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DISLOCATED QUASI b-METRIC SPACE AND FIXED POINT THEOREMS

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ABSTRACT. In this article, we have given the concept of dislocated quasi b-metric space, which is a generalization of dislocated quasi metric, partial b-metric, b-dislocated metric and b-metric spaces. Banach's contraction principle, Kannan and Chetterjae type fixed point results for self-mapping in such a space are established.

1. Introduction and Preliminaries

In 1906, Frechet introduced the notion of metric space, which is one of the cornerstones of not only mathematics but also several quantitative sciences. Due to its importance and application potential, this notion has been extended, improved and generalized in many different ways. An incomplete list of the results of such an attempt is the following: quasi metric space, symmetric space, partial metric space, cone metric space, G-metric space, b-metric space, dislocated metric space, dislocated quasi metric space and so on.

The Banach contraction theorem [12] is a very papular tool in solving existence problems in many branches of mathematical analysis. Many generalizations of Banach's contraction theorem in different types of generalizations of metric spaces are made as clear from the literature. Some problems particularly the problem of the convergence of measurable functions with respect to measure leads Czerwik [6] to the generalization of metric space and introduced the concept of b-metric space and proved Banach's contraction theorem in so called b-metric space. After Czerwik [6] many papers have been published containing fixed point results on b-metric spaces.

Cone metric space was introduced in [7], while studying cone metric spaces Khamsi [10] re-introduced the b-metric space to which he give the name of metric-type space. Several papers have been published in metric-type space contains fixed point results for single valued and multi-valued functions ([1], [8], [11]).

Alghamdi et al. [4] introduced the notation of b-metric like space which generalized the notation of b-metric space, where they proved some exciting new fixed point results in b-metric like space. Recently Shukla [13] introduced the concept

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of partial b-metric space and gave some fixed point results and examples in such a space.

In the current work the concept of dislocated quasi b-metric space or shortly (dq b-metric space) which generalized the notation of b-metric, partial b-metric and b-metric like spaces, has been studied. The famous Banach's contraction principle and many other well known results in so called dislocated quasi b-metric space(dq b-metric space) have been established .

Definition.([2], [15]). Let X be a non-empty set and let $d : X \times X \to [0, \infty)$ be a function satisfying the conditions

1) $d(x, x) = 0$;
2) $d(x, y) = d(y, x) = 0$ implies that $x = y$;
3) $d(x, y) = d(y, x)$;
4) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

If d satisfies the conditions from 1 to 4 then it is called metric on X, if d satisfies conditions 2 to 4 then it is called dislocated metric(d-metric)on X and if d satisfies conditions 2 and 4 only then it is called dislocated quasi metric (dq-metric space) on X.

In the view of above definition one can define the b-metric space [11] as following:

Definition. Let X be a non empty set, let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called b-metric if $\forall\ x, y, z \in X$,

1) $d(x, x) = 0$;
2) $d(x, y) = d(y, x) = 0$ implies that $x = y$;
3) $d(x, y) = d(y, x)$;
4) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

And the pair (X, d) is called b-metric space.

It is crystal clear from the above definition that b-metric is more generalization of (usual) metric.

Evidently every (usual) metric is b-metric but the converse is not true as shown by the following example.

Example.[11]. Let $X = \{0, 1, 2\}$ and
$d(0, 0) = d(1, 1) = d(2, 2) = 0$, $d(1, 2) = d(2, 1) = d(0, 1) = d(1, 0) = 1, d(2, 0) = d(0, 2) = m \geq 2$ for $k = \frac{m}{2}$ where $m \geq 2$
the function defined above is a b-metric but not a metric for $m > 2$.

For more examples of b-metric space (see [6, 11]). Also in [11] the notations such as Cauchy sequence, convergent sequence and completeness in b-metric spaces are defined in an obvious way.

A natural question arise in the mind whether one can find the contractive condition for which the unique fixed point in metric space wil be proved without continuity. Kannan [9] answered the above question in affirmative.

Theorem.[9]. Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping satisfying

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \tag{1}$$

for all $x, y \in X$ and for real number $\alpha \in [0, \frac{1}{2})$. Then T has a unique fixed point. The mapping satisfying the above condition is called Kannan type mapping. A similar type of contractive condition has been introduced by Chetterjea [5].

