Electronic Journal of Mathematical Analysis and Applications
Vol. 4(2) Jul. 2016, pp. 1-15.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

STABILITY BY CONTRACTION MAPPING IN NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH VARIABLE DELAY

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Abstract. The nonlinear neutral differential equation

$$x'(t) = -a(t)\, g(x(t-\tau(t))) + c(t)\, x'(t-\tau(t)),$$

with variable delay $\tau(t) \geq 0$ is investigated. We find suitable conditions for τ, a, c and g so that for a given continuous initial function ψ a mapping P for the above equation can be defined on a carefully chosen complete metric space S_{ψ}^{0} in which P possesses a unique fixed point. The final result is an asymptotic stability theorem for the zero solution with a necessary and sufficient condition. The obtained theorem improves and generalizes previous results due to Becker and Burton [6]. We end this work by giving an illustrative example.

1. Introduction

In 1892 Lyapunov published a major work on stability of ordinary differential equations based on positive definite functions and the chain rule. Lyapunov's work has been the foundation of stability and instability theory as we know today for a wide variety of ordinary, functional, partial differential and integro-differential equations. Nevertheless, the application of this method to problems of stability in differential and integro-differential equations with delays has encountered serious obstacles if the delays are unbounded or if the equation has unbounded terms [8]–[10]. Maybe this is due to the pointwise aspect of the method while real world-problems asks for averaging conditions. So, it does seem that the times are ripe to try other avenues. In recent years, several investigators in this field have tried stability by using a new technique. Particularly, Burton, Furumochi, Becker, Zhang and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1]–[21], [23]–[25]). The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's. The conditions of the former are often averages but those of the latter are usually pointwise (see [8]). Moreover, recent works have shown that the fixed point technique can be applied to problems

2010 *Mathematics Subject Classification.* 34K20, 34K30, 34K40.
Key words and phrases. Fixed points, Stability, Neutral differential equations, Variable delays.
Submitted Jun 6, 2015.

perturbed by stochastic terms and still yielding stability (see for example [20]). This is another important feature for applications to real-world problems.

In this paper we consider the nonlinear neutral differential equation with variable delay

$$x'(t) = -a(t)\, g(x(t-\tau(t))) + c(t)\, x'(t-\tau(t)), \tag{1}$$

with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(0), 0],$$

such that $\psi \in C([m(0), 0], \mathbb{R})$ where $m(0) = \inf\{t - \tau(t),\ t \geq 0\}$.
Here $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $\|\cdot\|$. Throughout this paper we assume that $a \in C(\mathbb{R}^+, \mathbb{R})$, $c : \mathbb{R}^+ \to \mathbb{R}$ is differentiable and $\tau : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable with $t - \tau(t) \to \infty$ as $t \to \infty$.

Special cases of *(1)* have been considered and investigated by many authors. For example the equation

$$x'(t) = -a(t)\, g(x(t-\tau(t))), \tag{2}$$

is of historical importance and has significant applications. The study of *(2)* go back to 1951 (cf. [23] and the references therein) and it has drawn the attention of a large number of investigators who have discussed in an enormous number of papers. To our knowledge, the most recent work about this equation is that of Becker and Burton in [6] who have studied this equation and have established and proved the following theorem.

Theorem 1 (Becker and Burton [6])**.** *Suppose a constant $l > 0$ exists such that g satisfies a Lipschitz condition on $[-l, l]$. Assume, further, that*

(i) the function $t - \tau(t) : [0, \infty) \to [m(0), \infty)$ is strictly increasing;

(ii) g is odd and strictly increasing on $[-l, l]$;

(iii) $x - g(x)$ is non-decreasing on $[0, l]$;

(iv) there is an $\alpha \in (0, 1)$ such that $2\int_{t-\tau(t)}^{t} a(u)\, du \leq \alpha$ for $t \geq t_1$, where t_1 is the unique solution of $t - \tau(t) = 0$, and if a continuous function $\widetilde{a} : [0, \infty) \to \mathbb{R}$ exists such that $a(t) = \widetilde{a}(t)(1 - \tau'(t))$ on $[0, \infty)$. Then a $\delta \in (0, l)$ exists such that, for each continuous function $\psi : [m(0), 0] \to (-\delta, \delta)$, there is a unique continuous function $x : [m(0), \infty) \to \mathbb{R}$ with $x(t) = \psi(t)$ on $[m(0), 0]$ that satisfies (2) on $[0, \infty)$. Moreover, x is bounded by l on $[m(0), \infty)$. Furthermore, the zero solution of (2) is stable at $t = 0$. If, in addition, g is continuously differentiable, $g'(0) \neq 0$ and

$$\int_0^t a(v)\, dv \to \infty \text{ as } t \to \infty, \tag{3}$$

then the zero solution is asymptotically stable.

Our purpose here is to improve Theorem 1 and extend it to investigate a wide class of nonlinear differential equation with variable delay of neutral type presented in (1). We point out here that the present work do not require that $t - \tau(t)$ be strictly increasing. Such an assumption has been at the heart of the methods used in the previous papers and here we propose a way free of further hypotheses on the inverse of delay $t - \tau(t)$, so that for a given continuous initial function ψ a mapping P for (1) is constructed in such a way to map a, carefully chosen, complete metric space S_ψ^0 into itself on which P is a contraction mapping possessing a fixed point.

This procedure will enable us to establish and prove by means of the contraction mapping theorem an asymptotic stability theorem for the zero solution of (1) with less restrictive conditions. The end results improve and generalize the main results in [6]. We also provide an example to illustrate our claim.

