COMPARING THE INTEGRAL MEANS FOR FUNCTIONS OF TWO VARIABLES WITH BOUNDED VARIATION

HÜSEYIN BUDAK, MEHMET ZEKI SARIKAYA

ABSTRACT. In this paper, we obtain an inequality for difference of the integral means using functions of two variables with bounded variation. An application to probability density functions is also given.

1. INTRODUCTION

Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) whoose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e. $\|f'\|_\infty := \sup\limits_{t \in (a,b)} |f'(t)| < \infty$. Then we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty, \qquad (1)$$

for all $x \in [a, b]$[23]. The constant $\frac{1}{4}$ is the best possible. This inequality is well known in the literature as the *Ostrowski inequality.*

In [15], Dragomir proved following Ostrowski type inequalities related functions of bounded variation:

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be a mapping of bounded variation on $[a, b]$. Then*

$$\left| \int_a^b f(t)dt - (b-a) f(x) \right| \leq \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right] \bigvee_a^b (f)$$

holds for all $x \in [a, b]$. The constant $\frac{1}{2}$ is the best possible.

Hwang and Dragomir gave the following inequality in [21].

Theorem 2. *Let $f : [a, b] \to \mathbb{R}$ be a mapping with bounded variation on $[a, b]$. Then, for $a \le x < y \le b$, we have the inequality*

$$\left| \frac{1}{(b-a)} \int_a^b f(s)ds - \frac{1}{(y-x)} \int_x^y f(s)ds \right| \tag{2}$$

$$\le \quad \frac{x-a}{b-a} \bigvee_a^x (f) + \max \left\{ \frac{x-a}{b-a}, \frac{b-y}{b-a} \right\} \bigvee_x^y (f) + \frac{b-y}{b-a} \bigvee_y^b (f)$$

The inequality (2) is sharp.

Definition 1. *(Vitali-Lebesque-Fréchet-de la Vallée Poussin)*[12]. *We introduce the notation*

$$\Delta_{11} f(x_i, y_j) = f(x_{i+1}, y_{j+1}) - f(x_{i+1}, y_j) - f(x_i, y_{j+1}) + f(x_i, y_j),$$

then, the function $f(x, y)$ is said tobe of bounded variation if the sum

$$\sum_{i=0 \, , \, j=0}^{m-1 \, , \, n-1} |\Delta_{11} f(x_i, y_j)|$$

is bounded for all nets.

Therefore, one can define the consept of total variation of a function of two variables, as follows:

Let f be of bounded variation on $Q = [a, b] \times [c, d]$, and let $\sum(P)$ denote the sum $\sum_{i=1}^{n} \sum_{j=1}^{m} |\Delta_{11} f(x_i, y_j)|$ corresponding to the partition P of Q. The number

$$\bigvee_Q (f) := \bigvee_c^d \bigvee_a^b (f) := \sup \left\{ \sum (P) : P \in P(Q) \right\},$$

is called the total variation of f on Q. Here $P([a, b])$ denotes the family of partitions of $[a, b]$.

In [20], Jawarneh and Noorani gave the following Lemmas for double Riemann-Stieltjes integral:

Lemma 1. *(Integrating by parts) If $f \in RS(\alpha)$ on Q, then $\alpha \in RS(f)$ on Q, and we have*

$$\int_c^d \int_a^b f(t, s) d_t d_s \alpha(t, s) + \int_c^d \int_a^b \alpha(t, s) d_t d_s f(t, s) \tag{3}$$

$$= \quad f(b, d)\alpha(b, d) - f(b, c)\alpha(b, c) - f(a, d)\alpha(a, d) + f(a, c)\alpha(a, c).$$

Lemma 2. *Assume that $g \in RS(\alpha)$ on Q and α is of bounded variation on Q, then*

$$\left| \int_c^d \int_a^b g(x, y) d_x d_y \alpha(x, y) \right| \le \sup_{(x,y) \in Q} |g(x, y)| \bigvee_Q (\alpha). \tag{4}$$