Theorem. Let (X, d) be a complete metric space and $T : X \to X$ be a self mapping satisfying

$$d(Tx, Ty) \leq \alpha[d(x, Ty) + d(y, Tx)] \tag{2}$$

for all $x, y \in X$ and for real number $\alpha \in [0, \frac{1}{2})$. Then T has a unique fixed point. The mapping satisfying the above condition is called Chetterjea type mapping.
Definition.[10]. A function d is said to be continuous in two variables if,

$$x_n \to x, y_n \to y \text{ in } (X, d) \Rightarrow d(x_n, y_n) \to d(x, y).$$

In the theory of symmetric spaces [14] the name given to the above property is "property (H_E)".
In [11] the authors generalized the Banach contraction theorem in b-metric space as following.
Theorem. Let (X, d) be a complete b-metric space with $k \geq 1$, then a contraction self mapping from X into itself with contraction constant $\alpha \in [0, 1)$ has a unique fixed point in X.

2. Main Results

To derive the main results we need the following definitions:
Definition. Let X be a non empty set, let $k \geq 1$ be a real number then a mapping $d : X \times X \to [0, \infty)$ is called dislocated quasi b-metric if $\forall\ x, y, z \in X$

(d_1) $d(x, y) = d(y, x) = 0$ implies that $x = y$;

(d_2) $d(x, y) \leq k[d(x, z) + d(z, y)]$.

The pair (X, d) is called dislocated quasi b-metric or shortly (dq b-metric) space.

Remark. In the definition of dislocated quasi b-metric space if $k = 1$ then it becomes (usual) dislocated quasi metric space. Therefore every dislocated quasi metric space is dislocated quasi b-metric space and every b-metric space is dislocated quasi b-metric space with same coefficient k and zero self distance. However, the converse is not true as clear from the following examples.
Example. Let $X = R^+$, for $p > 1$ and $d : X \times X \to [0, \infty)$ be defined s,

$$d(x, y) = |x - y|^p + |x|^p \text{ for all } x, y \in X.$$

Then (X, d) is a dislocated quasi b-metric space with $k = 2^p > 1$. But it is neither b-metric nor dislocated quasi metric.
Example. Let $X = \mathbb{R}$ and suppose

$$d(x, y) = |2x - y|^2 + |2x + y|^2.$$

Then (X, d) is a dislocated quasi$b$-metric space with the coefficient $k = 2$.
Remark. Like(Usual) dislocated quasi metric space in dislocated quasi b-metric space the distance between similar points need not to be zero necessarily as clear from the above examples.

Some more examples of dislocated quasi b-metric spaces can be constructed with the help of the following proposition.
Proposition. Let X be a non-empty set such that d^* is dq-metric and d^{**} is a b-metric with $k > 1$ on X. Then the function $d : X \times X \to [0, \infty)$ defined by

$$d(x, y) = d^*(x, y) + d^{**}(x, y) \quad \text{ for all } \quad x, y \in X$$

is dq b-metric on X.
proof. As (X, d^*) is dq-metric space and d^{**} is b-metric on X with $k > 1$, therefore (d_1) is clearly satisfied for the function d. To check (d_2), let $x, y \in X$ be arbitrary

elements, then we have

$$\begin{aligned} d(x,y) &= d^*(x,y)+d^{**}(x,y) \\ &\leq d^*(x,z)+d^*(z,y)+k[d^{**}(x,z)+d^{**}(z,y)] \\ &\leq k[d^*(x,z)+d^*(z,y)+d^{**}(x,z)+d^{**}(z,y)] \\ &= k[d(x,z)+d(z,y)]. \end{aligned}$$