2. Main results

For each $\psi \in C([m(0),0],\mathbb{R})$, a solution of (1) through $(0,\psi)$ is a continuous function $x:[m(0),T)\to\mathbb{R}$ for some positive constant $T>0$ such that x satisfies (1) on $[0,T)$ and $x=\psi$ on $[m(0),0]$. We denote such a solution by $x(t)=x(t,0,\psi)$. From the existence theory we can conclude that for each $\psi\in C([m(0),0],\mathbb{R})$, there exists a unique solution $x(t)=x(t,0,\psi)$ of (1) defined on $[0,\infty)$. We define $\|\psi\|=\max\{|\psi(t)|:m(0)\le t\le 0\}$. Stability definitions may be found in [8], for example. In our investigation we need that τ is twice differentiable and

$$\tau'(t)\neq 1 \text{ for all } t\in\mathbb{R}^+. \tag{4}$$

The absence of linear terms in (1) makes it difficult to obtain a fixed point mapping. So, to make equation (1) more tractable, we have to transform it.

Generally, the investigation of the stability of an equation using fixed point technic involves the construction of a suitable fixed point mapping. This can, in so many cases, be a difficult task. So, to construct our mapping, we begin by transforming (1) to a more tractable, but equivalent, equation, which we then invert to obtain an equivalent integral equation from which we derive a fixed point mapping. After then, we define prudently a suitable complete space, depending on the initial condition, so that the mapping is a contraction. Using Banach's contraction mapping principle [22], we obtain a solution for this mapping, and hence a solution for (1), which is asymptotically stable.

First, we have to transform (1) into an equivalent equation that possesses the same basic structure and properties to which we apply the variation of parameters to define a fixed point mapping.

Lemma 1. *Let $\psi:[m(0),0]\to\mathbb{R}$ be a given continuous initial function. If x is a solution of (1) on an interval $[0,T)$ with $x=\psi$ on $[m(0),0]$, then x is a solution of the integral equation*

$$\begin{aligned}
x(t) &= \left\{\psi(0)-\frac{c(0)}{1-\tau'(0)}\psi(-\tau(0))-\int_{-\tau(0)}^{0}H(s)g(\psi(s))\,ds\right\}e^{-\int_0^t H(v)dv}\\
&\quad+\frac{c(t)}{1-\tau'(t)}x(t-\tau(t))+\int_{t-\tau(t)}^{t}H(s)g(x(s))\,ds\\
&\quad-\int_0^t e^{-\int_s^t H(v)dv}H(s)\left(\int_{s-\tau(s)}^{s}H(u)g(x(u))\,du\right)ds\\
&\quad+\int_0^t e^{-\int_s^t H(v)dv}\{[-a(s)+H(s-\tau(s))(1-\tau'(s))]\,g(x(s-\tau(s)))\\
&\qquad\qquad -\mu(s)x(s-\tau(s))\}\,ds\\
&\quad+\int_0^t e^{-\int_s^t H(v)dv}H(s)\,[x(s)-g(x(s))]\,ds,
\end{aligned}\tag{5}$$

where

$$\mu(s)=\frac{[c'(s)+H(s)c(s)](1-\tau'(s))+\tau''(s)c(s)}{(1-\tau'(s))^2},$$

and $H:[m(0),\infty)\to\mathbb{R}$ *is an arbitrary continuous function. Conversely, if a continuous function* x *is equal to* ψ *on* $[m(0),0]$ *and is a solution of (5) on an interval* $[0,T)$*, then* x *is a solution of (1) on* $[0,T)$*.*

Proof. Let x be a solution of (1). Rewrite (1) as

$$\begin{aligned}x'(t)=&-H(t)x(t)+\frac{d}{dt}\int_{t-\tau(t)}^{t}H(s)g(x(s))\,ds+c(t)x'(t-\tau(t))\\&+[-a(t)+H(t-\tau(t))(1-\tau'(t))]\,g(x(t-\tau(t)))+H(t)[x(t)-g(x(t))].\end{aligned}\tag{6}$$

Multiplying both sides of (6) by the factor $e^{\int_0^t H(v)dv}$ and integrating from 0 to any $t\in[0,T)$, we obtain

$$\begin{aligned}x(t) &= e^{-\int_0^t H(v)dv}\psi(0)+\int_0^t e^{-\int_s^t H(v)dv}H(s)g(x(s))\,ds\\&\quad+\int_0^t e^{-\int_s^t H(v)dv}\frac{d}{ds}\int_{s-\tau(s)}^{s}H(u)g(x(u))\,du\,ds\\&\quad+\int_0^t e^{-\int_s^t H(v)dv}c(s)x'(s-\tau(s))ds\\&\quad+\int_0^t e^{-\int_s^t H(v)dv}\{-a(s)+H(s-\tau(s))(1-\tau'(s))\}\,g(x(s-\tau(s)))\,ds\\&\quad+\int_0^t e^{-\int_s^t H(v)dv}H(s)[x(s)-g(x(s))]\,ds.\end{aligned}$$

Then an integration by parts yields (5). Conversely, suppose that a continuous function x is equal to ψ on $[m(0),0]$ and satisfies (5) on an interval $[0,T)$. Then, x is differentiable on $[0,T)$. Differentiating x with the aid of Leibniz's rule, we obtain (1). □

From equation (5) we shall derive a fixed point mapping P for (1). But the challenge here is to choose a suitable metric space of functions on which the map P can be defined. Below a weighted metric on a specific space is defined. Let C be the set of real-valued bounded continuous functions on $[m(0),\infty)$ with the supremum norm $\|\cdot\|$, that is, for $\phi\in C$,

$$\|\phi\|:=\sup\{|\phi(t)|:t\in[m(0),\infty)\}.$$

In other words, we carry out our investigations in the complete metric space (C,d) where d denotes the supremum metric $d(\phi_1,\phi_2)=\|\phi_1-\phi_2\|$ for $\phi_1,\phi_2\in C$. For a given initial function $\psi:[m(0),0]\to[-l,l]$, $l>0$, define the set

$$S_\psi:=\{\phi:[m(0),\infty)\to\mathbb{R}\mid\phi\in C,\ \phi(t)=\psi(t)\ for\ t\in[m(0),0],|\phi(t)|\le l\}.$$

Since S_ψ is a closed subset of C, the metric space (S_ψ,d) is complete.

