Electronic Journal of Mathematical Analysis and Applications
Vol. 4(2) July 2016, pp. 107-128.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

GLOBAL EXISTENCE AND STABILITY OF SOLUTIONS FOR A NONLINEAR WAVE EQUATION

GIAI GIANG VO

Abstract. The paper is devoted to the study of a nonlinear wave equation with mixed nonhomogeneous boundary conditions. Existence of weak solutions is proved by applying the Galerkin method associated with a priori estimates, weak convergence and compactness techniques. Uniqueness and stability of the solutions are also established.

1. Introduction

In this paper, we consider the following nonlinear wave equation with mixed nonhomogeneous boundary conditions

$$u_{tt} - u_{xx} + f(x, u, u_t) = 0,\ 0 < x < 1,\ 0 < t < T, \tag{1.1}$$

$$u(1, t) = 0, \tag{1.2}$$

$$u_x(0, t) = \int_0^t g(t-s)u(0, s)ds + h(u(0, t)) + k(t), \tag{1.3}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.4}$$

where u_0, u_1, f, g, h, k are given functions satisfying conditions specified later.

The problems of wave equations have been studied by many authors, we can see in the works $[1, 2, 4-6, 8-15]$. Below are some typical works.

A. Dang and D. Alain [4] studied a special case of the problem (1.1)-(1.4) with $g = h = 0$ and $f(x, u, u_t) = |u_t|^{p-2}u_t,\ 1 < p < 2$.

In [13], J. Rivera and D. Andrade gave the global existence and exponential decay of the solutions of the wave equation with a viscoelastic boundary condition

$$u_{tt} - (\rho(u_x))_x + f(x, t) = 0, \tag{1.5}$$

$$u(0, t) = 0,\ u(1, t) = \int_0^t g(t-s)\rho(u_x(1, s))ds + k(t), \tag{1.6}$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \tag{1.7}$$

2010 *Mathematics Subject Classification.* 34A12, 34A30, 34D20.
Key words and phrases. Galerkin method, Global solution, Nonlinear wave equation, Integral equation, Stability of the solutions.
Submitted Nov. 10, 2015.

where u_0, u_1, f, g, k, ρ are given functions. In this case, the problem (1.5)-(1.7) is a mathematical model for a nonlinear one-dimensional motion of an elastic bar connected with a viscoelastic element at an end of the bar.

In [5], N.T. Le et al. considered the following problem

$$u_{tt} - (\rho(x,t)u_x)_x + f(u) + \lambda u_t = 0, \tag{1.8}$$

$$\rho(0,t)u_x(0,t) = \int_0^t g_0(t-s)u(0,s)ds + k_0(t), \tag{1.9}$$

$$-\rho(1,t)u_x(1,t) = \int_0^t g_1(t-s)u(1,s)ds + k_1(t), \tag{1.10}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.11}$$

where λ is a constant and u_0, u_1, f, g_0, g_1, k_0, k_1, ρ are given functions.

In [10], Nguyen and Giang Vo obtained the asymptotic expansion of the weak solution of the following problem in four small parameters $(K, \lambda, h, \varepsilon)$

$$u_{tt} - u_{xx} + Ku + \lambda u_t + f(x,t) = 0, \tag{1.12}$$

$$u(1,t) = 0, \tag{1.13}$$

$$u_x(0,t) = \int_0^t g(t-s)u(0,s)ds + hu(0,t) + \varepsilon u_t(0,t) + k(t), \tag{1.14}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.15}$$

where K, λ, h, ε are given positive constants and u_0, u_1, f, g, k are given functions.

M. Bergounioux et al. [2] established the global existence of weak solutions of the linear wave equation given by

$$u_{tt} - u_{xx} + Ku + \lambda u_t + f(x,t) = 0, \tag{1.16}$$

$$u_x(0,t) = v(t), \tag{1.17}$$

$$u_x(1,t) = -\delta u(1,t) - \varepsilon u_t(1,t), \tag{1.18}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \tag{1.19}$$

where K, λ, δ, ε are positive constants and u_0, u_1, f, k are given functions. Also, the unknown function $u(x,t)$ and the unknown boundary value $v(t)$ satisfy the following Cauchy problem for an ordinary differential equation

$$\begin{cases} v''(t) + \mu^2 v(t) = hu_{tt}(0,t), \\ v(0) = v_0, \quad v'(0) = v_1, \end{cases} \tag{1.20}$$

where $\mu > 0$, $h \geq 0$, v_0, v_1 are constants. It is worth noting that the equation (1.20) is equivalent to

$$v(t) = \int_0^t g(t-s)u(0,s)ds + hu(0,t) + k(t), \tag{1.21}$$

where

$$\begin{cases} g(t) = -h\mu \sin \mu t, \\ k(t) = [v_1 - hu_1(0)] \frac{\sin \mu t}{\mu} + [v_0 - hu_0(0)] \cos \mu t. \end{cases} \tag{1.22}$$

In this paper, we consider two main parts. In Part 1, we prove the existence and uniqueness of the weak solution of the problem (1.1)-(1.4). The proof is based on the Galerkin method and the weak compact method associated with a monotone operator. Finally, in Part 2, we prove that the solution of this problem is stable with respect to the data. These results are considered as the relative generalization of the works [4, 9, 14, 15].

2. The existence and uniqueness of solutions

Firstly, we denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively the scalar product and the norm in $L^2(0,1)$.

Let $u(t)$, $u'(t)=u_t(t)$, $u''(t)=u_{tt}(t)$, $u_x(t)$ and $u_{xx}(t)$ denote $u(x,t)$, $\frac{\partial u}{\partial t}(x,t)$, $\frac{\partial^2 u}{\partial t^2}(x,t)$, $\frac{\partial u}{\partial x}(x,t)$ and $\frac{\partial^2 u}{\partial x^2}(x,t)$ respectively.

Next, we define a closed subspace of the Sobolev space $H^1(0,1)$ as follows

$$H=\left\{u\in H^1(0,1):\, u(1)=0\right\}, \tag{2.1}$$

with the following scalar product and norm

$$\langle u,v\rangle_H=\langle u_x,v_x\rangle \quad \text{and} \quad \|u\|_H=\|u_x\|. \tag{2.2}$$

We have the following results

Lemma 2.1. *The embedding $H\hookrightarrow C^0([0,1])$ is compact and*

$$\|u\|_{C^0([0,1])}\le\|u\|_H\le\|u\|_{H^1(0,1)}\le\sqrt{2}\|u\|_H,\ \textit{for all}\ \ u\in H. \tag{2.3}$$

Lemma 2.2. *Let $\varepsilon>0$. Then*

$$\|v\|^2_{C^0([0,1])}\le\varepsilon\|v_x\|^2+\left(1+\frac{1}{\varepsilon}\right)\|v\|^2,\ \textit{for all}\ \ u\in H^1(0,1). \tag{2.4}$$

The proofs of these lemmas are simple, we omit the details.

We make the following assumptions

- (A_1) $u_0\in H$ and $u_1\in L^2(0,1)$,
- (A_2) $g,\,k\in W^{1,1}(0,T)$,
- (A_3) $h\in C^2(\mathbb{R})$ and there exist positive constants a, b such that
$$\int_0^u h(z)dz\ge -au^2-b,\ \text{for all}\ u\in\mathbb{R},$$
- (A_4) The function $f\in C^0([0,1]\times\mathbb{R}^2)$ satisfies the following condition
$$[f(x,u,v_1)-f(x,u,v_2)](v_1-v_2)\ge 0,\ \text{for all}\ x\in[0,1]\ \text{and}\ u,v_1,v_2\in\mathbb{R},$$
- (A_5) There exist a constant $p>1$ and positive functions $\widehat{f}\in C^0([0,1]\times\mathbb{R})$, $p_1,\,p_2\in C^0([0,1])$, $q_1,\,q_2,\,r_1\in L^1(0,1)$ such that
  - (i) $\int_0^u\widehat{f}(x,z)dz\ge -p_1(x)|u|^p-q_1(x)u^2-r_1(x)$, a.e. $x\in[0,1]$ and $u\in\mathbb{R}$,
  - (ii) $[f(x,u,v)-\widehat{f}(x,u)]v\ge p_2(x)|v|^p-q_2(x)$, a.e. $x\in[0,1]$ and $u,\,v\in\mathbb{R}$,
- (A_6) For every $M>0$, there exist positive functions $r_M\in C^0([0,1]\times\mathbb{R})$ and $(p_M,q_M)\in C^0([0,1])\times L^{p'}(0,1)$, $p'=p/(p-1)$, such that
  - (i) $|f(x,u,v)|\le p_M(x)|v|^{p-1}+q_M(x)$, a.e. $x\in[0,1]$, $u\in[-M,M]$ and $v\in\mathbb{R}$,
  - (ii) $|f(x,u_1,v)-f(x,u_2,v)|\le r_M(x,v)\,|u_1-u_2|$, a.e. $x\in[0,1]$, $u_i\in[-M,M]$ and $v\in\mathbb{R}$, $i=1,2$,
  - (iii) $r_M(x,v)\in L^1(0,T;L^2(0,1))$, for all $x\in[0,1]$ and $v\in L^\infty(0,T;L^2(0,1))$.

