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GLOBAL ANALYSIS OF A NON-AUTONOMOUS DIFFERENCE EQUATION WITH BOUNDED COEFFICIENT

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Abstract. In this paper, we investigate the boundedness character and the global behavior of positive solutions of the following non-autonomous difference equation.

$$x_{n+1} = A_n + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \cdots ,$$

where $k \in \mathbb{N}$ and $\{A_n\}$ is a bounded sequence of non-negative real numbers and the initial conditions $x_{-k}, \cdots, x_0$ are arbitrary positive real numbers.

1. Introduction

Difference equations, also referred to recursive sequence, is a hot topic. There has been an increasing interest in the study of qualitative analysis of difference equations and systems of difference equations. For example, see $[1-25]$ and the references cited therein. Difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, economics, physics, computer sciences and so on.

This paper studies the boundedness character and the global asymptotic behavior of positive solutions of the non-autonomous difference equation

$$x_{n+1} = A_n + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \cdots , \tag{1.1}$$

where $k \in \mathbb{N}$, the initial conditions $x_{-k}, \cdots, x_0$ are arbitrary positive numbers and $\{A_n\}$ is a positive bounded sequence of non-negative real numbers with

$$\liminf_{n\to\infty} A_n = p \geq 0 \text{ and } \limsup_{n\to\infty} A_n = q < \infty. \tag{1.2}$$

Eq.(1.1) was studied by many authors with $k = 1$.

In [15], [16] and [23] the authors independently studied the asymptotic behavior of positive solutions of the following difference equation

$$x_{n+1} = p_n + \frac{x_{n-1}}{x_n}, \quad n = 0, 1, \cdots , \tag{1.3}$$

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where $\{p_n\}$ is a two-periodic sequence. For the boundedness results, Kulenović et al. [16] and Stević [23] used nearly similar proofs. However, in order to obtain global attracting results of Eq.(1.3) these authors used different techniques in proving their results. For the proof by Kulenović et al. see [Theorem1, 16]. Stević, on the other hand, used the monotonicity of the $\{x_{2n}\}$ and $\{x_{2n+1}\}$ in his proof.

In [21] Papaschinopoulos et al. obtained analogous results for the difference equation (1.3), where $\{p_n\}$ is a three-periodic sequence and the initial conditions are positive.

In [24] Stević studied Eq.(1.3), where $\{p_n\}$ is a sequence of non-negative real numbers which converges to $p \geq 0$; and in [4] Devault et al. studied Eq.(1.1), where $\{p_n\}$ is a positive bounded sequence.

In [22] Papaschinopoulos et al. investigated the boundedness, the periodicity, the attractivity and the global asymptotic stability of positive solutions of Eq.(1.1) where k is an odd number, A_n is $(k+1)$-periodic sequence and the initial conditions are positive.

Recently, in [17], the author has studied Eq.(1.1) for the case $\{p_n\}$ is a two-periodic sequence.

Our goal in this paper is to extend some results obtained in [4] and improve the conditions of the results concerning the boundedness and the global behavior of positive solutions.

For the autonomous cases of Eq.(1.1) and Eq.(1.3), we can refer the reader to [2, 3] and [1] respectively.

2. Boundedness Character of Eq. (1.1)

In this section, we investigate the boundedness character of Eq. (1.1), assuming that Eq.(1.2) is satisfied.

The autonomous case of Eq.(1.1),

$$x_{n+1} = A + \frac{x_{n-k}}{x_n}, \quad n = 0, 1, \ldots.$$

where $A > 0$, has been thoroughly studied in [3].

The following lemma is given in [10] which will be useful in analysis of the boundedness character of solutions of Eq.(1.1).

Lemma 1. *Assume that all the roots of the polynomial*

$$P(t) = t^N - s_1 t^{N-1}, \ldots, s_N$$

where $s_1, s_2, \ldots, s_N \geq 0$ for $n = 0, 1, \ldots,$ have absolute value less than 1. *If $\{x_n\}$ is a non-negative solution of the inequality*

$$x_{n+N} \leq s_1 x_{n+N-1} + \ldots + s_N x_n + y_n$$

where $y_n \geq 0$ for $n = 0, 1, \ldots,$ then the following statements are true:

(i) If $\sum_{n=0}^{\infty} y_n$ coverges, then $\sum_{n=0}^{\infty} x_n$ coverges.

(ii) If $\{y_n\}$ is bounded, then $\{x_n\}$ is bounded.

(iii) If $\lim_{n\to\infty} y_n = 0$, then $\lim_{n\to\infty} x_n = 0$.