In [20], Jawarneh and Noorani proved the following Ostrowski type inequality or functions of two variables with bounded variation:

Theorem 3. *Let* $f : Q \to\to \mathbb{R}$ *be mapping of bounded variation on* Q. *Then for all* $(x, y) \in Q$, *we have inequality*

$$\left| (b-a)(d-c) f(x,y) - \int\limits_c^d \int\limits_a^b f(t,s) dt ds \right| \tag{5}$$

$$\leq \quad \left[\frac{1}{2}(b-a) + \left| x - \frac{a+b}{2} \right| \right]$$

$$\times \left[\frac{1}{2}(d-c) + \left| y - \frac{c+d}{2} \right| \right] \bigvee_Q (f)$$

where $\bigvee\limits_Q (f)$ *denotes the total (double) variation of* f *on* Q.

For more information and recent developments on integral inequalities for mappings of bounded variation (single variable and two variables), please refer to ([1]-[11], [13]-[22],[24]-[29]).

In this paper, we compare the integral means using functions of two variables with bounded variation. A application to probability density functions is also given.

2. MAIN RESULTS

First, we give the following notations to simplify presentetion of some intervals.

$$Q_1 \quad = \quad [a, x_1] \times [c, y_1], \ Q_2 = [a, x_1] \times [y_1, y_2], \ Q_3 = [a, x_1] \times [y_2, d],$$

$$Q_4 \quad = \quad [x_1, x_2] \times [c, y_1], \ Q_5 = [x_1, x_2] \times [y_1, y_2], \ Q_6 = [x_1, x_2] \times [y_2, d],$$

$$Q_7 \quad = \quad [x_2, b] \times [c, y_1], \ Q_8 = [x_1, x_2] \times [y_1, y_2], \ Q_9 = [x_2, b] \times [y_2, d].$$

Theorem 4. *If the function* $f : Q = [a, b] \times [c, d] \to R$ *is of bounded variation on* Q, *then, for* $a \leq x_1 < x_2 \leq b$ *and* $c \leq y_1 < y_2 \leq d$, *we have the inequality*

$$\left| \frac{1}{(b-a)(d-c)} \int\limits_a^b \int\limits_c^d f(t,s) ds dt - \frac{1}{(b-a)(y_2-y_1)} \int\limits_a^b \int\limits_{y_1}^{y_2} f(t,s) ds dt \right. \tag{6}$$

$$\left. - \frac{1}{(x_2-x_1)(d-c)} \int\limits_{x_1}^{x_2} \int\limits_c^d f(t,s) ds dt + \frac{1}{(x_2-x_1)(y_2-y_1)} \int\limits_{x_1}^{x_2} \int\limits_{y_1}^{y_2} f(t,s) ds dt \right|$$

$$\leq \quad \frac{[(b-a)-(x_2-x_1)][(d-c)-(y_2-y_1)]}{(b-a)(d-c)} \left[\frac{1}{2} + \left| \frac{\frac{a+b}{2} - \frac{x_2+x_1}{2}}{(b-a)-(x_2-x_1)} \right| \right]$$

$$\times \left[\frac{1}{2} + \left| \frac{\frac{d+c}{2} - \frac{y_2+y_1}{2}}{(d-c)-(y_2-y_1)} \right| \right] \bigvee_a^b \bigvee_c^d (f).$$

where $\bigvee\limits_a^b \bigvee\limits_c^d (f)$ *denotes the total (double) variation of* f *on* Q.