Theorem 2. *Let $H:[m(0),\infty)\to\mathbb{R}$ be a continuous function and define a mapping P on S_ψ as follows, for $\phi\in S_\psi$ $(P\phi)(t)=\psi(t)$ if $t\in[m(0),0]$; while, for $t>0$*

$$\begin{aligned}(P\phi)(t)&=\left\{\psi(0)-\frac{c(0)}{1-\tau'(0)}\psi(-\tau(0))-\int_{-\tau(0)}^{0}H(s)g(\psi(s))\,ds\right\}e^{-\int_0^t H(v)dv}\\&+\frac{c(t)}{1-\tau'(t)}\phi(t-\tau(t))+\int_{t-\tau(t)}^{t}H(s)g(\phi(s))\,ds\\&-\int_0^t e^{-\int_s^t H(v)dv}H(s)\left(\int_{s-\tau(s)}^{s}H(u)g(\phi(u))\,du\right)ds\\&+\int_0^t e^{-\int_s^t H(v)dv}\{[-a(s)+H(s-\tau(s))(1-\tau'(s))]\,g(\phi(s-\tau(s)))\\&\qquad-\mu(s)\phi(s-\tau(s))\}\,ds\\&+\int_0^t e^{-\int_s^t H(v)dv}H(s)[\phi(s)-g(\phi(s))]\,ds.\end{aligned}\tag{7}$$

Suppose that the following conditions are satisfied,

(i) there exists a constant $l>0$ such that g satisfies a Lipschitz condition on $[-l,l]$ and let L be the Lipschitz constant for both $g(x)$ and $x-g(x)$ on $[-l,l]$;

(ii) $H(t)\geq 0$ for $t\geq m(0)$.

Assume, further, that the following condition is satisfied for some constant $k>5$

$$k\left|\frac{c(t)}{1-\tau'(t)}\right|\leq 1.\tag{8}$$

Then there is a metric d_h for S_ψ such that (S_ψ,d_h) is complete and P is a contraction on (S_ψ,d_h) if P maps S_ψ into itself.

Proof. It is clear that $P\phi$ is continuous when ϕ is such. Now, for $t\in[m(0),\infty)$ and a constant $k>5$, define

$$h(t)=kL\int_0^t[H(v)+\omega(v)]dv,$$

where

$$\omega(v)=\begin{cases}0, & if\ v\in[m(0),0]\\ |-a(v)+H(v-\tau(v))(1-\tau'(v))|+\frac{|\mu(v)|}{L}, & if\ v\in[0,\infty).\end{cases}\tag{9}$$

Let S be the space of all continuous functions $\phi:[m(0),\infty)\to\mathbb{R}$ such that

$$|\phi|_h:=\sup\left\{|\phi(t)|\,e^{-h(t)}:t\in[m(0),\infty)\right\}<\infty.$$

Then $(S,|\cdot|_h)$ is a Banach space, which can be checked with Cauchy criterion for uniform convergence. Thus, (S,d_h) is a complete metric space where d_h denotes the induced metric $d_h(\phi,\eta)=|\phi-\eta|_h$ for $\phi-\eta\in S$. Being closed in S with this metric, the space $(S_\psi,|\cdot|_h)$ is also complete. Suppose now, that $P:S_\psi\to S_\psi$. We need to show that P defined by (7) is a contraction. Toward this end. Let $\phi,\eta\in S_\psi$, since,

by hypothesis (i), g satisfies a Lipschitz condition on $[-l,l]$, it follows that

$$\begin{aligned}
&|(P\phi)(t)-(P\eta)(t)|\,e^{-h(t)}\\
&\leq\left|\frac{c(t)}{1-\tau'(t)}\right||\phi(t-\tau(t))-\eta(t-\tau(t))|\,e^{-h(t)-h(t-\tau(t))+h(t-\tau(t))}\\
&+\int_{t-\tau(t)}^{t}H(s)L\,|\phi(s)-\eta(s)|\,e^{-h(t)+h(s)-h(s)}ds\\
&+\int_{o}^{t}e^{-\int_s^t H(v)dv}H(s)\int_{s-\tau(s)}^{s}H(u)L\,|\phi(u)-\eta(u)|\,e^{-h(t)+h(u)-h(u)}duds\\
&+\int_{0}^{t}e^{-\int_s^t H(v)dv}\left\{L\,|-a\,(s)+H(s-\tau(s))\,(1-\tau'(s))|+|\mu(s)|\right\}\\
&\times|\phi(s-\tau(s))-\eta(s-\tau(s))|\,e^{-h(t)+h(s-\tau(s))-h(s-\tau(s))}ds\\
&+\int_{0}^{t}e^{-\int_s^t H(v)dv}H(s)L\,|\phi(s)-\eta(s)|\,e^{-h(t)+h(s)-h(s)}ds, \qquad (10)
\end{aligned}$$

for $t>0$. There are five terms on the right hand side of inequality (10), denote them respectively by I_i, $i=1,2,...,5$. For $t-\tau(t)\leq s\leq v\leq t$, we have

$$-h(t)+h(s)=-kL\int_s^t[H(v)+\omega(v)]dv\leq-kL\int_s^t H(v)dv.$$

Consequently,

$$I_2\leq\int_{t-\tau(t)}^{t}e^{-kL\int_s^t H(v)dv}H(s)\,|\phi(s)-\eta(s)|\,e^{-h(s)}ds.$$