Remark 2.3. We consider the following functions

$$\begin{cases} f(x,u,v)=|v|^{p-2}v+\alpha(x)|u|^\delta|v|^{q-2}v+\beta(x)|u|^{r-2}u+\gamma(x),\\ \widehat{f}(x,u)=\beta(x)|u|^{r-2}u+\gamma(x),\end{cases}$$

where p, q, r, δ are constants, with $r \geq 2$, $\delta \geq 1$, $1 < q \leq \min\{p, 2\}$ and α, β, γ are nonnegative continuous functions on $[0, 1]$. Then

$$\begin{gathered}[f(x,u,v_1) - f(x,u,v_2)](v_1 - v_2) = (|v_1|^{p-2}v_1 - |v_2|^{p-2}v_2)(v_1 - v_2) \\ + \alpha(x)|u|^{\delta}(|v_1|^{q-2}v_1 - |v_2|^{q-2}v_2)(v_1 - v_2) \geq 0, \\ \int_0^u \widehat{f}(x,z)dz = \beta(x)\frac{|u|^r}{r} + \gamma(x)u \geq -\frac{1}{2}u^2 - \frac{1}{2}\gamma^2(x), \\ [f(x,u,v) - \widehat{f}(x,u)]v = |v|^p + \alpha(x)|u|^{\delta}|v|^q \geq |v|^p,\end{gathered}$$

for all $x \in [0, 1]$ and $u, v, v_1, v_2 \in \mathbb{R}$. Also, we have

$$\begin{gathered}|f(x,u,v)| \leq [M^{\delta}\alpha(x) + 1]|v|^{p-1} + [M^{\delta}\alpha(x) + M^{r-1}\beta(x) + \gamma(x)], \\ |f(x,u_1,v) - f(x,u_2,v)| \leq [\delta M^{\delta-1}\alpha(x)|v|^{q-1} + rM^{r-2}\beta(x)]\,|u_1 - u_2|\,,\end{gathered}$$

for all $x \in [0, 1]$, $u, u_1, u_2 \in [-M, M]$ and $v \in \mathbb{R}$.
Therefore f satisfies the assumptions (A_4)-(A_6).

Under the above assumptions, we obtain the following theorem

Theorem 2.4. *Let the assumptions* (A_1)-(A_6) *hold. Then the problem* (1.1)-(1.4) *has a unique weak solution* u *such that*

$$\begin{gathered}u \in L^{\infty}(0, T; H),\ u_t \in L^{\infty}(0, T; L^2(0, 1)) \cap L^p((0, 1) \times (0, T)), \\ u(0, \cdot) \in W^{1,\min\{p',2\}}(0, T),\ p' = p/(p - 1).\end{gathered} \tag{2.5}$$

Proof of Theorem 2.4. The procedure includes four steps as follows.
Step 1. Galerkin approximation(see [7]). We use a special orthonormal base of H

$$\varphi_k(x) = \sqrt{2/(1 + \mu_k^2)}\cos(\mu_i x), \quad \mu_k = (2k - 1)\frac{\pi}{2},\ k = 1,\ 2, ... \tag{2.6}$$

We find the approximate solution of the problem (1.1)-(1.4) in the form

$$u_m(x, t) = \sum_{k=1}^{m} \omega_{mk}(t)\varphi_k(x), \tag{2.7}$$

where the coefficient functions $\omega_{mk}(t)$ satisfy the following system of nonlinear differential equations

$$\langle u_m''(t), \varphi_k\rangle + \langle u_{mx}(t), \varphi_{kx}\rangle + v_m(t)\varphi_k(0) + \langle f(\cdot, u_m(t), u_m'(t)), \varphi_k\rangle = 0, \tag{2.8}$$

$$v_m(t) = \int_0^t g(t - s)u_m(0, s)ds + h(u_m(0, t)) + k(t),\ k = \overline{1, m}, \tag{2.9}$$

with the initial conditions

$$\begin{cases} u_m(0) = u_{0m} = \sum\limits_{k=1}^{m} a_{mk}\varphi_k \to u_0 \text{ strongly in } H^1(0, 1), \\ u_m'(0) = u_{1m} = \sum\limits_{k=1}^{m} b_{mk}\varphi_k \to u_1 \text{ strongly in } L^2(0, 1). \end{cases} \tag{2.10}$$

Therefore, the system of the equations is written in the form

$$\omega''_{mk}(t)+\mu_k^2\omega_{mk}(t)=-\frac{1}{\|\varphi_k\|^2}\left[v_m(t)\varphi_k(0)+\langle f(\cdot,u_m(t),u'_m(t)),\varphi_k\rangle\right], \tag{2.11}$$

$$v_m(t)=\int_0^t g(t-s)u_m(0,s)ds+h(u_m(0,t))+k(t), \tag{2.12}$$

$$\omega_{mk}(0)=a_{mk},\quad \omega'_{mk}(0)=b_{mk},\quad k=\overline{1,m}. \tag{2.13}$$

We shall require the following lemma

Lemma 2.5. *Consider the following differential equation*

$$\begin{cases} y''(t)+\mu^2 y(t)=\varphi(t),\ 0<t<T,\\ y(0)=y_0,\ \ y'(0)=y_1. \end{cases} \tag{2.14}$$

where $\mu>0$, y_0, y_1 *are constants and* $\varphi\in C^0([0,T])$. *Then the general solution is given by*

$$y(t)=y_0\cos\mu t+y_1\frac{\sin\mu t}{\mu}+\frac{1}{\mu}\int_0^t\sin\mu(t-s)\varphi(s)ds. \tag{2.15}$$

The proof of this lemma is simple, we omit the details.

Putting $\rho_k(t)=\sin(\mu_k t)/\mu_k$, we deduce Lemma 2.5 and (2.11)-(2.13) that

$$\begin{aligned}\omega_{mk}(t)&=a_{mk}\rho'_k(t)+b_{mk}\rho_k(t)-\frac{2}{\varphi_k(0)}\int_0^t ds\int_0^s\rho_k(t-s)g(s-\tau)u_m(0,\tau)d\tau\\&-\frac{2}{\varphi_k(0)}\int_0^t\rho_k(t-s)[h(u_m(0,s))+k(s)]ds\\&-\frac{2}{\varphi_k^2(0)}\int_0^t\rho_k(t-s)\left\langle f(\cdot,u_m(s),u'_m(s)),\varphi_k\right\rangle ds,\ k=\overline{1,m}.\end{aligned} \tag{2.16}$$

Applying the Schauder fixed-point theorem, then the system (2.16) has a solution $(\omega_{m1},\omega_{m2},\ldots,\omega_{mm})$ on an interval $[0,T_m]$. Hence, the system (2.8)-(2.10) exists a solution $u_m(t)$ on $[0,T_m]$. The following estimates allow us to take $T_m=T$, for all $m\in\mathbb{N}$ (see [3]).

Step 2. A priori estimates. Substituting (2.9) into (2.8), multiplying the jth equation of (2.8) by $\omega'_{mk}(t)$ and summing in k, then integrating with respect to time variable from 0 to t, we obtain

$$\begin{aligned}E_m(t)&=E_m(0)+2\int_0^1\int_0^{u_{0m}(x)}\widehat{f}(x,z)dzdx+2\int_0^{u_{0m}(0)}h(z)dz\\&-2\int_0^{u_m(0,t)}h(z)dz-2\int_0^t u'_m(0,s)dr\int_0^s g(s-\tau)u_m(0,\tau)d\tau\\&-2\int_0^t\Big\langle f(\cdot,u_m(s),u'_m(s))-\widehat{f}(\cdot,u_m(s)),u'_m(s)\Big\rangle ds\\&-2\int_0^t k(s)u'{}_m(0,s)ds-2\int_0^1\int_0^{u_m(x,t)}\widehat{f}(x,z)dzdx\\&=E_m(0)+2\int_0^1\int_0^{u_{0m}(x)}\widehat{f}(x,z)dzdx+2\int_0^{u_{0m}(0)}h(z)dz+\sum_{k=1}^5 I_k(t),\end{aligned} \tag{2.17}$$

where

$$E_m(t) = \|u_m'(t)\|^2 + \|u_{mx}(t)\|^2. \tag{2.18}$$

We shall estimate respectively the following terms on the right-hand side of (2.17). Estimating $I_1(t)$. Using the following inequality

$$2ab \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2, \text{ for all } a,\, b \in \mathbb{R},\ \varepsilon > 0, \tag{2.19}$$

and Lemma 2.2, it follows from (2.17) and the assumption (A_3) that

$$I_1(t) = -2\int_0^{u_m(0,t)} h(z)dz \le 2a\varepsilon\|u_{mx}(t)\|^2 + 2a\left(1+\frac{1}{\varepsilon}\right)\|u_m(t)\|^2 + 2b. \tag{2.20}$$

On the other hand

$$\|u_m(t)\|^2 = \left\|u_{0m} + \int_0^t u_m'(s)ds\right\|^2 \le 2\|u_{0m}\|^2 + 2t\int_0^t E_m(s)ds. \tag{2.21}$$

Therefore

$$I_1(t) \le 2\varepsilon a E_m(t) + 4at\left(1+\frac{1}{\varepsilon}\right)\int_0^t E_m(s)ds + 4a\left(1+\frac{1}{\varepsilon}\right)\|u_{0m}\|^2 + 2b. \tag{2.22}$$