Proof. First, we define the mappings $K_{x_1,x_2}(t)$ and $L_{y_1,y_2}(s)$ by

$$K_{x_1,x_2}(t) = \begin{cases} \frac{a-t}{b-a}, & \text{if } t \in [a, x_1] \\ \frac{t-x_1}{x_2-x_1} + \frac{a-t}{b-a}, & \text{if } t \in (x_1, x_2] \\ \frac{b-t}{b-a}, & \text{if } t \in (x_2, b] \end{cases}$$

and

$$L_{y_1,y_2}(s) = \begin{cases} \frac{c-s}{d-c}, & \text{if } s \in [c, y_1] \\ \frac{s-y_1}{y_2-y_1} + \frac{c-s}{d-c}, & \text{if } s \in (y_1, y_2] \\ \frac{d-s}{d-c}, & \text{if } s \in (y_2, d] \end{cases} \quad.$$

Using the kernels $K_{x_1,x_2}(t)$ and $L_{y_1,y_2}(s)$, we have

$$\int_a^b \int_c^d K_{x_1,x_2}(t) L_{y_1,y_2}(s) d_s d_t f(t,s) \tag{7}$$

$$= \int_a^{x_1} \int_c^{y_1} \frac{a-t}{b-a} \frac{c-s}{d-c} d_s d_t f(t,s) + \int_a^{x_1} \int_{y_1}^{y_2} \frac{a-t}{b-a} \left[\frac{s-y_1}{y_2-y_1} + \frac{c-s}{d-c} \right] d_s d_t f(t,s)$$

$$+ \int_a^{x_1} \int_{y_2}^d \frac{a-t}{b-a} \frac{d-s}{d-c} d_s d_t f(t,s) + \int_{x_1}^{x_2} \int_c^{y_1} \left[\frac{t-x_1}{x_2-x_1} + \frac{a-t}{b-a} \right] \frac{c-s}{d-c} d_s d_t f(t,s)$$

$$+ \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left[\frac{t-x_1}{x_2-x_1} + \frac{a-t}{b-a} \right] \left[\frac{s-y_1}{y_2-y_1} + \frac{c-s}{d-c} \right] d_s d_t f(t,s)$$

$$+ \int_{x_1}^{x_2} \int_{y_2}^d \left[\frac{t-x_1}{x_2-x_1} + \frac{a-t}{b-a} \right] \frac{d-s}{d-c} d_s d_t f(t,s) + \int_{x_2}^b \int_c^{y_1} \frac{b-t}{b-a} \frac{c-s}{d-c} d_s d_t f(t,s)$$

$$+ \int_{x_2}^b \int_{y_1}^{y_2} \frac{b-t}{b-a} \left[\frac{s-y_1}{y_2-y_1} + \frac{c-s}{d-c} \right] d_s d_t f(t,s) + \int_{x_2}^b \int_{y_2}^d \frac{b-t}{b-a} \frac{d-s}{d-c} d_s d_t f(t,s)$$

$$= K_1 + K_2 + ... + K_9.$$

Integrating the by parts, we have

$$K_1 = \int_a^{x_1} \int_c^{y_1} \frac{a-t}{b-a} \frac{c-s}{d-c} d_s d_t f(t,s) \tag{8}$$

$$= \frac{1}{(b-a)(d-c)} \left[(a-x_1)(c-y_1) f(x_1, y_1) - \int_a^{x_1} \int_c^{y_1} f(t,s) ds dt \right].$$

Similarly, we have

$$
K_2 \;=\; \int\limits_{a}^{x_1}\int\limits_{y_1}^{y_2} \frac{a-t}{b-a}\left[\frac{s-y_1}{y_2-y_1}+\frac{c-s}{d-c}\right]d_s d_t f(t,s) \tag{9}
$$

$$
=\; \frac{a-x_1}{b-a}f(x_1,y_2)+\frac{1}{(b-a)(y_2-y_1)}\int\limits_{a}^{x_1}\int\limits_{y_1}^{y_2}f(t,s)dsdt
$$

$$
+\frac{1}{(b-a)(d-c)}\Big[(a-x_1)(c-y_2)f(x_1,y_2)
$$

$$
-(a-x_1)(c-y_1)f(x_1,y_1)-\int\limits_{a}^{x_1}\int\limits_{y_1}^{y_2}f(t,s)dsdt\Big],
$$

$$
K_3 \;=\; \int\limits_{a}^{x_1}\int\limits_{y_2}^{d} \frac{a-t}{b-a}\frac{d-s}{d-c}d_s d_t f(t,s) \tag{10}
$$

