, The results show the modified method is efficiency compared with other versions of the VIM in fractional calculus. Unlike the ADM, the modified VIM is free from the need to use Adomian polynomials. In contrast to homotopy analysis method, its efficiency is very much depended on choosing auxiliary parameter. And this modified VIM can also be used to solve the fractional equations of Riemann-Liouville type.

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References

[1] I. Podlubny, Fractional differential equations, Academic Press, 1999.
[2] R. Hilfer, Applications of fractional calculus in physics, World scientific, 2000.
[3] R. Metzler and J. Klafter, The random walk??s guide to anomalous diffusion: a fractional dynamics approach, Phys. Rep. Vol., 339, 1-7, 2000.
[4] A.M. Wazwaz and S. M. El-Sayed, A new modification of the Adomian decomposition method for linear and nonlinear operators, Appl.Math. Comput., Vol. 122, 393-405, 2001.
[5] J.H. He, Variational iteration method- a kind of nonlinear analytical technique: some examples, Int. J. Nonlin. Mech., Vol 34, 609-708, 1999.
[6] A.M. Wazwaz, The variational iteration method for analytic treatment for linear and nonlinear ODEs, Appl. Math. Comput., Vol 212, 120-134, 2009.
[7] J.H. He,variational iteration method-some recent results and new interpretations, ,J. Comput. Appl. Math., Vol 207, 3-17, 2007.
[8] V. S. Erturk, S. Momani and Z. Odibat: Application of generalized differential transfom method to multi-order fractional differential equations, Commun. Nonlinear. Sci., Vol 13, 1642-1654, 2008.
[9] M. Madani, Yasir Khan and Gh. Mahmodi et al, Application of homotopy perturbation and numerical methods to the circular porous slider, Int. J. Num. Meth. Heat & Fl. Flow, Vol 22, 705-717, 2012.

[10] P.K. Gupta, A. Yildirim and K. N. Rai, Application of He's homotopy perturbation method for multi-dimensional fractional Helmholtz equation, Int. J. Num. Meth. Heat & Fl. Flow, Vol 22, 424-435, 2012.
[11] Z. Odibat and S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, Chaos Soliton Fract., Vol 36, 167-174, 2008.
[12] X.Y. Jiang and M.Y. Xu, Analysis of fractional anomalous diffusion caused by an instantaneous point source in disordered fractal media, Int. J. Nonlinear Mech., Vol 41, 156-165, 2006.
[13] J.H. Ma and Y.Q. Liu, Exact solutions for a generalized nonlinear fractional Fokker-Planck equation, Nonlinear Anal: Real World Appl., Vol 11, 515-521, 2010.
[14] Y.Q. Liu and J.H. Ma, Exact solutions of a generalized multi-fractional nonlinear diffusion equation in radial symmetry, ommun. Theor. Phys., Vol 52, 857-861, 2009.
[15] Y.Q. Liu:, Approximate solutions of fractional nonlinear equations using homotopy perturbation transformation method, Abstr. Appl. Anal., Vol 2012, Article ID 752869, 14 pages.
[16] Y.Q. Liu: Study on space-time fractional nonlinear biological equation in radial symmetry, Math. Probl. Eng., Vol 2013, Article ID 654759, 6 pages.
[17] Y.Q. Liu: Variational homotopy perturbation method for solving fractional initial boundary value problems, Abstr. Appl. Anal., Vol 2012, Article ID 727031, 10 pages.
[18] S.M. Guo, L.Q. Mei and Y. Li: Fractional variational homotopy perturbation iteration method and its application to a fractional diffusion equation, Appl. Math. Comput., Vol 219, 5909-5917,2013.
[19] R. Molliq Yulita, M.S.M. Noorani and I.Hashim: Variational iteration method for fractional heat-and wave equations, Nonlinear Anal: Real World Appl., Vol 10, 1854-1869, 2009.
[20] G.C. Wu and D. Baleanu: Vatiational iteration method for fractional calculus-a universal approach by Laplace tranform, Adv. Difference Equa., Vol 18, 1-9, 2013.
[21] G.C. Wu: Laplace tranform overcoming principle drawbacks in applicaiton of the variation iteration method to fractional heat equations, Ther. Sci., Vol 16, 1257-1261, 2012.
[22] O. Zaid, M. Shaher: The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics, Comput. Math. Appl., Vol 58, 2199-2208, 2009.
[23] M. A. Noor and S. T. Mohyud-Din, Modified variational iteration method for heat and wave-like equaitons, Acta. Appl. Math., Vol 104, 257-269, 2008.

Yanqin Liu
College of Mathematical Sciences, Dezhou University, Dezhou, China
E-mail address: `yanqinliu@dzu.edu.cn`

Xiuling Yin
College of Mathematical Sciences, Dezhou University, Dezhou, China
E-mail address: `yinxiuling@dzu.edu.cn`

Linlin Zhao
College of Mathematical Sciences, Dezhou University, Dezhou, China
E-mail address: `zhaolinlin0635@163.com`