Estimating $I_2(t)$. By using integration by parts, gives

$$\begin{aligned} I_2(t) &= -2\int_0^t u_m'(0,s)ds\int_0^s g(s-\tau)u_m(0,\tau)d\tau \\ &= -2u_m(0,t)\int_0^t g(t-s)u_m(0,s)ds \\ &\quad + 2\int_0^t u_m(0,s)\left[g(0)u_m(0,s) + \int_0^s g'(s-\tau)u_m(0,\tau)d\tau\right]ds \\ &\le 2\,|g(0)|\int_0^t E_m(s)ds + 2\sqrt{E_m(t)}\int_0^t |g(t-s)|\sqrt{E_m(s)}ds \\ &\quad + 2\int_0^t \sqrt{E_m(s)}ds\int_0^s |g'(s-\tau)|\sqrt{E_m(\tau)}d\tau \\ &= 2\,|k(0)|\int_0^t E_m(s)ds + J_1(t) + J_2(t). \end{aligned} \tag{2.23}$$

By the inequality (2.19) and the Cauchy-Schwartz inequality, we estimate without diffculty the following integrals in the right-hand side of (2.23) as follows

$$\begin{aligned} J_1(t) &= 2\sqrt{E_m(t)}\int_0^t |g(t-s)|\sqrt{E_m(s)}ds \\ &\le \varepsilon E_m(t) + \frac{1}{\varepsilon}\|g\|^2_{L^2(0,T)}\int_0^t E_m(s)ds, \end{aligned} \tag{2.24}$$

$$
\begin{aligned}
J_2(t) &= 2\int_0^t \sqrt{E_m(s)}ds\int_0^s |g'(s-\tau)|\sqrt{E_m(\tau)}d\tau \\
&\le \int_0^t E_m(s)ds + \|g'\|_{L^1(0,T)}\int_0^t ds\int_0^s |g'(s-\tau)|\,E_m(\tau)d\tau \\
&= \int_0^t E_m(s)ds + \|g'\|_{L^1(0,T)}\int_0^t d\tau\int_\tau^t |g'(s-\tau)|\,E_m(\tau)ds \\
&\le \left(\|g'\|^2_{L^1(0,T)}+1\right)\int_0^t E_m(s)ds.
\end{aligned} \tag{2.25}
$$

It follows from the estimates $J_1(t)$ and $J_2(t)$ that

$$
I_2(t) \le \varepsilon E_m(t) + \left(\frac{1}{\varepsilon}\|g\|^2_{L^2(0,T)} + \|g'\|^2_{L^1(0,T)} + 2\,|g(0)| + 1\right)\int_0^t E_m(s)ds. \tag{2.26}
$$

Estimating $I_3(t)$. Setting with $\delta = \min_{x\in[0,1]} p_2(x) > 0$. From assumption (A_5)-(ii), it is clear to see that

$$
\begin{aligned}
I_3(t) &= -2\int_0^t \left\langle f(\cdot,u_m(s),u_m'(s)) - \widehat{f}(\cdot,u_m(s)),u_m'(s)\right\rangle ds \\
&\le -2\delta\int_0^t \|u_m'(s)\|^p_{L^p(0,1)}\,ds + T\|q_2\|_{L^1(0,1)},
\end{aligned} \tag{2.27}
$$

Estimating $I_4(t)$. It follows from assumption (A_2) that

$$
\begin{aligned}
I_4(t) &= -2\int_0^t k(s)u_m'(0,s)ds \\
&\le \varepsilon E_m(t) + \int_0^t |k'(s)|\,E_m(s)ds \\
&\quad + \frac{1}{\varepsilon}\|k\|^2_{L^\infty(0,T)} + \|k'\|_{L^1(0,T)} + 2\,|k(0)u_{0m}(0)|\,.
\end{aligned} \tag{2.28}
$$

Estimating $I_5(t)$. By the assumption (A_5)-(i) then

$$
\begin{aligned}
I_5(t) &= -2\int_0^1\int_0^{u_m(x,t)} \widehat{f}(x,z)dzdx \\
&\le \int_0^1 p_1(x)|u_m(x,t)|^p dx + \int_0^1 q_1(x)|u_m(x,t)|^2 dx + \|r_1\|_{L^1(0,1)}.
\end{aligned} \tag{2.29}
$$

Applying the following inequalities

$$
(a+b)^p \le 2^{p-1}(a^p+b^p),\ \text{for all } a,\, b\ge 0, \tag{2.30}
$$

$$
ab \le \varepsilon a^p + C(\varepsilon)b^{p'},\ \text{for all } a,\, b\ge 0,\ \varepsilon>0, \tag{2.31}
$$

where $p' = \frac{p}{p-1},\ C(\varepsilon) = \frac{1}{p'}(\varepsilon p)^{-\frac{p'}{p}}$, we arrive at

$$
\begin{aligned}
\|u_m(t)\|^p_{L^p(0,1)} &\le \left(\|u_m(0)\|_{L^p(0,1)} + \left\|\int_0^t u_m'(s)ds\right\|_{L^p(0,1)}\right)^p \\
&\le 2^{p-1}\left(\|u_{0m}\|^p_H + T^{p-1}\int_0^t \|u_m'(s)\|^p_{L^p(0,1)}ds\right),
\end{aligned} \tag{2.32}
$$

$$\begin{aligned}\|u_m(t)\|^p_{L^p(0,1)} &\leq \|u_{0m}\|^p_H + p\int_0^t \|u'_m(s)\|_{L^p(0,1)} \left\| |u_m(s)|^{p-1} \right\|_{L^{p'}(0,1)} ds \\ &\leq \|u_{0m}\|^p_H + \varepsilon \int_0^t \|u'_m(s)\|^p_{L^p(0,1)}\, ds + p^{p'} C(\varepsilon)\int_0^t \|u_m(s)\|^p_{L^p(0,1)}\, ds.\end{aligned} \tag{2.33}$$

Consequently

$$\begin{aligned}\|u_m(t)\|^p_{L^p(0,1)} \leq \left[1 + 2^{p-1}p^{p'}TC(\varepsilon)\right] \|u_{0m}\|^p_H + \varepsilon \int_0^t \|u'_m(s)\|^q_{L^q(0,1)}\, ds \\ + 2^{p-1}p^{p'}T^{p-1}C(\varepsilon)\int_0^t ds \int_0^s \|u'_m(\tau)\|^p_{L^p(0,1)} d\tau.\end{aligned} \tag{2.34}$$

Using Lemma 2.2, it follows from (2.21) that

$$\begin{aligned}\int_0^1 q_1(x)|u_m(x,t)|^2 dx \leq 2t\left(1+\frac{1}{\varepsilon}\right)\|q_1\|_{L^1(0,1)} \int_0^t E_m(s) ds \\ + \varepsilon \|q_1\|_{L^1(0,1)} E_m(t) + 2\left(1+\frac{1}{\varepsilon}\right)\|u_{0m}\|^2 \|q_1\|_{L^1(0,1)}.\end{aligned} \tag{2.35}$$

By (2.29), (2.34) and (2.35), it follows that

$$\begin{aligned}I_5(t) \leq M_T^1 \int_0^t ds \int_0^s \|u'_m(\tau)\|^p_{L^p(0,1)} d\tau + \varepsilon\|p_1\|_{L^\infty(0,T)} \int_0^t \|u'_m(s)\|^q_{L^q(0,1)}\, ds \\ + M_T^2 \int_0^t E_m(s)ds + \varepsilon\|q_1\|_{L^1(0,1)} E_m(t) + M_m,\end{aligned} \tag{2.36}$$

where

$$\begin{cases} M_T^1 = 2^{q-1}p^{p'}T^{q-1}C(\varepsilon)\|p_1\|_{L^\infty(0,T)}, \\ M_T^2 = 2T\left(1+\frac{1}{\varepsilon}\right)\|q_1\|_{L^1(0,1)}, \\ M_m = [1+2^{p-1}p^{p'}TC(\varepsilon)]\|p_1\|_{L^\infty(0,T)}\|u_{0m}\|^p_H \\ \qquad + 2\left(1+\dfrac{1}{\varepsilon}\right)\|u_{0m}\|^2\|q_1\|_{L^1(0,1)} + \|r_1\|_{L^1(0,1)}. \end{cases} \tag{2.37}$$

By the assumptions (A_1)-(A_3), (A_5) and the imbedding $H^1(0,1) \hookrightarrow C^0([0,1])$, there exists a positive constant M such that

$$\begin{aligned}E_m(0) + 2\int_0^1 \int_0^{u_{0m}(x)} \widehat{f}(x,z)dzdx + 2\int_0^{u_{0m}(0)} h(z)dz \\ + 4a\left(1+\frac{1}{\varepsilon}\right)\|u_{0m}\|^2 + 2\left|k(0)u_{0m}(0)\right| + M_m \leq M.\end{aligned} \tag{2.38}$$

Combining (2.17), (2.18), (2.22), (2.26)-(2.28) and (2.36)-(2.38), we obtain after some rearrangements

$$\begin{aligned}&\left(1-\varepsilon M_T^3\right)E_m(t) + \left(2\delta - \varepsilon\|p_1\|_{L^\infty(0,T)}\right)\int_0^t \|u'_m(s)\|^p_{L^p(0,1)}\, ds \\ &\leq \int_0^t M_T^4(s)E_m(s)ds + M_T^1\int_0^t ds\int_0^s \|u'_m(\tau)\|^p_{L^p(0,1)}d\tau + M_T^5.\end{aligned} \tag{2.39}$$

where

$$\begin{cases} M_T^3 = \|q_1\|_{L^1(0,1)} + 2a + 2, \\ M_T^4(s) = |k'(s)| + \left(T + \frac{1}{\varepsilon}\right) \|g\|_{H^1(0,T)}^2 + 2\,|g(0)| + 4aT\left(1 + \frac{1}{\varepsilon}\right) + M_T^2 + 1, \\ M_T^5 = \frac{1}{\varepsilon}\,\|k\|_{L^\infty(0,T)}^2 + \|k'\|_{L^1(0,T)} + T\|q_2\|_{L^1(0,1)} + M + 2b. \end{cases} \tag{2.40}$$