We now present the following results about the boundedness character of Eq.(1.1).

Lemma 2. *Consider Eq.*(1.1) *and suppose that $k \in \mathbb{N}$. Assume that* (1.2) *is satisfied and $\{x_n\}$ be a solution of Eq.*(1.1). *Then the following statements are true:*

(i) If $p>0$, then $\{x_n\}$ persists.
(ii) If $p>1$, then $\{x_n\}$ is bounded.

Proof. (i) Since $x_{n+1}=A_n+\frac{x_{n-k}}{x_n}>A_n$, we have $\liminf_{n\to\infty} x_n \geq \liminf_{n\to\infty} A_n = p>0$ which completes the proof of part (i).

(ii) Let $\varepsilon>0$, such that $p-\varepsilon>1$, then for sufficiently large n

$$x_n \geq A_{n-1} \geq p-\varepsilon \quad \text{and} \quad x_{n+1} \leq A_n + \frac{x_{n-k}}{p-\varepsilon}.$$

Since $\{A_n\}$ is bounded and from Lemma 1, it follows that $\{x_n\}$ is also bounded. □

The following result is essentially proved in [10] for $k=1$. It is clear that the result is satisfied when k is odd and its proof will be omitted.

Lemma 3. *Consider Eq.*(1.1) *and suppose that k is odd. Then the following statements are true.*

(i) *Suppose that there exists $0<b<1$ such that $0<A_{2n+1}\leq b$. Choose*

$$x_{-k},\ x_{-k+2},...,x_{-1} > \frac{1}{(1-b)}$$

and

$$0<x_{-k+1},\ x_{-k+3},...,x_0<1.$$

Then

$$x_{2n-1} > \frac{1}{(1-b)} \quad \textit{and} \quad 0<x_{2n}<1 \quad \textit{for all } n\geq 0.$$

(ii) *Suppose that there exists $0<b<1$ such that $0<A_{2n}\leq b$. Choose*

$$x_{-k+1},\ x_{-k+3},...,x_0 > \frac{1}{(1-b)}$$

and

$$0<x_{-k},\ x_{-k+2},...,x_{-1}<1.$$

Then

$$x_{2n} > \frac{1}{(1-b)} \quad \textit{and} \quad 0<x_{2n-1}<1 \quad \textit{for all } n\geq 0.$$

The following result, when k is odd, demonstrates the existence of unbounded solutions of Eq.(1.1).

Lemma 4. *Consider Eq.*(1.1) *when k is odd. Suppose that either*

$$0<A_{2n+1}<1 \quad \textit{and} \quad \lim_{n\to\infty} A_{2n+1}=0 \quad \textit{or} \quad 0<A_{2n}<1 \quad \textit{and} \quad \lim_{n\to\infty} A_{2n}=0.$$

Then, there exists positive solutions of Eq. (1.1) *that are unbounded.*

Theorem 1. *Consider Eq.*(1.1) *and suppose that k is odd. Suppose that $0<A_{2n}<1$ and there exists $0<b<1$ such that either*

$$A_{2n+1}\leq b \quad \textit{or } A_{2n}\leq b.$$

Then, there exists positive solutions of Eq. (1.1) *that are unbounded.*

3. Global Attractivity of Eq.(1.1)

In this section, we study the global attractivity of positive solutions of Eq.(1.1). Let $\{\overline{x}_n\}$ be an arbitrary positive solution of Eq.(1.1). We will find sufficient conditions such that $\{\overline{x}_n\}$ attracts all positive solutions of Eq.(1.1).

We define the sequence $\{y_n\}$ to be

$$y_n = \frac{x_n}{\overline{x}_n}, \quad n = -k, ..., 0, 1, ... \tag{3.1}$$

Then, Eq.(1.1) reduces

$$\overline{x}_{n+1} y_{n+1} = A_n + \frac{\overline{x}_{n-k} y_{n-k}}{\overline{x}_n y_n}$$

or

$$y_{n+1} = \frac{A_n + \frac{\overline{x}_{n-k} y_{n-k}}{\overline{x}_n y_n}}{A_n + \frac{\overline{x}_{n-k}}{\overline{x}_n}}. \tag{3.2}$$

To prove the global attractivity result of Eq.(1.1), we need the following lemmas.