For $s-\tau(s)\leq u\leq v\leq s$, we have

$$-h(t)+h(u)=-kL\int_u^t[H(v)+\omega(v)]dv\leq-kL\int_u^s H(v)dv.$$

So

$$I_3\leq\int_0^t e^{-\int_s^t H(v)dv}H(s)\int_{s-\tau(s)}^{s}e^{-kL\int_u^s H(v)dv}H(u)L\,|\phi(u)-\eta(u)|\,e^{-h(u)}duds$$

Similarly, using (9), we obtain for $s-\tau\,(s)\leq v\leq t$

$$-h\,(t)+h\,(s-\tau(s))=-kL\int_{s-\tau(s)}^{t}[H(v)+\omega(v)]dv\leq-kL\int_s^t\omega(v)dv$$

Thus

$$I_4\leq\int_0^t e^{-kL\int_s^t\omega(v)dv}\omega\,(s)\,L\,|\phi(s-\tau(s))-\eta(s-\tau(s))|\,e^{-h(s-\tau(s))}ds$$

Consequently, inequality (10), became

$$\begin{aligned}
&|(P\phi)(t)-(P\eta)(t)|\,e^{-h(t)}\\
&\leq e^{-kL\int_{t-\tau(t)}^{t}[H(v)+\omega(v)]dv}\left|\frac{c(t)}{1-\tau'(t)}\right||\phi(t-\tau(t))-\eta(t-\tau(t))|\,e^{-h(t-\tau(t))}\\
&+\int_{t-\tau(t)}^{t}e^{-kL\int_{u}^{t}H(v)dv}H(s)\,L\,|\phi(s)-\eta(s)|\,e^{-h(s)}ds\\
&+\int_{0}^{t}e^{-\int_{s}^{t}H(v)dv}H(s)\int_{s-\tau(s)}^{s}e^{-kL\int_{u}^{s}H(v)dv}H(u)\,|\phi(u)-\eta(u)|\,e^{-h(u)}duds\\
&+\int_{0}^{t}e^{-kL\int_{s}^{t}\omega(v)dv}\omega(s)\,|\phi(s-\tau(s))-\eta(s-\tau(s))|\,e^{-h(s-\tau(s))}ds\\
&+\int_{0}^{t}e^{-(kL+1)\int_{s}^{t}H(v)dv}H(s)L\,|\phi(s)-\eta(s)|\,e^{-h(s)}ds.
\end{aligned}$$

Consequently, by using (8), we obtain

$$|(P\phi)(t)-(P\eta)(t)|\,e^{-h(t)}\leq\left[\frac{1}{k}+\left(\frac{1}{kL}+\frac{1}{kL}+\frac{1}{kL}+\frac{1}{kL+1}\right)L\right]|\phi-\eta|_{h},$$

for all $t>0$. Since $P\phi$ and $P\eta$ agree on $[m(0),0]$, the last inequality holds for all $t\geq m(0)$. Thus

$$|P\phi-P\eta|_{h}\leq\frac{5}{k}|\phi-\eta|_{h}.$$

Since $k>5$, we conclude that P is a contraction on (S_{ψ},d_{h}). □

Now, by choosing $\|\psi\|$ sufficiently small, we establish the existence and uniqueness of solutions by showing that $P:S_{\psi}\to S_{\psi}$.

Definition 1. *The zero solution of (1) is said to be stable at $t=0$ if, for every $\varepsilon>0$, there exists a $\delta>0$ such that $\psi:[m(0),0]\to(-\delta,\delta)$ implies that $|x(t)|<\varepsilon$ for $t\geq m(0)$.*

Theorem 3. *Suppose g, H satisfy conditions (i)–(ii) in Theorem 2, (8) holds and suppose further that*

(i) g is odd and strictly increasing on $[-l,l]$;

(ii) $x-g(x)$ is non-decreasing on $[0,l]$;

(iii) there exists an $\alpha\in(0,1)$ such that, for $t\geq0$

$$\begin{aligned}
&l\left(\left|\frac{c(t)}{1-\tau'(t)}\right|+\int_{0}^{t}e^{-\int_{s}^{t}H(v)dv}|\mu(s)|\,ds\right)\\
&+g(l)\left(\int_{t-\tau(t)}^{t}H(s)\,ds+\int_{0}^{t}e^{-\int_{s}^{t}H(v)dv}H(s)\int_{s-\tau(s)}^{s}H(u)duds\right.\\
&\left.+\int_{0}^{t}e^{-\int_{s}^{t}H(v)dv}|-a(s)+H(s-\tau(s))(s-\tau'(s))|\,ds\right)\\
&\leq\alpha g(l).
\end{aligned}$$

Then a $\delta\in(0,l)$ exists such that for each initial continuous function $\psi:[m(0),0]\to(-\delta,\delta)$, there is a unique continuous function $x:[m(0),\infty)\to\mathbb{R}$ with $x=\psi$ on $[m(0),0]$, which is a solution of (1) on $[0,\infty)$. Moreover, x is bounded by l on $[m(0),\infty)$. Furthermore, the zero solution of (1) is stable at $t=0$.

Proof. Since g is odd and satisfies the Lipschitz condition on $[-l,l]$, $g(0)=0$ and g is uniformly continuous on $[-l,l]$. So we can choose a δ that satisfies

$$\delta\left(1+\left|\frac{c(0)}{1-\tau'(0)}\right|\right)+g(\delta)\int_{-\tau(0)}^{0}H(s)ds\leq(1-\alpha)\,g(l). \tag{11}$$