Choosing $0 < \varepsilon \le \min\left\{ 1/2M_T^3,\ \delta/\|p_1\|_{L^\infty(0,T)} \right\}$, by the Gronwall inequality, then

$$E_m(t) + 2\delta \int_0^t \|u_m'(s)\|_{L^p(0,1)}^p\, ds \le 2M_T^5 \exp\left(\int_0^T \left[2M_T^4(s) + M_T^1/\delta\right] ds \right) = M_T. \tag{2.41}$$

Next, we need an estimate on the term $\int_0^t |u_m'(0,s)|^q ds,\ q = \min\{p', 2\}$.
We set

$$g_m(t) = \sum_{k=1}^m \frac{\sin(\mu_k t)}{\mu_k}, \tag{2.42}$$

$$\begin{aligned} h_m(t) = &\sum_{k=1}^m \varphi_k(0) \left[a_{mk} \cos(\mu_k t) + b_{mk} \frac{\sin(\mu_k t)}{\mu_k} \right] \\ &- 2 \sum_{k=1}^m \frac{1}{\varphi_k(0)} \int_0^t \frac{\sin[\mu_k(t-s)]}{\mu_k} \left\langle f(\cdot, u_m(s), u_m'(s)), \varphi_k \right\rangle ds. \end{aligned} \tag{2.43}$$

In view of (2.7), (2.9), (2.16), (2.42) and (2.43), $u_m(0,t)$ can be rewritten as follows

$$u_m(0,t) = h_m(t) - 2\int_0^t g_m(t-s) v_m(s) ds. \tag{2.44}$$

In connection with $h_m(t)$, we have the following lemma

Lemma 2.6. *There exist a positive constant C_T and a positive continuous function φ on the interval $[0,T]$ such that*

$$\int_0^t |h_m'(s)|^q ds \le \varphi(t) \int_0^t \|f(\cdot, u_m(s), u_m'(s))\|_{L^{p'}(0,1)}^{p'} ds + C_T, \tag{2.45}$$

for all $t \in [0,T]$.

Proof of Lemma 2.6. Since $\varphi_k(2-z) = -\varphi_k(z)$, for all $z \in \mathbb{R},\ k = \overline{1,m}$, without loss of generality, we can suppose that $0 < T \le 1$. Putting

$$h_m'(t) = p_m(t) - q_m(t) - r_m(t), \tag{2.46}$$

with

$$p_m(t) = \sum_{k=1}^m \varphi_k(0) b_{mk} \cos(\mu_k t), \tag{2.47}$$

$$q_m(t) = \sum_{k=1}^m \varphi_k(0) a_{mk} \mu_k \sin(\mu_k t), \tag{2.48}$$

$$r_m(t) = 2\sum_{k=1}^m \frac{1}{\varphi_k(0)} \int_0^t \cos[\mu_k(t-s)] \left\langle f(\cdot, u_m(s), u_m'(s)), \varphi_k \right\rangle ds. \tag{2.49}$$

On the other hand, using the following inequalities

$$(a+b+c)^r \le 3^{r-1}(a^r+b^r+c^r), \text{ for all } a,\, b,\, c,\, d \ge 0,\ r \ge 1, \tag{2.50}$$

$$a^s \le \frac{s}{r}a^r + 1 - \frac{s}{r}, \text{ for all } a \ge 0,\ r \ge s > 0, \tag{2.51}$$

then

$$\begin{aligned}\int_0^t |h'_m(s)|^q ds &\le 3\left(\int_0^t |p_m(s)|^q ds + \int_0^t |q_m(s)|^q ds + \int_0^t |r_m(s)|^q ds\right)\\ &\le 3\left(\int_0^t |p_m(s)|^2 ds + \int_0^t |q_m(s)|^2 ds + \int_0^t |r_m(s)|^{p'} ds\right) + 9T.\end{aligned} \tag{2.52}$$

Now, we estimate each term on the right-hand side of this inequality
Estimating $J_1(t) = \int_0^t |p_m(s)|^2 ds + \int_0^t |q_m(s)|^2 ds$. From (2.10), it follows that

$$J_1(t) = \int_0^t |u_{1m}(s)|^2 ds + \int_0^t |u'_{0m}(s)|^2 ds \le \|u_{1m}\|^2 + \|u_{0m}\|_H^2 \le C. \tag{2.53}$$

Estimating $J_2(t) = \int_0^t |r_m(s)|^{p'} ds$. We note that

$$\begin{aligned} r_m(t) &= 2\sum_{k=1}^m \frac{1}{\varphi_k(0)} \int_0^t \cos[\mu_k(t-s)] \left\langle f(\cdot, u_m(s), u'_m(s)), \varphi_k\right\rangle ds\\ &= 2\int_0^t \sum_{k=1}^m \frac{\cos \mu_k\theta}{\varphi_k(0)} \left\langle f_m(t-\theta), \varphi_k\right\rangle d\theta\\ &= \int_0^t \left[\int_0^1 \sum_{k=1}^m f_m(x, t-\theta)\cos\mu_k(x+\theta)dx\right] d\theta\\ &\quad + \int_0^t \left[\int_0^1 \sum_{k=1}^m f_m(x, t-\theta)\cos\mu_k(x-\theta)dx\right] d\theta\\ &= \int_0^t \delta_m^{(1)}(t,\theta)d\theta + \int_0^t \delta_m^{(2)}(t,\theta)d\theta, \end{aligned} \tag{2.54}$$

with $f_m(x,t) = f(x, u_m(x,t), u'_m(x,t))$. Now we require the following lemma

Lemma 2.7. *Let $q > 1$. We have*

$$\int_{\pi\theta/2}^{\pi(\theta+1)/2} \frac{dz}{\sin^p z} \le \sum_{k=1}^{p_*} \frac{2^k}{k} \frac{1}{\sin^k \pi\theta}, \text{ for all } \theta \in (0,1), \tag{2.55}$$

where $p_* = p-1$ if $p \in \mathbb{N}$, $p_* = [p]$ if $p \notin \mathbb{N}$, with $[p]$ is an integer part of p.

Proof of Lemma 2.7. We consider two cases for p.
Case 1: $p \in \mathbb{N}$. We prove (2.55) by induction.
It is easy to see that $\int_{\pi\theta/2}^{\pi(\theta+1)/2} \frac{dz}{\sin^2 z} = \frac{2}{\sin\pi\theta}$. Hence, (2.55) holds for $p = 2$.
Suppose that (2.55) holds for $p-1$, we prove that (2.55) holds for p. Indeed, by means of integration by parts, we get

$$\int_{\pi\theta/2}^{\pi(\theta+1)/2} \frac{dz}{\sin^p z} = \frac{2^{p-1}}{p-1} \frac{\sin^p(\pi\theta/2) + \cos^p(\pi\theta/2)}{\sin^{p-1}\pi\theta} + \frac{p-2}{p-1} \int_{\pi\theta/2}^{\pi(\theta+1)/2} \frac{dz}{\sin^{p-2} z}. \tag{2.56}$$

Since $|\sin^p z + \cos^p z| \le 1$, for all $z \in \mathbb{R}$, we deduce from (2.56) that

$$\begin{aligned}\int_{\pi\theta/2}^{\pi(\theta+1)/2} \frac{dz}{\sin^p z} &\le \frac{2^{p-1}}{p-1}\frac{1}{\sin^{p-1}\pi\theta} + \frac{p-2}{p-1}\int_{\pi\theta/2}^{\pi(\theta+1)/2}\frac{dz}{\sin^{p-1} z} \\ &\le \frac{2^{p-1}}{p-1}\frac{1}{\sin^{p-1}\pi\theta} + \frac{p-2}{p-1}\sum_{k=1}^{q-2}\frac{2^k}{k}\frac{1}{\sin^k \pi\theta} \\ &\le \sum_{k=1}^{p-1}\frac{2^k}{k}\frac{1}{\sin^k\pi\theta}.\end{aligned} \tag{2.57}$$

Case 2: $p \notin \mathbb{N}$. Applying inequality (2.55) with p replaced by $[p]+1$, then

$$\int_{\pi\theta/2}^{\pi(\theta+1)/2} \frac{dz}{\sin^p z} \le \int_{\pi\theta/2}^{\pi(\theta+1)/2} \frac{dz}{\sin^{[p]+1} z} \le \sum_{k=1}^{[p]}\frac{2^k}{k}\frac{1}{\sin^k\pi\theta}. \tag{2.58}$$