Lemma 5. *We assume that* $\liminf\limits_{n\to\infty} A_n = p > 1$. *Let* $\{x_n\}$ *be a solution of Eq.*(1.1), *if*

$$\lambda = \liminf_{n\to\infty} x_n \text{ and } \mu = \limsup_{n\to\infty} x_n, \tag{3.3}$$

then

$$\frac{\mu}{\lambda} \le \frac{(q-1)}{(p-1)}. \tag{3.4}$$

Proof. Using (1.2), (3.3) and Eq.(1.1) we obtain

$$\lambda \ge p + \frac{\lambda}{\mu} \text{ and } \mu \le q + \frac{\mu}{\lambda}$$

and

$$\lambda\mu \ge p\mu + \lambda \text{ and } \mu\lambda \le q\lambda + \mu.$$

So, we have

$$p\mu + \lambda \le q\lambda + \mu,$$

thus,

$$\mu(p-1) \le \lambda(q-1)$$

and so relation (3.4) is true. □

Lemma 6. *Let* $\{\overline{x}_n\}$ *be a positive solution of Eq.*(1.1). *The following statements are true.*

(*i*) *Eq.*(3.2) *has a positive equilibrium solution* $\overline{y} = 1$.

(*ii*) *If* $y_{n-k} < y_n$ *for some* n, *then* $y_{n+1} < 1$. *Similarly, if* $y_{n-k} \ge y_n$ *for some* n, *then* $y_{n+1} \ge 1$.

(*iii*) *Let* $\{y_n\}$ *be a solution to Eq.*(3.2). *Then, either* $\{y_n\}$ *consists of a single semicycle or* $\{y_n\}$ *oscillates about the equilibrium* $\overline{y} = 1$ *with semicycles having at most* k *terms.*

Proof. (*i*) It is clear from the equilibrium definition.

(*ii*) Let be $y_{n-k} < y_n$. Then, $(y_{n-k})/y_n < 1$ and

$$y_{n+1} = \frac{A_n + \frac{\overline{x}_{n-k} y_{n-k}}{\overline{x}_n y_n}}{A_n + \frac{\overline{x}_{n-k}}{\overline{x}_n}} < \frac{A_n + \frac{\overline{x}_{n-k}}{\overline{x}_n}}{A_n + \frac{\overline{x}_{n-k}}{\overline{x}_n}} = 1.$$

The other case is similar and will be omitted.

(*iii*) Let $\{y_n\}$ be an eventually oscillatory solution of Eq.(3.2) such that the positive semicycle beginning with the term y_{n+1} has k terms. Then, $y_n < 1 \leq y_{n+k}$ and so, from part (*ii*) it follows that $y_{n+k+1} < 1$. Therefore, the positive semicycle has exactly at most k terms. The proof for the negative semicycle is similar and will be omitted. □

Theorem 2. *Every non-oscillatory solution of Eq.*(3.2) *converges to* 1.

Proof. Let $\{y_n\}$ be a non-oscillatory solution of Eq.(3.2). Without loss of generality, we may assume that $y_n < 1$ for $n \geq N_0$. Thus, we have $y_{n+1} > y_{n+1-k}$ for $n \geq N$. Otherwise, there exists $l > N$ such that $y_l \leq y_{l-k}$, and by Lemma 6 (*ii*), it follows that $y_{l+1} \geq 1$, which is impossible. Hence, $\lim_{m\to\infty} y_{mk+i}$ exists for each $i \in \{0, 1, ..., k-1\}$. Let

$$\lim_{m\to\infty} y_{mk+i} = \alpha_i \text{ for } i = 0, 1, ..., k-1.$$

Clearly, $0 < \alpha_i \leq 1$ for $i = 0, 1, ..., k-1$. We must show that $\alpha_i = 1$ for $i = 0, 1, ..., k-1$. Without lossing of generality, since for $i = 0$

$$\lim_{m\to\infty} \frac{y_{mk-1}}{y_{(m+1)k-1}} = 1$$

for $\varepsilon > 0$ and m sufficiently large, we have

$$\left| \frac{y_{mk-1}}{y_{(m+1)k-1}} - 1 \right| < \varepsilon.$$

Thus,

$$\begin{aligned}
\left| y_{(m+1)k} - 1 \right| &= \left| \frac{A_{(m+1)k-1} + \frac{\overline{x}_{mk-1} y_{mk-1}}{\overline{x}_{(m+1)k-1} y_{(m+1)k-1}}}{A_{(m+1)k-1} + \frac{\overline{x}_{mk-1}}{\overline{x}_{(m+1)k-1}}} - 1 \right| \\
&= \left| \frac{\frac{\overline{x}_{mk-1}}{\overline{x}_{(m+1)k-1}}}{A_{(m+1)k-1} + \frac{\overline{x}_{mk-1}}{\overline{x}_{(m+1)k-1}}} \right| \left| \frac{y_{mk-1}}{y_{(m+1)k-1}} - 1 \right| \\
&< \left| \frac{y_{mk-1}}{y_{(m+1)k-1}} - 1 \right| \\
&< \varepsilon.
\end{aligned}$$