Therefore (d_2) is satisfied and hence (X,d) is a dq b- metric space.
Now, we define Cauchy sequence, convergent sequence and completeness in the context of dislocated quasi b-metric spaces which can be carried over directly from conventional b-metric space.
Definition. A sequence $\{x_n\}$ is called dq $b-$convergent in X if for $n \geq N$ we have $d(x_n,x) < \epsilon$ where $\epsilon > 0$ then x is called the dq b-limit of the sequence $\{x_n\}$.
Definition. A sequence $\{x_n\}$ in dq $b-$metric space is called Cauchy sequence if for $\epsilon > 0$ there exist $n_0 \in N$, such that for $m,n \geq n_0$, we have $d(x_m,x_n) < \epsilon$.
Definition. A dq $b-$metric space (X,d) is said to be complete if every Cauchy sequence in X converge to a point of X.
Now, we begin with the following simple but useful lemmas.
Lemma. Limit of convergent sequence in dislocated quasi b-metric space is unique.
Proof. Let $\{x_n\}$ be a convergent sequence in dq b-metric space. Let $\{x_n\}$ have two limits x and y with $x \neq y$ by, using (d_2) in the definition of dq b-metric space we have,

$$d(x,y) \leq k[d(x,x_n)+d(x_n,y)]$$

taking limit $n \to \infty$ since x and y are limits of a convergent sequence $\{x_n\}$ therefore $\lim_{n\to\infty} d(x_n,x) = 0$ and $\lim_{n\to\infty} d(x_n,y) = 0$ which implies that $d(x,y) = 0$, similarly we can show that $d(y,x) = 0$ so by (d_1) in the definition of dq b-metric space $x = y$.
Hence limit in dislocated quasi b-metric space is unique.
Lemma. Let (X,d) be a dislocated quasi b-metric space and $\{x_n\}$ be a sequence in dqb-metric space such that

$$d(x_n,x_{n+1}) \leq \alpha d(x_{n-1},x_n) \tag{3}$$

for $n = 1,2,3,...$ and $0 \leq \alpha k < 1, \alpha \in [0,1)$, and k is defined in d_q b-metric space. Then $\{x_n\}$ is a Cauchy sequence in X.
Proof. Let for $n,m > 0$ and $m > n$. Applying (d_2) to triplets, (x_n,x_{n+1},x_m), (x_{n+1},x_{n+2},x_m),........., we obtain,

$$\begin{aligned} d(x_n,x_m) &\leq k[d(x_n,x_{n+1})+d(x_{n+1},x_{n+2}).................] \\ &\leq kd(x_n,x_{n+1})+k^2[d(x_{n+1},x_{n+2})+d(x_{n+2},x_{n+3}).........] \\ &\leq kd(x_n,x_{n+1})+k^2d(x_{n+1},x_{n+2})+k^3d(x_{n+2},x_{n+3})........... \end{aligned}$$

Now using (3) we get the following

$$\begin{aligned} d(x_n,x_m) &\leq k\alpha^n d(x_0,x_1)+k^2\alpha^{n+1}d(x_0,x_1)+k^3\alpha^{n+2}d(x_0,x_1)+........... \\ &\leq (1+k\alpha+(k\alpha)^2+........)k\alpha^n d(x_0,x_1). \\ &= k\alpha^n\left(\frac{1}{1-k\alpha}\right)d(x_0,x_1). \end{aligned}$$

Taking limit $m,n \to \infty$ we have

$$\lim_{m,n\to\infty} d(x_n,x_m) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in dislocated quasi b-metric space X.
Remark. In the above lemma if we put $k = 1$ then the lemma can be effectively used for (usual) dislocated quasi metric spaces.
Now we prove the famous Banach contraction principle in complete dislocated quasi b-metric space.
Theorem. Let (X, d) be a complete dislocated quasi b-metric space. Let $T : X \to X$ be a continuous contraction with $\alpha \in [0, 1)$ and $0 \leq k\alpha < 1$ where $k \geq 1$, then T has a unique fixed point in X.
Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n$$

consider,

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

Now clearly by above lemma $\{x_n\}$ is a Cauchy sequence in complete dq b-metric space so there exist $z \in X$ such that $\lim_{n\to\infty} x_n = z$, since every contraction is continuous so,

$$\lim_{n\to\infty} Tx_n = Tz$$

$$\lim_{n\to\infty} x_{n+1} = Tz.$$

Thus $Tz = z$. Hence z is the fixed point of T.