The proof of Lemma 2.7 is complete. □

By using the sum

$$2\sin(\pi z/2)\sum_{k=1}^{n}\cos(\mu_k z) = \sin(n\pi z), \text{ for all } z \in \mathbb{R}, \tag{2.59}$$

and the inequality

$$\left(\sum_{k=1}^{n} a_k\right)^r \le \sum_{k=1}^{n} a_j^r, \text{ for all } a_1,\, a_2, \ldots, a_n \ge 0,\ 0 < r \le 1, \tag{2.60}$$

we deduce from Lemma 2.7 that

$$\begin{aligned}\left|\delta_m^{(1)}(t,\theta)\right| &= \frac{1}{2}\left|\int_0^1 f_m(x,t-\theta)\frac{\sin m\pi(x+\theta)}{\sin[\pi(x+\theta)/2]}dx\right| \\ &\le \frac{1}{2}\left(\frac{2}{\pi}\right)^{1/p}\|f_m(t-\theta)\|_{L^{p'}(0,1)}\left(\int_{\pi\theta/2}^{\pi(\theta+1)/2}\frac{dz}{\sin^p z}\right)^{1/p} \\ &\le \frac{1}{2}\left(\frac{2}{\pi}\right)^{1/p}\sum_{k=1}^{p_*}\left(\frac{2^k}{k}\right)^{1/p}\frac{1}{\sin^{k/p}\pi\theta}\|f_m(t-\theta)\|_{L^{p'}(0,1)} \\ &= \psi(\theta)\|f_m(t-\theta)\|_{L^{p'}(0,1)}.\end{aligned} \tag{2.61}$$

$\delta_m^{(2)}(t,\theta)$ verifies a similar inequality to (2.61). Finally, we obtain

$$\begin{aligned}|r_m(t)|^{p'} &\le C\left(\int_0^t \psi(\theta)\|f_m(t-\theta)\|_{L^{p'}(0,1)}d\theta\right)^{p'} \\ &\le C\left(\int_0^t \psi(\theta)d\theta\right)^{p'-1}\int_0^t \psi(\theta)\,\|f_m(t-\theta)\|_{L^{p'}(0,1)}^{p'}\,d\theta,\end{aligned} \tag{2.62}$$

where C is a positive constant. So using the Fubini theorem, we obtain

$$\begin{aligned} J_2(t) &\leq C\left(\int_0^t \psi(\theta)d\theta\right)^{p'-1} \int_0^t \int_0^s \psi(\theta) \left\| f_m(s-\theta)\right\|_{L^{p'}(0,1)}^{p'} d\theta ds \\ &= C\left(\int_0^t \psi(\theta)d\theta\right)^{p'-1} \int_0^t \int_\theta^t \psi(\theta) \left\| f_m(s-\theta)\right\|_{L^{p'}(0,1)}^{p'} dsd\theta \\ &\leq C\left(\int_0^t \psi(\theta)d\theta\right)^{p'} \int_0^t \left\| f_m(s)\right\|_{L^{p'}(0,1)}^{p'} ds \\ &= \varphi(t) \int_0^t \left\| f(\cdot, u_m(s), u_m'(s))\right\|_{L^{p'}(0,1)}^{p'} ds. \end{aligned} \tag{2.63}$$

From (2.52), (2.53) and (2.63), we get Lemma 2.6. □

Remark 2.8. Lemma 3 in [4] is a special case of Lemma 2.6 with $g = h = 0$, $f(x, u, u_t) = |u_t|^{p-2}u_t,\ 1 < p < 2$.

To estimate $\int_0^t |u_m'(0, s)|^q ds$, we need the following lemma

Lemma 2.9. *There exist positive constants C_T and D_T such that*

$$\int_0^t \left| \int_0^s g_m'(s-\tau)v_m(\tau)d\tau\right|^q ds \leq C_T \int_0^t \int_0^s |u_m'(0,\tau)|^q d\tau ds + D_T, \tag{2.64}$$

for all $t \in [0, T]$.

Proof of Lemma 2.9. Using integration by parts, we have

$$\int_0^s g_m'(s-\tau)v_m(\tau)d\tau = g_m(s)v_m(0) + \int_0^s g_m(s-\tau)v_m'(\tau)d\tau. \tag{2.65}$$

By the inequality (2.30), we deduce from (2.65) that

$$\begin{aligned} &\left|\int_0^s g_m'(s-\tau)v_m(\tau)d\tau\right|^q \\ &\leq 2|v_m(0)|^q|g_m(s)|^q + 2\left(\int_0^s |g_m(\tau)|^{q'} d\tau\right)^{q-1} \int_0^s |v_m'(\tau)|^q d\tau, \end{aligned} \tag{2.66}$$

with $q' = q/(q-1)$. Moreover

$$\|v\|_{L^q(0,T)} \leq T^{(q'-q)/q'q} \|v\|_{L^{q'}(0,T)}, \text{ for all } v \in L^{q'}(0,T). \tag{2.67}$$

Thus

$$\begin{aligned} &\int_0^t \left|\int_0^s g_m'(s-\tau)v_m(\tau)d\tau\right|^q ds \\ &\leq 2|v_m(0)|^q \int_0^t |g_m(s)|^q ds + 2\left(\int_0^t |g_m(s)|^{q'} ds\right)^{q-1} \int_0^t \int_0^s |v_m'(\tau)|^q d\tau ds \\ &\leq 2\left\|g_m\right\|_{L^{q'}(0,T)}^q \left[T^{(q'-q)/q'}(|h(u_{0m}(0))| + |k(0)|)^q + \int_0^t \int_0^s |v_m'(\tau)|^q d\tau ds\right]. \end{aligned} \tag{2.68}$$

Note that

$$v_m'(t) = \int_0^t g'(t-s)u_m(0,s)ds + h'(u_m(0,t))u_m'(0,t) + g(0)u_m(0,t) + k'(t). \tag{2.69}$$

Using (2.10), (2.69) and the assumptions (A_2), (A_3), then

$$\begin{aligned} |v_m'(t)| &\leq \Big(\|g'\|_{L^1(0,T)} + |g(0)|\Big) \|u_m\|_{L^\infty(0,T;H)} \\ &\quad + \|h'\|_{C^0([-M_T,M_T])} |u_m'(0,t)| + |k'(t)| \\ &\leq C_T \left(|u_m'(0,t)| + |k'(t)| + 1\right), \end{aligned} \tag{2.70}$$

where C_T is a positive constant depending only on T.
Using (2.50), (2.51) and (2.70), we get

$$\int_0^s |v_m'(\tau)|^q d\tau \leq C_T \left(\int_0^s |u_m'(0,\tau)|^q d\tau + \|k'\|^2_{L^2(0,T)} + 1 \right). \tag{2.71}$$

By the imbedding $H^1(0,T) \hookrightarrow C^0([0,T])$, it follows from (2.10), (2.68) and (2.71) that

$$\int_0^t \left| \int_0^s g_m'(s-\tau) v_m(\tau) d\tau \right|^q ds \leq \|g_m\|^q_{L^{q'}(0,T)} \left(C_T \int_0^t \int_0^s |u_m'(0,\tau)|^q d\tau ds + D_T \right). \tag{2.72}$$

Also, we have following lemma

Lemma 2.10. *Let $m \in \mathbb{N}$. Then*

$$|g_m(t)| = \left| \sum_{k=1}^m \frac{\sin(\mu_k t)}{\mu_k} \right| \leq 1 + \frac{4}{\pi}, \text{ for all } t \in \mathbb{R}. \tag{2.73}$$

The proof of Lemma 2.10 is straightforward, we omit the details.

By Lemma 2.10, then $\{g_m\}$ is a bounded sequence in $L^{q'}(0,T)$. Using (2.72), we get (2.64). Lemma 2.9 is completely proved. □

By Lemma 2.6, 2.9 and the inequality (2.30), we deduce from (2.44) that

$$\begin{aligned} \int_0^t |u_m'(0,s)|^q ds &\leq 2\varphi(t) \int_0^t \|f(\cdot, u_m(s), u_m'(s))\|^{p'}_{L^{p'}(0,1)} ds \\ &\quad + C_T \int_0^t \int_0^s |u_m'(0,\tau)|^q d\tau ds + D_T. \end{aligned} \tag{2.74}$$

On account of the assumption (A_6)-(i), we have

$$|f(x, u_m(t), u_m'(t))| \leq p_{M_T}(x) |u_m'(t)|^{p-1} + q_{M_T}(x). \tag{2.75}$$

Therefore

$$\begin{aligned} &\|f(\cdot, u_m, u_m')\|_{L^{p'}((0,1)\times(0,T))} \\ &\leq \|p_{M_T}\|_{L^\infty(0,1)} \|u_m'\|^{p-1}_{L^p((0,1)\times(0,T))} + T^{1/p'} \|q_{M_T}\|_{L^{p'}(0,1)} \leq C_T. \end{aligned} \tag{2.76}$$

Using (2.74) and (2.76), it follows that

$$\int_0^t |u_m'(0,s)|^q ds \leq C_T \int_0^t \int_0^s |u_m'(0,\tau)|^q d\tau ds + D_T. \tag{2.77}$$

By the Gronwall inequality, then

$$\int_0^t |u_m'(0,s)|^q ds \leq D_T \exp(TC_T) = \widehat{D}_T. \tag{2.78}$$

Step 3. Limiting process. Due to (2.41), (2.76) and (2.78), by the Banach-Alaoglu theorem, we can extract a subsequence of sequence $\{u_m\}$, still labeled by the same notations, such that

$$\begin{cases} u_m \to u & \text{weakly* in } L^\infty(0,T;H^1(0,1)), \\ u'_m \to u' & \text{weakly* in } L^\infty(0,T;L^2(0,1)) \text{ and} \\ & \text{weakly in } L^p((0,1)\times(0,T)), \\ u_m(0,\cdot) \to u(0,\cdot) & \text{weakly in } W^{1,\min\{p',2\}}(0,T), \\ f(\cdot,u_m,u'_m) \to \chi & \text{weakly in } L^{p'}((0,1)\times(0,T)). \end{cases} \tag{2.79}$$