Let $\psi:[m(0),0]\to(-\delta,\delta)$ be a continuous function. Note that (11) implies $\delta<l$ since $g(l)\leq l$ by condition (ii). Thus, $|\psi(t)|\leq l$ for $m(0)\leq t\leq 0$. Now we show that for such a ψ the mapping $P:S_\psi\to S_\psi$. Indeed, consider (7). For an arbitrary $\phi\in S_\psi$, it follows from conditions (i) and (ii) that

$$\begin{aligned}
&|(P\phi)(t)|\\
&\leq\delta\left(1+\left|\frac{c(0)}{1-\tau'(0)}\right|\right)+g(\delta)\int_{-\tau(0)}^{0}H(s)ds+l\left|\frac{c(t)}{1-\tau'(t)}\right|\\
&+g(l)\int_{t-\tau(t)}^{t}H(s)ds+g(l)\int_{o}^{t}e^{-\int_s^t H(v)dv}H(s)\left(\int_{s-\tau(s)}^{s}H(u)du\right)ds\\
&+\int_0^t e^{-\int_s^t H(v)dv}\left\{g(l)\left|-a(s)+H(s-\tau(s))\left(1-\tau'(s)\right)\right|+l\left|\mu(s)\right|\right\}ds\\
&+(l-g(l))\int_0^t e^{-\int_s^t H(v)dv}H(s)ds,
\end{aligned}$$

for $t>0$. By applying (iii) and (11), we see that

$$\begin{aligned}
|(P\phi)(t)| &\leq \delta\left(1+\left|\frac{c(0)}{1-\tau'(0)}\right|\right)+g(\delta)\int_{-\tau(0)}^{0}H(s)ds+\alpha g(l)+l-g(l)\\
&\leq (1-\alpha)\,g(l)+(\alpha-1)\,g(l)+l=l.
\end{aligned}$$

Hence, $|(P\phi)(t)|\leq l$ for $t\in[m(0),\infty)$ because $|(P\phi)(t)|=|\psi(t)|\leq l$ for $t\in[m(0),0]$. Therefore, $P\phi\in S_\psi$. By Theorem 2, P is a contraction on the complete metric space (S_ψ,d_h). Then P has a unique fixed point $x\in S_\psi$. Thus $|x(t)|\leq l$ for all $t\geq m(0)$ and is a solution of (1) on $[0,\infty)$ by Lemma 1. Hence x is the only continuous function satisfying (1) such that $x(t)=\psi(t)$ for $t\in[m(0),0]$.
To prove the stability at $t=0$, let $\varepsilon>0$ be given and choose $r>0$ such that $r<\min\{\varepsilon,l\}$. Replacing l with r beginning with (11), we see that there is a $\delta>0$ such that $\|\psi\|<\delta$ implies that the unique continuous solution x agreeing on $[m(0),0]$ with ψ, satisfies $|x(t)|\leq r<\varepsilon$ for all $t\geq m(0)$. □

Supposing that the conditions in Theorem 2 and Theorem 3 hold for some $l>0$, we investigate asymptotic stability with a necessary and sufficient condition by shifting our attention to the subset of functions in S_ψ that are tend to zero as $t\to\infty$, namely,

$$S_\psi^0=\{\phi\in S_\psi\mid\phi(t)\to 0\text{ as }t\to\infty\}.$$

Since S_ψ^0 is a closed subset of S_ψ, the metric space $\left(S_\psi^0,d\right)$ is complete. Under the conditions of the next theorem, the zero solution of (1) is asymptotically stable.

Definition 2. *The zero solution of (1) is asymptotically stable if it is stable at $t=0$ and a $\delta>0$ exists such that for any continuous functions $\psi:[m(0),0]\to(-\delta,\delta)$, the solution x with $x=\psi$ on $[m(0),0]$ tends to zero as $t\to\infty$.*

Recall here that B. Zhang was the first to establish necessary and sufficient condition for the stability of solutions of functional differential equations by the fixed point theory. The necessity of condition (12) below for the main stability result is due to him (see [25]).

Theorem 4. *Suppose all of the conditions in Theorem 2 and Theorem 3 hold. Furthermore, suppose g is continuously differentiable on $[-l,l]$ and $g'(0)\neq 0$. Then the zero solution of (1) is asymptotically stable if and only if*

$$\int_0^t H(v)\,dv \to \infty \text{ as } t\to\infty. \tag{12}$$

Proof. First, suppose that (1) holds. We set

$$K=\sup_{t\geq 0}\left\{e^{-\int_0^t H(v)dv}\right\}. \tag{13}$$

We argue that $P:S_\psi^0\to S_\psi^0$, the conditions of Theorems 2 and 3 hold, and $\|\psi\|$ is sufficiently small. For $\delta>0$ satisfying (11), let $\psi:[m(0),0]\to(-\delta,\delta)$ be a continuous function. Let $\phi\in S_\psi^0$. The proof of Theorem 3 shows that $\delta<l$ and $|(P\phi)(t)|\leq l$ for $t\in[m(0),\infty)$. Hence, $(P\phi)(t)\to 0$ would imply that P maps S_ψ^0 into itself. To show that this is the case, consider $|(P\phi)(t)|$. But first note for any $\phi\in S_\psi^0$ that

$$|g(\phi(t))|\leq L|\phi(t)| \text{ and } |\phi(t)-g(\phi(t))|\leq L|\phi(t)|,$$

since $g(x)$ and $x-g(x)$ satisfy a Lipschitz condition on $[-l,l]$ with a common Lipschitz constant L and $g(0)=0$. Because of this and (iii) of Theorem 3, it follows from (7) that

$$\begin{aligned}
&|(P\phi)(t)|\\
&\leq\left\{|\psi(0)|+\left|\frac{c(0)}{1-\tau'(0)}\right||\psi(-\tau(0))|+\int_{-\tau(0)}^0 H(s)|\psi(s)|\,ds\right\}e^{-\int_0^t H(v)dv}\\
&+\left|\frac{c(t)}{1-\tau'(t)}\right||\phi(t-\tau(t))|+\int_{t-\tau(t)}^t H(s)|\phi(s)|\,ds\\
&+\int_0^t e^{-\int_s^t H(v)dv}H(s)\left(\int_{s-\tau(s)}^s H(u)|\phi(u)|\,du\right)ds\\
&+\int_0^t e^{-\int_s^t H(v)dv}\{|-a(s)+H(s-\tau(s))(1-\tau'(s))||g(\phi(s-\tau(s)))|\\
&\qquad\qquad+|\mu(s)||\phi(s-\tau(s))|\}\,ds\\
&+\int_0^t e^{-\int_s^t H(v)dv}H(s)|\phi(s)-g(\phi(s))|\,ds.
\end{aligned}\tag{14}$$