$$
=\; \frac{1}{(b-a)(d-c)}\left[-(a-x_1)(d-y_2)f(x_1,y_2)-\int\limits_{a}^{x_1}\int\limits_{y_2}^{d}f(t,s)dsdt\right],
$$

$$
K_4 \;=\; \int\limits_{x_1}^{x_2}\int\limits_{c}^{y_1} \left[\frac{t-x_1}{x_2-x_1}+\frac{a-t}{b-a}\right]\frac{c-s}{d-c}d_s d_t f(t,s) \tag{11}
$$

$$
=\; \frac{c-y_1}{d-c}f(x_2,y_1)+\frac{1}{(x_2-x_1)(d-c)}\int\limits_{x_1}^{x_2}\int\limits_{c}^{y_1}f(t,s)dsdt
$$

$$
+\frac{1}{(b-a)(d-c)}\Big[(a-x_2)(c-y_1)f(x_2,y_1)
$$

$$
-(a-x_1)(c-y_1)f(x_1,y_1)-\int\limits_{x_1}^{x_2}\int\limits_{c}^{y_1}f(t,s)dsdt\Big],
$$

$$
\tag{12}
$$

$$
K_5 \;=\; \int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2} \left[\frac{t-x_1}{x_2-x_1}+\frac{a-t}{b-a}\right]\left[\frac{s-y_1}{y_2-y_1}+\frac{c-s}{d-c}\right]d_s d_t f(t,s)
$$

$$
=\; f(x_2,y_2)-\frac{1}{(x_2-x_1)(y_2-y_1)}\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}f(t,s)dsdt
$$

$$+\frac{c-y_2}{d-c}f(x_2,y_2) - \frac{c-y_1}{d-c}f(x_2,y_1) + \frac{1}{(x_2-x_1)(d-c)}\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}f(t,s)dsdt$$

$$+\frac{a-x_2}{b-a}f(x_2,y_2) - \frac{a-x_1}{b-a}f(x_1,y_2) + \frac{1}{(b-a)(y_2-y_1)}\int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}f(t,s)dsdt$$

$$+\frac{1}{(b-a)(d-c)}\left[(a-x_2)(c-y_2)f(x_2,y_2) - (a-x_1)(c-y_2)f(x_1,y_2)\right.$$

$$\left.-(a-x_2)(c-y_1)f(x_2,y_1) + (a-x_1)(c-y_1)f(x_1,y_1) - \int\limits_{x_1}^{x_2}\int\limits_{y_1}^{y_2}f(t,s)dsdt\right],$$

$$K_6 = \int\limits_{x_1}^{x_2}\int\limits_{y_2}^{d}\left[\frac{t-x_1}{x_2-x_1}+\frac{a-t}{b-a}\right]\frac{d-s}{d-c}d_sd_tf(t,s) \tag{13}$$

$$= -\frac{d-y_2}{d-c}f(x_2,y_2) + \frac{1}{(x_2-x_1)(d-c)}\int\limits_{x_1}^{x_2}\int\limits_{y_2}^{d}f(t,s)dsdt$$

$$+\frac{1}{(b-a)(d-c)}\left[-(a-x_2)(d-y_2)f(x_2,y_2)\right.$$

$$\left.+(a-x_1)(d-y_2)f(x_1,y_2) - \int\limits_{x_1}^{x_2}\int\limits_{y_2}^{d}f(t,s)dsdt\right],$$

$$K_7 = \int\limits_{x_2}^{b}\int\limits_{c}^{y_1}\frac{b-t}{b-a}\frac{c-s}{d-c}d_sd_tf(t,s) \tag{14}$$

$$= \frac{1}{(b-a)(d-c)}\left[-(b-x_2)(c-y_1)f(x_2,y_1) - \int\limits_{x_2}^{b}\int\limits_{c}^{y_1}f(t,s)dsdt\right],$$

$$K_8 = \int\limits_{x_2}^{b}\int\limits_{y_1}^{y_2}\frac{b-t}{b-a}\left[\frac{s-y_1}{y_2-y_1}+\frac{c-s}{d-c}\right]d_sd_tf(t,s) \tag{15}$$