Thanks to the compactness of the imbedding $W^{1,\min\{p',2\}}(0,T) \hookrightarrow C^0([0,T])$ and Lemma of J.L. Lions [7], (2.79) leads to the existence of a subsequence still denoted by $\{u_m\}$, such that

$$\begin{cases} u_m \to u & \text{strongly in } L^2((0,1)\times(0,T)) \text{ and a.e. in } (0,1)\times(0,T), \\ u_m(0,\cdot) \to u(0,\cdot) & \text{strongly in } C^0([0,T]). \end{cases} \tag{2.80}$$

By means of the following inequality

$$|h(u_m(0,t)) - h(u(0,t))| \le \|h'\|_{C^0([-\sqrt{M_T},\sqrt{M_T}])} |u_m(0,t) - u(0,t)|. \tag{2.81}$$

Hence

$$h(u_m(0,t)) \to h(u(0,t)) \text{ strongly in } C^0([0,T]). \tag{2.82}$$

From (2.9), $(2.80)_2$ and (2.82), we get

$$v_m(t) \to \int_0^t g(t-s)u(0,s)ds + h(u(0,t)) + k(t) = v(t), \tag{2.83}$$

strongly in $C^0([0,T])$.
Thus passing to the limit in (2.8) by $(2.79)_{1,2,4}$ and (2.83) leads to

$$\frac{d}{dt}\langle u'(t),\varphi\rangle + \langle u_x(t),\varphi_x\rangle + v(t)\varphi(0) + \langle \chi(t),\varphi\rangle = 0, \tag{2.84}$$

for all $\varphi \in H$, a.e. $t \in [0,T]$.
Furthermore, it is easy to show a similar way to [4] that

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{2.85}$$

To prove the existence of solutions of the problem (1.1)-(1.4), we have to show that $\chi = f(\cdot,u,u')$. We need the following lemma

Lemma 2.11. *Let u be the weak solution of the following problem*

$$\begin{cases} u_{tt} - u_{xx} + F = 0,\ 0<x<1,\ 0<t<T, \\ u(1,t) = 0, \quad u_x(0,t) = v(t), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \\ u \in L^\infty(0,T;H),\ \ u_t \in L^\infty(0,T;L^2(0,1)), \\ u_t(0,\cdot) \in L^r(0,T),\ \ v \in L^{r'}(0,T),\ \ r' = r/(r-1),\ r>1, \\ F \in L^1(0,T;L^2(0,1)). \end{cases} \tag{2.86}$$

Then we have

$$\begin{aligned} &\|u_t(t)\|^2 + \|u_x(t)\|^2 + 2\int_0^t v(s)u_t(0,s)ds + 2\int_0^t \langle F(\cdot,s), u_t(s)\rangle\, ds \\ &\geq \|u_1\|^2 + \|u_{0x}\|^2, \ \textit{a.e. } t \in [0,T]. \end{aligned} \tag{2.87}$$

Equality holds in case $u_0 = u_1 = 0$.

The proof of Lemma 2.11 is the same as Lemma 2.4 in [11].

We now return to the proof of the existence of a solution of the problem (1.1)-(1.4). It follows from (2.8) and (2.9) that

$$\begin{aligned} &2\int_0^t \langle f(\cdot, u_m(s), u_m'(s)), u_m'(s)\rangle\, ds \\ &= \|u_{1m}\|^2 + \|u_{0mx}\|^2 - \|u_m'(t)\|^2 - \|u_{mx}(t)\|^2 - 2\int_0^t v_m(s)u_m'(0,s)ds. \end{aligned} \tag{2.88}$$

Applying Lemma 2.11, we deduce from (2.8)-(2.10), (2.79) and (2.83) that

$$\begin{aligned} &2\limsup_{m\to\infty}\int_0^t \langle f(\cdot, u_m(s), u_m'(s)), u_m'(s)\rangle\, ds \\ &\leq \|u_1\|^2 + \|u_{0x}\|^2 - \liminf_{m\to\infty}\|u_m'(t)\|^2 - \liminf_{m\to\infty}\|u_{mx}(t)\|^2 - 2\int_0^t v(s)u'(0,s)ds \\ &\leq \|u_1\|^2 + \|u_{0x}\|^2 - \|u'(t)\|^2 - \|u_x(t)\|^2 - 2\int_0^t v(s)u'(0,s)ds \\ &\leq 2\int_0^t \langle \chi(s), u'(s)\rangle\, ds. \end{aligned} \tag{2.89}$$

Noting that

$$\varphi_m(t) = \int_0^t \langle f(\cdot, u_m(s), u_m'(s)) - f(\cdot, u_m(s), v(s)), u_m'(s) - v(s)\rangle\, ds \geq 0, \tag{2.90}$$

for all $v \in L^p((0,1)\times(0,T))$.
Besides, we deduce from the dominated convergence theorem that

$$f(u_m, v) \to f(u, v) \ \text{ strongly in } L^{p'}((0,1)\times(0,T)), \tag{2.91}$$

where $v \in L^p((0,1)\times(0,T))$. Therefore, from $(2.79)_{2,4}$ and (2.89)-(2.91) that

$$0 \leq \limsup_{m\to\infty}\varphi_m(t) \leq \int_0^t \langle \chi(s) - f(u(s), v(s)), u'(s) - v(s)\rangle\, ds. \tag{2.92}$$

In (2.92), choosing $v(s) = u'(s) + \varepsilon w(s)$, with $\varepsilon > 0$, $w \in L^p((0,1)\times(0,T))$ and using the same arguments of Minty and Browder [7], we arrive at $\chi = f(\cdot, u, u')$. The existence of solutions is proved.
Step 4. Uniqueness of the weak solutions. Let u_1 and u_2 be two weak solutions of the problem (1.1)-(1.4) such that

$$\begin{cases} u_i \in L^\infty(0,T;H), \ u'_i \in L^\infty(0,T;L^2(0,1)) \cap L^p((0,1)\times(0,T)), \\ u_i(0,\cdot) \in W^{1,\min\{p',2\}}(0,T), \quad p' = p/(p-1), \ i = 1,2. \end{cases} \tag{2.93}$$

Then $u = u_1 - u_2$ is a weak solution of the following problem

$$\begin{cases} u_{tt} - u_{xx} + f(\cdot, u_1, u_1') - f(\cdot, u_2, u_2') = 0, \\ u(1,t) = 0, \quad u_x(0,t) = v(t), \\ u(x,0) = u_t(x,0) = 0, \\ v(t) = \int_0^t g(t-s)u(0,t)ds + \sum\limits_{i=1}^{2} (-1)^{i-1} h(u_i(0,t)). \end{cases} \tag{2.94}$$

Applying Lemma 2.11 with $u_0 = u_1 = 0,\ F = f(\cdot, u_1, u_1') - f(\cdot, u_2, u_2')$, we have

$$\begin{aligned} E(t) = &-2\int_0^t \langle f(\cdot, u_1(s), u_1'(s)) - f(\cdot, u_2(s), u_2'(s)), u'(s)\rangle\, ds \\ &- 2\int_0^t u'(0,s) \sum_{i=1}^{2} (-1)^{i-1} h(u_i(0,s))ds \\ &- 2\int_0^t u'(0,s)ds \int_0^s k(s-\tau)u(0,\tau)d\tau = \sum_{i=1}^{3} J_i(t). \end{aligned} \tag{2.95}$$

where $E(t) = \|u'(t)\|^2 + \|u_x(t)\|^2$.
Estimating $J_1(t)$. Using the assumptions (A_4) and (A_6)-(ii), we obtain

$$\begin{aligned} J_1(t) &= -2\int_0^t \langle f(\cdot, u_1(s), u_1'(s)) - f(\cdot, u_2(s), u_2'(s)), u'(s)\rangle\, ds \\ &\le -2\int_0^t \langle f(\cdot, u_1(s), u_2'(s)) - f(\cdot, u_2(s), u_2'(s)), u'(s)\rangle\, ds \\ &\le 2\int_0^t \|r_M(\cdot, u_2'(s))\|\, \|u_x(s)\|\, \|u'(s)\|\, ds \\ &\le \int_0^t \|r_M(\cdot, u_2'(s))\|\, E(s)ds, \end{aligned} \tag{2.96}$$

with $M = \|u_1\|_{L^\infty(0,T;H)} + \|u_2\|_{L^\infty(0,T;H)}$.
Estimating $J_2(t)$. Putting $c_M = \|h\|_{C^2([-M,M])}$. Integrating by parts, then

$$\begin{aligned} J_2(t) &= -2\int_0^t u'(0,s) \sum_{i=1}^{2} (-1)^{i-1} h(u_i(0,s))ds \\ &= -2\int_0^t u'(0,s)ds \int_0^1 \frac{d}{d\theta} h(u_2(0,s) + \theta u(0,s))d\theta \\ &= -u^2(0,t)\int_0^1 h'(u_2(0,t) + \theta u(0,t))d\theta \\ &\quad + \int_0^t u^2(0,s)ds \int_0^1 h''(u_2(0,s) + \theta u(0,s))(u_2'(0,s) + \theta u'(0,s))d\theta \\ &\le c_M \left[u^2(0,t) + \int_0^t (|u_1'(0,s)| + |u_2'(0,s)|)\, u^2(0,s)ds \right]. \end{aligned} \tag{2.97}$$