Denote the six terms on the right hand side of (14) by $I_1,I_2,...,I_6$, respectively. It is obvious that the first term I_1 tends to zero as $t\to\infty$, by condition (12). Also, due to the facts that $\phi(t)\to 0$ and $t-\tau(t)\to\infty$ as $t\to\infty$, the second term I_2 in (14) tends to zero as $t\to\infty$. What is left to show is that each of the remaining terms in (14) tends to zero as $t\to\infty$.

Let $\phi\in S_\psi^0$ be fixed. For a given $\epsilon>0$, we choose $T_0>0$ large enough such that $t-\tau(t)\geq T_0$, implies $|\phi(s)|<\epsilon$ if $s\geq t-\tau(t)$. Therefore, the third term I_3 in

(14) satisfies

$$\begin{aligned} I_3 &\leq \int_{t-\tau(t)}^{t} H(s)L\left|\phi(s)\right| ds \\ &\leq L\epsilon \int_{t-\tau(t)}^{t} H(s)ds \leq L\alpha\epsilon. \end{aligned}$$

Thus, $I_3 \to 0$ as $t \to \infty$. Now consider I_4. For the given $\epsilon > 0$, there exists a $T_1 > 0$ such that $s \geq T_1$ implies $|\phi(s-\tau(s))| < \epsilon$. Thus, for $t \geq T_1$, the term I_4 in (14) satisfies

$$\begin{aligned} I_4 \leq & \int_o^{T_1} e^{-\int_s^t H(v)dv} H(s) \left(\int_{s-\tau(s)}^{s} H(u)L\left|\phi(s)\right| du \right) ds \\ & + \int_{T_1}^{t} e^{-\int_s^t H(v)dv} H(s) \left(\int_{s-\tau(s)}^{s} H(u)L\left|\phi(s)\right| du \right) ds \\ & \leq \sup_{\sigma \geq m(0)} \left|\phi(\sigma)\right| L \int_o^{T_1} e^{-\int_s^t H(v)dv} H(s) \left(\int_{s-\tau(s)}^{s} H(u)du \right) ds \\ & + L\epsilon \int_{T_1}^{t} e^{-\int_s^t H(v)dv} H(s) \left(\int_{s-\tau(s)}^{s} H(u)du \right) ds. \end{aligned}$$

By (12), there exists $T_2 > T_1$ such that $t \geq T_2$ implies

$$\begin{aligned} & \sup_{\sigma \geq m(0)} \left|\phi(\sigma)\right| L \int_o^{T_1} e^{-\int_s^t H(v)dv} H(s) \left(\int_{s-\tau(s)}^{s} H(u)du \right) ds \\ & = \sup_{\sigma \geq m(0)} \left|\phi(\sigma)\right| L e^{-\int_{T_2}^t H(v)dv} \int_o^{T_1} e^{-\int_s^{T_2} H(v)dv} H(s) \left(\int_{s-\tau(s)}^{s} H(u)du \right) ds \\ & < L\epsilon. \end{aligned}$$

Now, apply condition (iii) in Theorem 3 to have $I_4 < L\epsilon + L\alpha\epsilon < 2L\epsilon$. Thus, $I_4 \to 0$ as $t \to \infty$. Similarly, by using condition (iii) in Theorem 3, then, if $t \geq T_2$ then the terms I_5 and I_6 in (14) satisfy

$$\begin{aligned} I_5 \leq & \sup_{\sigma \geq m(0)} \left|\phi(\sigma)\right| e^{-\int_{T_2}^t H(v)dv} \\ & \times \int_o^{T_1} e^{-\int_s^{T_2} H(v)dv} \left\{ L\left|-a(s) + H(s-\tau(s))\left(1-\tau'(s)\right)\right| + \left|\mu(s)\right| \right\} ds \\ & + \epsilon \int_{T_1}^{t} e^{-\int_s^t H(v)dv} \left\{ L\left|-a(s) + H(s-\tau(s))\left(1-\tau'(s)\right)\right| + \left|\mu(s)\right| \right\} ds \\ < & L\epsilon + L\alpha\epsilon + \frac{g(l)}{l}\alpha\epsilon < \left(2L + \frac{g(l)}{l} \right) \epsilon, \end{aligned}$$

and

$$\begin{aligned} I_6 &\leq \sup_{\sigma \geq m(0)} |\phi(\sigma)| \, L e^{-\int_{T_2}^{t} H(v)dv} \int_{o}^{T_1} e^{-\int_{s}^{T_2} H(v)dv} H(s)ds \\ &\quad + L\epsilon \int_{T_1}^{t} e^{-\int_{s}^{t} H(v)dv} H(s)ds \\ &< L\epsilon + L\epsilon = 2L\epsilon. \end{aligned}$$

Thus, $I_5, I_6 \to 0$ as $t \to \infty$. In conclusion $(P\phi)(t) \to 0$ as $t \to \infty$, as required. Hence P maps S_ψ^0 into S_ψ^0.