Uniqueness: Let z, t are two fixed points of T and T is contraction therefore for $z \neq t$ we have

$$d(z, t) = d(Tz, Tt) \leq \alpha d(z, t)$$

since $\alpha \in [0, 1)$ so the above inequality is possible if $d(z, t) = 0$. Similarly we can show that $d(t, z) = 0$. So by using (d_1) we have $z = t$. Therefore fixed point of T in X is unique.
Remark. The above theorem generalize the result of [11] in complete dq b-metric space.
Proposition. Let (X, d) be a complete dq b-metric space $T : X \to X$ be a self mapping and $T^n : X \to X$ is a contraction and all other conditions of Banach contraction theorem holds, then T has a unique fixed point in X.
Proof. Since $T^n : X \to X$ is contraction so by Banach contraction Theorem T^n has a unique fixed point in X i.e $T^n x = x$ for $x \in X$, we have

$$TT^n x = Tx$$

$$T^{n+1}x = Tx$$

$$T^n Tx = Tx.$$

Which implies that Tx is the fixed point of T^n but T^n has a unique fixed point. Therefore $Tx = x$. Hence T has a unique fixed point in X.
Remark. The above proposition generalizes some results in [3].
Kannan and Chetterjea type mappings ([], []) are not continuous in metric spaces. So they are not continuous in dq b-metric space. Therefore we enforce the condition of continuity in order to hold these theorems in dq b-metric space.
Theorem. Let (X, d) be a complete dq b-metric space with $k \geq 1$, let $T : X \to X$ be a continuous self mapping for $\delta \in [0, \frac{1}{2})$ satisfying the condition

$$d(Tx, Ty) \leq \delta[d(x, Tx) + d(y, Ty)] \tag{4}$$

$\forall \quad x, y \in X$. Then T has a unique fixed point in X.
proof. Let x_0 be arbitrary in X, we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n$$

consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n).$$

By using condition (4) we have

$$\begin{aligned} d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) &\leq \delta[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] \\ d(x_n, x_{n+1}) &\leq \delta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ d(x_n, x_{n+1}) &\leq \frac{\delta}{1-\delta}.d(x_{n-1}, x_n) \end{aligned}$$

since $\delta \in [0, \frac{1}{2})$, therefore let $h = \frac{\delta}{1-\delta} < 1$. Hence

$$\begin{aligned} d(x_n, x_{n+1}) &\leq h.d(x_{n-1}, x_n) \\ d(x_n, x_{n+1}) &\leq h^n.d(x_0, x_1) \end{aligned}$$

by lemma proved above we can say that $\{x_n\}$ is a Cauchy sequence in complete dq b-metric space X. So there exist $z \in X$ such that $\lim_{n\to\infty} x_n = z$ also since T is continuous so

$$\begin{aligned} \lim_{n\to\infty} Tx_n &= Tz \\ Tz &= z. \end{aligned}$$

Thus z is the fixed point of T in X. Now it is an easy job to see that the fixed point z is unique.
Our next theorem is about Chetterjea type fixed point theorem in the context of dislocated quasi b-metric space.
Theorem. Let (X, d) be a complete dq b-metric space and let $T : X \to X$ be a continuous self mapping with $k\beta \in [0, \frac{1}{4})$ and $k \geq 1$ such that

$$d(Tx, Ty) \leq \beta[d(x, Ty) + d(y, Tx)] \tag{5}$$

$\forall\ x, y \in X$, then T has a unique fixed point in X.
Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule

$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n$$

consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$

now by using equation (5) we have the following inequality

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})] \\ &\leq \beta d(x_{n-1}, x_{n+1}) + \beta d(x_n, x_n) \end{aligned}$$

using triangular inequality in both terms we have

$$d(x_n, x_{n+1}) \leq 2k\beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})].$$

which implies that

$$d(x_n, x_{n+1}) \leq \left(\frac{2k\beta}{1-2k\beta}\right) d(x_{n-1}, x_n)$$

since $k\beta < \frac{1}{4}$ therefore $\frac{2k\beta}{1-2k\beta} < 1$ let

$$h = \frac{2k\beta}{1-2k\beta}.$$

So we have
$$d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n).$$
Therefore by lemma proved above we can say that $\{x_n\}$ is a Cauchy sequence in complete dislocated quasi b-metric space so there exist $t \in X$ such that
$$\lim_{n\to\infty} x_n = t$$
also since T is continuous therefore
$$T \lim_{n\to\infty} x_n = Tt \Rightarrow \lim_{n\to\infty} Tx_n = Tt$$
$$\lim_{n\to} x_{n+1} = Tt \Rightarrow Tt = t.$$
Therefore t is the fixed point of T in X.