On the other hand

$$\|u(t)\|^2 \le t\int_0^t \|u'(s)\|^2 ds \le t\int_0^t E(s)ds. \tag{2.98}$$

So applying Lemma 2.2, we arrive at

$$\begin{aligned} u^2(0,t) &\le \|u(t)\|^2_{C^0([0,1])} \le \varepsilon \|u(t)\|^2_H + \left(1+\frac{1}{\varepsilon}\right)\|u(t)\|^2 \\ &\le \varepsilon E(t) + t\left(1+\frac{1}{\varepsilon}\right)\int_0^t E(s)ds, \text{ for all } \varepsilon > 0. \end{aligned} \tag{2.99}$$

Thus, it follows from (2.97) and (2.99) that

$$J_2(t) \le \varepsilon c_M E(t) + \int_0^t d(s)E(s)ds, \tag{2.100}$$

with

$$\begin{aligned} d(s) &= \varepsilon c_M \left(|u_1'(0,s)| + |u_2'(0,s)|\right) \\ &\quad + (1+\tfrac{1}{\varepsilon})Tc_M\left(\|u_1'(0,\cdot)\|_{L^1(0,T)} + \|u_2'(0,\cdot)\|_{L^1(0,T)} + 1\right). \end{aligned} \tag{2.101}$$

Estimating $J_3(t)$. It is easy to see that

$$\begin{aligned} J_3(t) &= -2\int_0^t u'(0,s)ds\int_0^s g(s-\tau)u(0,\tau)d\tau \\ &= -2u(0,t)\int_0^t g(t-s)u(0,s)ds \\ &\quad + 2\int_0^t u(0,s)\left[g(0)u(0,s) + \int_0^s g'(s-\tau)u(0,\tau)d\tau\right]ds \\ &\le \varepsilon E(t) + \left(\frac{1}{\varepsilon}\|g\|^2_{L^2(0,T)} + \|g'\|^2_{L^1(0,T)} + 2|g(0)| + 1\right)\int_0^t E(s)ds. \end{aligned} \tag{2.102}$$

Combining (2.95), (2.96) and (2.100)-(2.102), we obtain

$$E(t) \le \varepsilon(c_M+1)E(t) + \int_0^t \widehat{d}(s)E(s)ds, \tag{2.103}$$

where

$$\widehat{d}(s) = d(s) + \|r_M(\cdot, u_2'(s))\| + \frac{1}{\varepsilon}\|g\|^2_{L^2(0,T)} + \|g'\|^2_{L^1(0,T)} + 2|g(0)| + 1. \tag{2.104}$$

Choosing $0 < \varepsilon(c_M+1) < 1$, then using Gronwall's lemma, we obtain $E(t) = 0$, i.e. $u_1 = u_2$. This completes the proof of Theorem 2.4. □

Remark 2.12. Theorem 2.4 still holds if the assumptions (A_5)-(ii) and (A_6)-(iii) are replaced by the following assumptions
(A_5')-(ii) There exist a constant $p > 1$ and positive functions $p_2 \in C^0([0,1])$ and q_2, $r_2 \in L^1(0,1)$ such that

$$[f(x,u,v) - \widehat{f}(x,u)]v \ge p_2(x)|v|^p - q_2(x)v^2 - r_2(x), \text{ a.e. } u, v \in \mathbb{R} \text{ and } x \in [0,1],$$

(A_6')-(iii) For every $M > 0$, there exists a positive function $r_M \in C^0([0,1]\times\mathbb{R})$ such that

$$r_M(x,v) \in L^1(0,T;L^2(0,1)), \text{ for all } v \in L^\infty(0,T;L^2(0,1)) \cap L^p((0,1)\times(0,T))$$

and $x \in [0,1]$.

3. Stability of the weak solutions

In this section, we assume that a, b are fixed constants and u_0, u_1, f are fixed functions satisfying the assumptions (A_1) and (A_3)-(A_6). Let g, h and k satisfy the assumptions (A_2) and (A_3). Applying Theorem 2.4, the problem (1.1)-(1.4) has a unique weak solution $u = u(g, h, k)$ (depending on g, h and k).
Then the stability of the solutions of the problems (1.1)-(1.4) are given by

Theorem 3.1. *Let (A_1)-(A_6) hold. Then the solutions of the problems (1.1)-(1.4) are stable with respect to the data (g, h, k) in the sense:*
If (g^j, h^j, k^j) and (g, h, k) satisfy the assumptions $(A_2)$and (A_3) such that

$$(g^j, k^j) \to (g, k) \text{ strongly in } [W^{1,1}(0, T)]^2, \tag{3.1}$$

$$h^j \to h \text{ strongly in } C^1([-M, M]), \tag{3.2}$$

as $j \to \infty$, for all $M > 0$.
Then

$$\left(u^j, u_t^j, u^j(0, \cdot)\right) \to (u, u_t, u(0, \cdot)), \tag{3.3}$$

strongly in $L^\infty(0, T; H^1(0, 1)) \times L^\infty(0, T; L^2(0, 1)) \times C^0([0, T])$, as $j \to \infty$, where $u^j = u(g^j, h^j, k^j)$, $u = u(g, h, k)$.

Proof of Theorem 3.1. First, we can assume that the data (g^j, g, k^j, k) satisfy

$$\left\|g^j\right\|_{W^{1,1}(0,T)} + \|g\|_{W^{1,1}(0,T)} + \left\|k^j\right\|_{W^{1,1}(0,T)} + \|k\|_{W^{1,1}(0,T)} \le C_*, \tag{3.4}$$

where C_* are fixed positive constant. Then the a priori estimates of the sequence $\{u_m\}$ in the proof of Theorem 2.4 satisfy

$$\|u_{mt}(t)\|^2 + \|u_{mx}(t)\|^2 + \int_0^t |u_{mt}(0, s)|^{\min\{p',2\}} ds \le C_T, \text{ for all } t \in [0, T], \tag{3.5}$$

where C_T is a constant depending only on T, C_*, f, u_0, u_1.
Due to (2.79) and (3.5), we conclude that

$$\begin{aligned} C_T &\ge \liminf_{m\to\infty} \left(\|u_{mt}(t)\|^2 + \|u_{mx}(t)\|^2 + \int_0^t |u_{mt}(0, s)|^{\min\{p',2\}} ds \right) \\ &\ge \liminf_{m\to\infty} \|u_{mt}(t)\|^2 + \liminf_{m\to\infty} \|u_{mx}(t)\|^2 + \liminf_{m\to\infty} \int_0^t |u_{mt}(0, s)|^{\min\{p',2\}} ds \\ &\ge \|u_t(t)\|^2 + \|u_x(t)\|^2 + \int_0^t |u_t(0, s)|^{\min\{p',2\}} ds. \end{aligned} \tag{3.6}$$

In addition, we can prove in similar way above, then the solution u^j of the problem (1.1)-(1.4) corresponding to the data (g^j, h^j, k^j) also satisfies

$$\left\|u_t^j(t)\right\|^2 + \left\|u_x^j(t)\right\|^2 + \int_0^t \left|u_t^j(0, s)\right|^{\min\{p',2\}} ds \le \widehat{C}_T, \text{ for all } t \in [0, T], \tag{3.7}$$

with $\widehat{C}_T$ is a constant depending only on T, C_*, f, u_0, u_1.
We put

$$\widehat{g_j} = g^j - g, \; \widehat{h_j} = h^j - h, \; \widehat{k_j} = k^j - k. \tag{3.8}$$

Then $v^j = u^j - u$ satisfies the following problem

$$\begin{cases} v^j_{tt} - v^j_{xx} + F^j = 0,\ 0 < x < 1,\ 0 < t < T, \\ v^j(1,t) = 0, \quad v^j_t(0,t) = w^j(t), \\ v^j(x,0) = v^j_t(x,0) = 0, \end{cases} \tag{3.9}$$

where

$$F^j(x,t) = f(x, u^j(t), u^j_t(t)) - f(x, u(t), u_t(t)), \tag{3.10}$$

$$\begin{aligned} w^j(t) &= \widehat{k}_j(t) + \widehat{h}_j(u^j(0,t)) + [h(u^j(0,t)) - h(u(0,t))] \\ &\quad + \int_0^t \widehat{g}_j(t-s)u^j(0,s))ds + \int_0^t g(t-s)v^j(0,s)ds. \end{aligned} \tag{3.11}$$

Applying Lemma 2.11 again with $u_0 = u_1 = 0,\ F = F^j,\ v = w^j$, we see that

$$\begin{aligned} E_j(t) &= -2\int_0^t \left\langle f(\cdot, u^j(s), u^j_t(s)) - f(\cdot, u(s), u_t(s)), v^j_t(s) \right\rangle ds \\ &\quad - 2\int_0^t \widehat{k}_j(s) v^j_t(0,s)ds - 2\int_0^t \widehat{h}_j(u^j(0,s))v^j_t(0,s)ds \\ &\quad - 2\int_0^t [h(u^j(0,s)) - h(u(0,s))]v^j_t(0,s)ds \\ &\quad - 2\int_0^t v^j_t(0,s)ds \int_0^s \widehat{g}_j(s-\tau)u^j(0,\tau)d\tau \\ &\quad - 2\int_0^t v^j_t(0,s)ds \int_0^s g(s-\tau)v^j(0,\tau)d\tau \\ &= K_1(t) + K_2(t) + \cdots + K_6(t). \end{aligned} \tag{3.12}$$

where

$$E_j(t) = \left\| v^j_t(t) \right\|^2 + \left\| v^j_x(t) \right\|^2. \tag{3.13}$$