We verify that P is a contraction on $\left(S_\psi^0, d\right)$. For $\zeta \in [0, l]$, define

$$q(\zeta) := \min\{g'(x) : |x| \leq \zeta\} \text{ and } Q(\zeta) := \max\{g'(x) : |x| \leq \zeta\}.$$

As we shall see, the mapping P defined by (7) will be a contraction on $\left(S_\psi^0, d\right)$ provided

$$\alpha Q(l) < q(l), \tag{15}$$

where α is given by condition (iii) in Theorem 3. We may assume (15) holds for if it does not, we merely decrease the value of $l > 0$ until it does. To see this, first notice from (iii) in Theorem 3 that $\alpha < 1$, thus,

$$\alpha = 1 - \epsilon, \tag{16}$$

for some $\epsilon \in (0, 1)$. Clearly, $\lim_{\zeta \to 0} q(\zeta) = \lim_{\zeta \to 0} Q(\zeta) = g'(0)$. Since g is strictly increasing, $g'(0) \neq 0$, and g' is continuous, a neighborhood of $x = 0$ exists in which $g'(x) > 0$. Consequently, $Q(\zeta) > 0$ for $0 \leq \zeta \leq l$. This and $\lim_{\zeta \to 0} Q(\zeta) \neq 0$ imply

$$\lim_{\zeta \to 0} \frac{q(\zeta)}{Q(\zeta)} = \frac{\lim_{\zeta \to 0} q(\zeta)}{\lim_{\zeta \to 0} Q(\zeta)} = 1.$$

Hence, there is a $\varsigma \in (0, l]$ such that

$$\left| \frac{q(\zeta)}{Q(\zeta)} - 1 \right| < \epsilon,$$

for $0 < \zeta < \varsigma$. Choosing a value for ζ from $(0, \varsigma)$, we have $(1 - \epsilon) Q(\zeta) < q(\zeta)$. This, along with (16), yields

$$\alpha Q(\zeta) = (1 - \epsilon) Q(\zeta) < q(\zeta).$$

Replacing the original value of l with $l = \zeta$, we obtain (15). Note that the conditions in Theorems 2 and 3 will still hold with this smaller l.

In the ensuing argument, bounds on the derivatives of $g(x)$ and $b(x) := x - g(x)$ on the interval $[-l, l]$ are used. By (i) in Theorem 3 and the definition of Q, $0 \leq g'(x) \leq Q(l)$. By (i) and (ii) in Theorem 3, b is non-decreasing on $[-l, l]$. This and $g'(x) \geq q(l)$ imply $0 \leq b'(x) \leq 1 - q(l)$. Let $\phi, \eta \in S_\psi^0$. By (7) and the Mean

Value Theorem, we have

$$\begin{aligned}
&|(P\phi)(t)-(P\eta)(t)|\\
&\leq \left|\frac{c(t)}{1-\tau'(t)}\right| |\phi(t-\tau(t))-\eta(t-\tau(t))|\\
&+\int_{t-\tau(t)}^{t} H(s)\left|g(\phi(s))-g(\eta(s))\right| ds\\
&+\int_{0}^{t} e^{-\int_s^t H(v)dv} H(s)\left(\int_{s-\tau(s)}^{s} H(u)\left|g(\phi(u))-g(\eta(u))\right| du\right) ds\\
&+\int_{0}^{t} e^{-\int_s^t H(v)dv}\left\{\left|-a(s)+H(s-\tau(s))(1-\tau'(s))\right|\right.\\
&\times\left|g(\phi(s-\tau(s)))-g(\eta(s-\tau(s)))\right|+|\mu(s)|\,|\phi(s-\tau(s))-\eta(s-\tau(s))|\Big\} ds\\
&+\int_{0}^{t} e^{-\int_s^t a(v)dv} a(s)\left|b(\phi(s))-b(\eta(s))\right| ds\\
&\leq [\alpha Q(l)+(1-q(l))]\,\|\phi-\eta\| = \rho\,\|\phi-\eta\|,
\end{aligned}$$

for all $t>0$, where $\rho=\alpha Q(l)+(1-q(l))$. This implies $\|P\phi-P\eta\|\leq\rho\,\|\phi-\eta\|$. Note $\rho\in(0,1)$. Consequently, for a continuous $\psi:[m(0),0]\to(-\delta,\delta)$, P has a unique fixed point $x\in S_\psi^0$. By Lemma 1, x is the unique continuous solution of (1) with $x(t)=\psi(t)$ on $[m(0),0]$. By virtue of $x\in S_\psi^0$, x tends to 0 as $t\to\infty$. By Theorem 3, the zero solution is stable at $t=0$. This shows that the zero solution of (1) is asymptotically stable if (12) holds.

Conversely, suppose (12) fails. Then, there exists a sequence $\{t_n\}$, $t_n\to\infty$ as $n\to\infty$ such that $\lim_{n\to\infty}\int_0^{t_n} H(v)\,dv=\beta$ for some $\beta\in\mathbb{R}^+$. We may also choose a positive constant J satisfying

$$-J\leq\int_0^{t_n} H(v)\,dv\leq J,$$

for all $n\geq 1$. To simplify our expressions, we define

$$\begin{aligned}
\omega(s)=l\,|\mu(s)|+g(l)\Big\{&|-a(s)+H(s-\tau(s))|\,|1-\tau'(s)|\\
&+H(s)\left(\int_{s-\tau(s)}^{s} H(u)\,du\right)\Big\}+LH(s),
\end{aligned}$$

for all $s\geq 0$. By by condition (iii) in Theorem 3, we have

$$\int_0^{t_n} e^{-\int_s^{t_n} H(v)dv}\omega(s)\,ds\leq\alpha g(l)+L.$$

This yields

$$\int_0^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds\leq(\alpha g(l)+L)\,e^{\int_0^{t_n} H(v)dv}\leq(g(l)+L)\,e^{J}.$$

The sequence $\left\{\int_0^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds\right\}$ is bounded, so there exists a convergent subsequence. For brevity of notation, we may assume that