Uniqueness: Let t, u are two distinct fixed points of T in X then by condition (5) we have
$$d(t, u) = d(Tt, Tu) \leq \beta[d(t, Tu) + d(u, Tt)] = \beta d(t, Tu) + \beta d(u, Tt) \tag{6}$$
$$d(u, t) \leq \beta d(u, Tt) + \beta d(t, Tu) \tag{7}$$
subtracting (7) from (6) we have
$$|d(t, u) - d(u, t)| \leq |\beta - \beta|.|d(t, u) - d(u, t)|$$
which implies that
$$|d(t, u) - d(u, t)| = 0.$$
Thus
$$d(t, u) = d(u, t) \tag{8}$$
using (8) in (6) and (7) we have
$$d(t, u) = d(u, t) = 0 \Rightarrow t = u.$$
Therefore fixed point of T in X is unique.

Remark. In order to valid Chetterjea type fixed point theorem in dislocated quasi b-metric space we used the restriction $k\beta \in [0, \frac{1}{4})$.

In all of above theorems if z be the fixed point of T then $d(z, z) = 0$.

If $\{x_n\}$ is a convergent sequence in complete dq b-metric space having limit point z then
$$\lim_{n\to\infty} d(x_n, z) = \lim_{n\to\infty} d(x_n, x_m) = d(z, z) = 0.$$

Now we have to prove Banach fixed point theorem for strict contraction. An example follow showing that the additional condition of sequential compactness(compactness) cannot be omitted in such a case.

Theorem. Let (X, d) be a sequentially compact(compact) dq b-metric space and d satisfy (**??**) if $T : X \to X$ is a self mapping satisfying
$$d(Tx, Ty) < d(x, y) \quad \forall \quad x, y \in X, x \neq y \tag{9}$$
Then T has a unique fixed point in X.

Proof. Let x_0 be arbitrary in X we define a sequence $\{x_n\}$ in X by the rule,
$$x_0, x_1 = Tx_0, x_2 = Tx_1, ..., x_{n+1} = Tx_n$$
$$d_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) < d(x_n, x_{n+1}) = d_n. \tag{10}$$
Therefore d_n is a decreasing sequence bounded below by 0.

Hence there exist d_0, such that $d_n \to d_0$ as $n \to \infty$ now by using sequential compactness of X choosing a sub-sequence $\{x_{n_i}\}$ of $\{x_n\}$ that converge to some point $z \in X$, the continuity of T implies that

$$Tx_{n_i} \to Tz, \quad T^2 x_{n_i} \to T^2 z \quad as\ i \to \infty.$$

Now using definition of the above section which implies that

$$d(Tx_{n_i}, x_{n_i}) \to d(Tz, z) \quad as \quad i \to \infty$$

which implies that $d(Tx_{n_i}, x_{n_i}) = d_{n_i} \to d_0 = d(Tz, z)$. Now to show that $Tz = z$. Suppose $Tz \neq z$ then by (10),

$$d_0 = \lim_{i\to\infty} d_{n_i+1} = \lim_{i\to\infty} d(T^2 x_{n_i}, Tx_{n_i}) = d(T^2 z, Tz) < d(Tz, z) = d_0$$

which is a contradiction hence $Tz = z$.

Uniqueness: Let z, t be two fixed points of T in X then we have,

$$d(z, t) = d(Tz, Tt) < d(z, t)$$

which is a contradiction. Therefore the above inequality is possible if $d(z, t) = 0$, similarly we can show that $d(t, z) = 0$. Thus $d(z, t) = 0 = d(t, z) \Rightarrow \quad z = t$. Hence fixed point of T is unique.

Remark. The Theorem **??** implies that the fixed point of T if exist then it must be unique but the condition of sequential compactness(compactness) is necessary otherwise the fixed point does not exist as clear from the following example.

Example. Let $X = [1, \infty)$ and $d : X \times X \to [0, \infty)$ be defined by $d(x, y) = k|x - y|$. Then (X, d) is a dq b-metric space for $k \geq 1$. Let $T : X \to X$ be defined by $Tx = x + \frac{1}{x}$. Hence T satisfy (9) but it has no fixed point because (X, d) is not (sequentially) compact.

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