It is clear that $\lim_{m\to\infty} y_{mk} = 1$. This completes the proof. □

Theorem 3. *Consider Eq.*(1.1) *when* $k \in \mathbb{N}$. *Suppose that*

$$p > 1 \text{ and } q < p(p-1) + 1 \tag{3.5}$$

and let $\{\overline{x}_n\}$ *be a particular positive solution of Eq.*(1.1). *Then for all positive solutions* $\{x_n\}$ *of Eq.*(1.1),

$$x_n \sim \overline{x}_n. \tag{3.6}$$

Proof. Since (3.6) is equivalent to

$$\lim_{n\to\infty} y_n = 1 \tag{3.7}$$

where $\{y_n\}$ satisfies Eq.(3.2), it suffices to show that (3.7) holds. In Theorem 2, it was shown that (3.7) holds for all non-oscillatory solutions $\{y_n\}$ of Eq.(3.2). So, we will assume that $\{y_n\}$ oscillates about the equilibrium 1. Consider the function

$$F(a,b,c) = \frac{a+bc}{a+b}, \tag{3.8}$$

for $a, b, c > 0$. Therefore, we have

(i) For $c > 1$, $F(a,b,c)$ is decreasing in a and increasing in b.

(ii) For $c < 1$, $F(a,b,c)$ is increasing in a and decreasing in b.

Since all semicycles, except for perhaps the first, having at most k terms, we may assume, without lossing generality, that there exists an integer m such that

$$y_{2n} < 1 \text{ and } y_{2n-1}, y_{2n-2}, ..., y_{2n-k} \geq 1 \text{ for } n \geq m. \tag{3.9}$$

Let

$$s = \liminf_{n\to\infty} y_n \text{ and } S = \limsup_{n\to\infty} y_n. \tag{3.10}$$

From Eq.(3.2) and (3.8) we have

$$\begin{aligned} y_{2n+1} &= F\left(A_{2n}, \frac{\overline{x}_{2n-k}}{\overline{x}_{2n}}, \frac{y_{2n-k}}{y_{2n}}\right), \\ y_{2n+2} &= F\left(A_{2n+1}, \frac{\overline{x}_{2n-k+1}}{\overline{x}_{2n+1}}, \frac{y_{2n-k+1}}{y_{2n+1}}\right). \end{aligned} \tag{3.11}$$

Since (3.9) holds, by Lemma 5, we obtain $y_{2n-k+1} < 1$ and $y_{2n+1} > 1$, and so we have

$$\frac{y_{2n-k}}{y_{2n}} \geq 1, \ \frac{y_{2n-k+1}}{y_{2n+1}} < 1.$$

Using (3.7), (3.9), (3.10), (3.11) and monotonicity properties of F, we have

$$\begin{aligned} S &\leq F\left(p, \frac{\mu}{\lambda}, \frac{S}{s}\right) = \frac{p + \frac{\mu}{\lambda}\frac{S}{s}}{p + \frac{\mu}{\lambda}}, \\ s &\geq F\left(p, \frac{\mu}{\lambda}, \frac{s}{S}\right) = \frac{p + \frac{\mu}{\lambda}\frac{s}{S}}{p + \frac{\mu}{\lambda}}. \end{aligned}$$

or

$$Ss \leq \frac{ps + \frac{\mu}{\lambda}S}{p + \frac{\mu}{\lambda}} \text{ and } Ss \geq \frac{pS + \frac{\mu}{\lambda}s}{p + \frac{\mu}{\lambda}}.$$

Then we get

$$\frac{pS + \frac{\mu}{\lambda}s}{p + \frac{\mu}{\lambda}} \leq Ss \leq \frac{ps + \frac{\mu}{\lambda}S}{p + \frac{\mu}{\lambda}}.$$

Hence, we obtain

$$pS + \frac{\mu}{\lambda}s \leq ps + \frac{\mu}{\lambda}S$$

and so

$$p(S-s) \leq \frac{\mu}{\lambda}(S-s).$$

Thus from (3.4), we have

$$p(S-s) \leq \frac{\mu}{\lambda}(S-s) \leq \frac{(q-1)}{(p-1)}(S-s).$$

and

$$[p(p-1) - (q-1)](S-s) \leq 0.$$

Therefore, from (3.5) we obtain

$$S = s.$$

Hence $\lim_{n\to\infty} y_n = 1$, and the proof is complete. □

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