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s H(v)dv}\omega(s)\,ds=\gamma,$$

for some $\gamma \in \mathbb{R}^+$ and choose a positive integer m so large that

$$\int_{t_m}^{t_n} e^{\int_0^s H(v)dv} \omega(s)\, ds < \delta_0/4K,$$

for all $n \geq m$, where $\delta_0 > 0$ satisfies

$$\left\{ \delta_0 \left(1 + \left| \frac{c(t_m)}{1-\tau'(t_m)} \right| \right) + g(\delta_0) \int_{t_m-\tau(t_m)}^{t_m} H(s)ds \right\} Ke^J \leq (1-\alpha)\, g(l).$$

If we replace l by 1 in the proof of Theorem 3, we have $|x(t)| \leq 1$ for $t \geq t_m$. Now we consider the solution $x(t) = x(t, t_m, \psi)$ of (1) with $\psi(t_m) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_m$. We may choose ψ so that

$$\psi(t_m) - \frac{c(t_m)}{1-\tau'(t_m)} \psi(t_m - \tau(t_m)) - \int_{t_m-\tau(t_m)}^{t_m} H(s)\, g(\psi(s))\, ds \geq \frac{1}{2}\delta_0.$$

It follows from (7) with $x(t) = (Px)(t)$ that for $n \geq m$

$$\begin{aligned}
&\left| x(t_n) - \frac{c(t_n)}{1-\tau'(t_n)} x(t_n - \tau(t_n)) - \int_{t_n-\tau(t_n)}^{t_n} H(s)\, g(x(s))\, ds \right| \\
&\geq \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(v)dv} - \int_{t_m}^{t_n} e^{-\int_s^{t_n} H(v)dv} \omega(s)\, ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_m}^{t_n} H(v)dv} - e^{-\int_0^{t_n} H(v)dv} \int_{t_m}^{t_n} e^{\int_0^s H(v)dv} \omega(s)\, ds \\
&= e^{-\int_{t_m}^{t_n} H(v)dv} \left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_m} H(v)dv} \int_{t_m}^{t_n} e^{\int_0^s H(v)dv} \omega(s)\, ds \right) \\
&\geq e^{-\int_{t_m}^{t_n} H(v)dv} \left(\frac{1}{2}\delta_0 - K \int_{t_m}^{t_n} e^{\int_0^s H(v)dv} \omega(s)\, ds \right) \\
&\geq \frac{1}{4}\delta_0 e^{-\int_{t_m}^{t_n} H(v)dv} \geq \frac{1}{4}\delta_0 e^{-2J} > 0.
\end{aligned} \tag{17}$$

On the other hand, if the zero solution of (1) is asymptotically stable, then $x(t) = x(t, t_m, \psi) \to 0$ as $t \to \infty$. Since $t_n - \tau(t_n) \to \infty$ as $n \to \infty$ and condition (iii) in Theorem 3 holds, we have

$$x(t_n) - \frac{c(t_n)}{1-\tau'(t_n)} x(t_n - \tau(t_n)) - \int_{t_n-\tau(t_n)}^{t_n} H(s)\, g(x(s))\, ds \to 0 \text{ as } n \to \infty,$$

which contradicts (17). Hence condition (12) is necessary for the asymptotic stability of the zero solution of (1). The proof is complete. □

Remark 1. *Obviously, if $c(t) = 0$, Theorem 4 extends Theorem C.*

3. An example

In this section, we give an example to illustrate the application of Theorem 4.

Example 1. *Consider the following nonlinear neutral differential equation with variable delay*

$$x'(t) = -a(t)\, g(x(t - \tau(t))) + c(t)\, x'(t - \tau(t)), \tag{18}$$

where $\tau(t) = 0.172t$, $a(t) = 0.391/(0.828t + 1)$, $g(x) = \sin x$, $c(t) = 0.032$. Then the zero solution of (18) is asymptotically stable.

Proof. Choosing $l = \pi/3$, $k = 6$ and $H(t) = 1/(t+1)$, clearly, condition (12) holds. Furthermore, we have $L = 2$, $g(0) = 0$, $g(l) = \sqrt{3}/2$, $g'(0) = 1$, g is odd and strictly increasing on $[-\pi/3, \pi/3]$, $x - g(x)$ is non-decreasing on $[0, \pi/3]$,

$$\left|\frac{c(t)}{1-\tau'(t)}\right| = \frac{0.032}{0.828} < 0.039,$$

$$\int_0^t e^{-\int_s^t H(v)dv} |\mu(s)| \, ds = \int_0^t e^{-\int_s^t \frac{1}{1+v}dv} \frac{0.032 \times 0.828}{1+s} ds < 0.027,$$

$$\int_{t-\tau(t)}^t H(s) \, ds = \int_{0.828t}^t \frac{1}{s+1} ds = \ln\left(\frac{t+1}{0.828t+1}\right) < 0.189,$$

$$\int_0^t e^{-\int_s^t H(v)dv} H(s) \left(\int_{s-\tau(s)}^s H(u)du\right) ds \; < \; 0.189,$$

and

$$\begin{aligned}
&\int_0^t e^{-\int_s^t H(v)dv} |-a(s) + H(s-\tau(s))(1-\tau'(s))| \, ds \\
&= \int_0^t e^{-\int_s^t \frac{1}{v+1}dv} \left|-\frac{0.391}{0.828s+1} + \frac{0.828}{0.828s+1}\right| ds \\
&< \frac{0.437}{0.828} \int_0^t e^{-\int_s^t \frac{1}{v+1}dv} \frac{1}{s+1} ds < 0.528.
\end{aligned}$$

It is easy to see that all the conditions of Theorems 2, 3 and 4 hold for $\alpha = \left[(\pi/3)/\left(\sqrt{3}/2\right)\right](0.066) + 0.189 + 0.189 + 0.528 \simeq 0.986 < 1$. Thus, Theorem 4 implies that the zero solution of (18) is asymptotically stable. □

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