$$= -\frac{b-x_2}{b-a}f(x_2,y_2) + \frac{1}{(b-a)(y_2-y_1)}\int\limits_{x_2}^{b}\int\limits_{y_1}^{y_2}f(t,s)dsdt$$

$$+\frac{1}{(b-a)(d-c)}\left[-(b-x_2)(c-y_2)f(x_2,y_2)\right.$$

$$\left.+(b-x_2)(c-y_1)f(x_2,y_1)-\int_{x_2}^{b}\int_{y_1}^{y_2}f(t,s)dsdt\right]$$

and

$$K_9 = \int_{x_2}^{b}\int_{y_2}^{d}\frac{b-t}{b-a}\frac{d-s}{d-c}d_sd_tf(t,s) \tag{16}$$

$$= \frac{1}{(b-a)(d-c)}\left[(b-x_2)(d-y_2)f(x_2,y_2)-\int_{x_2}^{b}\int_{y_2}^{d}f(t,s)dsdt\right].$$

If we add the equalities (8)-(16) in (7), then we have

$$\int_{a}^{b}\int_{c}^{d}K_{x_1,x_2}(t)L_{y_1,y_2}(s)d_sd_tf(t,s)$$

$$= \frac{1}{(b-a)(y_2-y_1)}\int_{a}^{b}\int_{y_1}^{y_2}f(t,s)dsdt + \frac{1}{(x_2-x_1)(d-c)}\int_{x_1}^{x_2}\int_{c}^{d}f(t,s)dsdt$$

$$-\frac{1}{(b-a)(d-c)}\int_{a}^{b}\int_{c}^{d}f(t,s)dsdt - \frac{1}{(x_2-x_1)(y_2-y_1)}\int_{x_1}^{x_2}\int_{y_1}^{y_2}f(t,s)dsdt.$$

On the other hand, using Lemma 2, we get

$$\left|\int_{a}^{b}\int_{c}^{d}K_{x_1,x_2}(t)L_{y_1,y_2}(s)d_sd_tf(t,s)\right|$$

$$\leq \left|\int_{a}^{x_1}\int_{c}^{y_1}\frac{a-t}{b-a}\frac{c-s}{d-c}d_sd_tf(t,s)\right| + \left|\int_{a}^{x_1}\int_{y_1}^{y_2}\frac{a-t}{b-a}\left[\frac{s-y_1}{y_2-y_1}+\frac{c-s}{d-c}\right]d_sd_tf(t,s)\right|$$

$$+\left|\int_{a}^{x_1}\int_{y_2}^{d}\frac{a-t}{b-a}\frac{d-s}{d-c}d_sd_tf(t,s)\right| + \left|\int_{x_1}^{x_2}\int_{c}^{y_1}\left[\frac{t-x_1}{x_2-x_1}+\frac{a-t}{b-a}\right]\frac{c-s}{d-c}d_sd_tf(t,s)\right|$$

$$+\left|\int_{x_1}^{x_2}\int_{y_1}^{y_2}\left[\frac{t-x_1}{x_2-x_1}+\frac{a-t}{b-a}\right]\left[\frac{s-y_1}{y_2-y_1}+\frac{c-s}{d-c}\right]d_sd_tf(t,s)\right|$$

$$+\left|\int_{x_1}^{x_2}\int_{y_2}^{d}\left[\frac{t-x_1}{x_2-x_1}+\frac{a-t}{b-a}\right]\frac{d-s}{d-c}d_sd_tf(t,s)\right| + \left|\int_{x_2}^{b}\int_{c}^{y_1}\frac{b-t}{b-a}\frac{c-s}{d-c}d_sd_tf(t,s)\right|$$

$$+\left|\int_{x_2}^{b}\int_{y_1}^{y_2}\frac{b-t}{b-a}\left[\frac{s-y_1}{y_2-y_1}+\frac{c-s}{d-c}\right]d_sd_tf(t,s)\right|+\left|\int_{x_2}^{b}\int_{y_2}^{d}\frac{b-t}{b-a}\frac{d-s}{d-c}d_sd_tf(t,s)\right|$$

$$\leq\quad\frac{x_1-a}{b-a}\frac{y_1-c}{d-c}\bigvee_{Q_1}(f)+\frac{x_1-a}{b-a}\max\left\{\frac{y_1-c}{d-c},\frac{d-y_2}{d-c}\right\}\bigvee_{Q_2}(f)$$

$$+\frac{x_1-a}{b-a}\frac{d-y_2}{d-c}\bigvee_{Q_3}(f)+\max\left\{\frac{x_1-a}{b-a},\frac{b-x_1}{b-a}\right\}\frac{y_1-c}{d-c}\bigvee_{Q_4}(f)$$

$$+\max\left\{\frac{x_1-a}{b-a},\frac{b-x_1}{b-a}\right\}\max\left\{\frac{y_1-c}{d-c},\frac{d-y_2}{d-c}\right\}\bigvee_{Q_5}(f)$$

$$+\max\left\{\frac{x_1-a}{b-a},\frac{b-x_1}{b-a}\right\}\frac{d-y_2}{d-c}\bigvee_{Q_6}(f)+\frac{b-x_1}{b-a}\frac{y_1-c}{d-c}\bigvee_{Q_7}(f)$$

$$+\frac{b-x_1}{b-a}\max\left\{\frac{y_1-c}{d-c},\frac{d-y_2}{d-c}\right\}\bigvee_{Q_8}(f)+\frac{b-x_1}{b-a}\frac{d-y_2}{d-c}\bigvee_{Q_9}(f)$$

$$\leq\quad\max\left\{\frac{x_1-a}{b-a},\frac{b-x_1}{b-a}\right\}\max\left\{\frac{y_1-c}{d-c},\frac{d-y_2}{d-c}\right\}\bigvee_{a}^{b}\bigvee_{c}^{d}(f)$$

$$=\quad\frac{[(b-a)-(x_2-x_1)][(d-c)-(y_2-y_1)]}{(b-a)(d-c)}\left[\frac{1}{2}+\left|\frac{\frac{a+b}{2}-\frac{x_2+x_1}{2}}{(b-a)-(x_2-x_1)}\right|\right]$$

$$\times\left[\frac{1}{2}+\left|\frac{\frac{d+c}{2}-\frac{y_2+y_1}{2}}{(d-c)-(y_2-y_1)}\right|\right]\bigvee_{a}^{b}\bigvee_{c}^{d}(f)$$

which completes the proof. □

2.1. **Applications for Probability Density Functions.** Let X and Y be two continuous random variables with $f:[a,b]\times[c,d]\to\mathbb{R}^+$ is a joint probability density function and $F:[a,b]\times[c,d]\to\mathbb{R}^+$, $F(t,s)=\int_{a}^{t}\int_{c}^{s}f(t,s)dsdt$ is its cumulative distribution function.

Proposition 1. *Let f and F be as above. Then we have*

$$\left|F(t,s)-\frac{t-a}{b-a}\int_{a}^{b}\int_{c}^{s}f(\xi,\eta)d\eta d\xi-\frac{s-c}{d-c}\int_{a}^{t}\int_{c}^{d}f(\xi,\eta)d\eta d\xi+\frac{(t-a)(s-c)}{(b-a)(d-c)}\right|$$

$$\leq\quad\frac{(b-t)(t-a)(d-s)(s-c)}{(b-a)(d-c)}\bigvee_{a}^{b}\bigvee_{c}^{d}(f).$$

Proof. Taking $x_1=a$, $x_2=t$, $y_1=c$, $y_2=s$ and using the fact that $\int_{a}^{b}\int_{c}^{d}f(\xi,\eta)d\eta d\xi=1$, we have the required result. □

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Hüseyin BUDAK, Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: hsyn.budak@gmail.com

Mehmet Zeki SARIKAYA, Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: sarikayamz@gmail.com