Let $M = \sqrt{C_T} + \sqrt{\widehat{C}_T}$. Now we can estimate eight integrals in the right-hand side of (3.13) as follows.
Estimating $K_1(t)$. From the assumptions (A_4) and (A_6)-(ii), it yields

$$\begin{aligned} K_1(t) &= -2\int_0^t \left\langle f(\cdot, u^j(s), u^j_t(s)) - f(\cdot, u(s), u_t(s)), v^j_t(s) \right\rangle ds \\ &\le -2\int_0^t \left\langle f(\cdot, u^j(s), u_t(s)) - f(\cdot, u(s), u_t(s)), v^j_t(s) \right\rangle ds \\ &\le 2\int_0^t \| r_M(\cdot, u_t(s)) \| \left\| v^j_x(s) \right\| \left\| v^j_t(s) \right\| ds \\ &\le \int_0^t \| r_M(\cdot, u_t(s)) \| E(s) ds. \end{aligned} \tag{3.14}$$

Estimating $K_2(t)$. Applying the Cauchy-Schwartz inequality, then

$$\begin{aligned} K_2(t) &= 2\int_0^t \widehat{k}_j(s) v^j_t(0,s)ds \\ &\le \varepsilon E_j(t) + \frac{1}{\varepsilon}\left| \widehat{k}_j(t) \right|^2 + \left\| \widehat{k}_j{}' \right\|_{L^1(0,T)} + \int_0^t \left| \widehat{k}_j{}'(s) \right| E_j(s) ds, \end{aligned} \tag{3.15}$$

for all $\varepsilon > 0$.
Moreover, applying the imbedding $W^{1,1}(0,T) \hookrightarrow C^0([0,T])$, there exists a positive constant d_T such that

$$\|v\|_{C^0([0,T])} \le d_T \|v\|_{W^{1,1}(0,T)}, \text{ for all } v \in W^{1,1}(0,T). \tag{3.16}$$

Therefore

$$K_2(t) \le \varepsilon E_j(t) + \frac{d_T^2}{\varepsilon} \left\|\widehat{k_j}\right\|^2_{W^{1,1}(0,T)} + \left\|\widehat{k_j}\right\|_{W^{1,1}(0,T)} + \int_0^t \left|\widehat{k_j}'(s)\right| E_j(s)ds. \tag{3.17}$$

Estimating $K_3(t)$. Since $v_j^2(0,t) \le E_j(t)$, we deduce from the assumption (A_3) that

$$\begin{aligned} K_3(t) &= -2\int_0^t \widehat{h}_j(u^j(0,s)) v_t^j(0,s)ds \\ &\le \varepsilon E(t) + \int_0^t \left|u_t^j(0,s)\right| E_j(s)ds \\ &\quad + \frac{1}{\varepsilon}\left\|\widehat{h}_j\right\|^2_{C^0([-M,M])} + \left\|u_t^j(0,\cdot)\right\|_{L^1(0,T)} \left\|\widehat{h}_j{}'\right\|^2_{C^0([-M,M])}. \end{aligned} \tag{3.18}$$

We remark that

$$\begin{aligned} \left\|u_t^j(0,\cdot)\right\|_{L^1(0,T)} &\le T^{1-1/\min\{p',2\}} \left\|u_t^j(0,\cdot)\right\|_{L^{\min\{p',2\}}(0,T)} \\ &\le T^{1-1/\min\{p',2\}} M^{2/\min\{p',2\}} = D_T. \end{aligned} \tag{3.19}$$

Then it follows that

$$K_3(t) \le \varepsilon E(t) + \int_0^t \left|u_t^j(0,s)\right| E_j(s)ds + \left(\frac{1}{\varepsilon} + D_T\right)\left\|\widehat{h}_j\right\|^2_{C^1([-M,M])}. \tag{3.20}$$

Estimating $K_4(t)$. Proving in a similar way to (2.100), we have

$$\begin{aligned} K_4(t) &= -2\int_0^t [h(u^j(0,s)) - h(u(0,s))] v_t^j(0,s)ds \\ &\le \varepsilon E_j(t)\|h\|_{C^2([-M,M])} + \int_0^t d_j(s)E(s)ds, \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} d_j(s) = \|h\|_{C^2([-M,M])} &\left[\varepsilon\left(\left|u_t^j(0,s)\right| + |u_t(0,s)|\right)\right. \\ &\left. + \left(1 + \frac{1}{\varepsilon}\right) T\left(\left\|u_t^j(0,\cdot)\right\|_{L^1(0,T)} + \|u_t(0,\cdot)\|_{L^1(0,T)} + 1\right)\right]. \end{aligned} \tag{3.22}$$

Using the inequality (3.19) again, this implies

$$d_j(s) \le \|h\|_{C^2([-M,M])}\left[\varepsilon\left(\left|u_t^j(0,s)\right| + |u_t(0,s)|\right) + \left(1 + \frac{1}{\varepsilon}\right) T\left(2D_T + 1\right)\right] = \widehat{d}_j(s). \tag{3.23}$$

It implies

$$K_4(t) \le \varepsilon E_j(t)\|h\|_{C^2([-M,M])} + \int_0^t \widehat{d}_j(s)E(s)ds. \tag{3.24}$$

Estimating $K_5(t)$. By reusing (3.16) and Lemma 2.1, then

$$\begin{aligned} K_5(t) &= -2\int_0^t v_t^j(0,s)ds\int_0^s \widehat{g}_j(s-\tau)u^j(0,\tau)d\tau \\ &\leq \varepsilon E_j(t) + \int_0^t E_j(s)ds + \left[T(M+d_T)^2 + \frac{M^2}{\varepsilon}\right]\|\widehat{g}_j\|^2_{W^{1,1}(0,T)}. \end{aligned} \tag{3.25}$$

Estimating $K_6(t)$. From the assumptions (A_2), we are easy to show that

$$\begin{aligned} K_6(t) &= -2\int_0^t v_t^j(0,s)ds\int_0^s g(s-\tau)v^j(0,\tau)d\tau \\ &\leq \varepsilon E_j(t) + \left(\frac{1}{\varepsilon}\|g\|^2_{L^2(0,T)} + \|g'\|^2_{L^1(0,T)} + 2|g(0)| + 1\right)\int_0^t E_j(s)ds. \end{aligned} \tag{3.26}$$

Combining (3.12), (3.14), (3.17), (3.20) and (3.24)-(3.26), we see that

$$\begin{aligned} E(t) \leq &\int_0^t \left[\left(\varepsilon\|h\|_{C^2([-M,M])}+1\right)\left|u_t^j(0,s)\right| + \left|\widehat{k}_j{}'(s)\right| + M_T^3(s)\right]E(s)ds \\ &+ \varepsilon M_T^1 E_j(t) + M_T^2\left(\|\widehat{g}_j\|^2_{W^{1,1}(0,T)} + \left\|\widehat{h}_j\right\|^2_{C^1([-M,M])} + \sum_{i=1}^2\left\|\widehat{k}_j\right\|^i_{W^{1,1}(0,T)}\right). \end{aligned} \tag{3.27}$$

where

$$M_T^1 = \|h\|_{C^2([-M,M])} + 4, \tag{3.28}$$

$$M_T^3 = \frac{d_T^2}{\varepsilon} + T(M+d_T)^2 + \frac{M^2+1}{\varepsilon} + D_T + 1, \tag{3.29}$$

$$\begin{aligned} M_T^3(s) = &\left[\left(1+\frac{1}{\varepsilon}\right)T\left(2D_T+1\right) + \varepsilon\left|u_t(0,s)\right|\right]\|h\|_{C^2([-M,M])} \\ &+ \|r_M(\cdot,u_t(s))\| + \frac{1}{\varepsilon}\|g\|^2_{L^2(0,T)} + \|g'\|^2_{L^1(0,T)} + 2|g(0)| + 2. \end{aligned} \tag{3.30}$$

Choosing $\varepsilon M_T^1 = \frac{1}{2}$, then applying the Gronwall inequality, we deduce from (3.4), (3.19) and (3.27)-(3.30) that

$$\begin{aligned} E_j(t) \leq\ &2M_T^2\exp\left[2\left\|M_T^3\right\|_{L^1(0,T)} + 2\left(\varepsilon\|h\|_{C^2([-M,M])}+1\right)D_T + 2C_*\right] \\ &\times\left(\|\widehat{g}_j\|^2_{W^{1,1}(0,T)} + \left\|\widehat{h}_j\right\|^2_{C^1([-M,M])} + \sum_{i=1}^2\left\|\widehat{k}_j\right\|^i_{W^{1,1}(0,T)}\right). \end{aligned} \tag{3.31}$$

This shows that

$$\begin{aligned} &\left\|v_t^j\right\|^2_{L^\infty(0,T;L^2(0,1))} + \left\|v^j\right\|^2_{L^\infty(0,T;H^1(0,1))} \\ &\leq M_T\left(\|\widehat{g}_j\|^2_{W^{1,1}(0,T)} + \left\|\widehat{h}_j\right\|^2_{C^1([-M,M])} + \sum_{i=1}^2\left\|\widehat{k}_j\right\|^i_{W^{1,1}(0,T)}\right), \end{aligned} \tag{3.32}$$

where M_T is a constant depending only on T, C_*, f, h, u_0, u_1.
On the other hand, by Lemma 2.2, it follows that

$$\|v(0,\cdot)\|_{C^0([0,T])} \leq \sqrt{2}\|v\|_{L^\infty(0,T;H^1(0,1))},\ \text{for all } v \in L^\infty(0,T;H^1(0,1)). \tag{3.33}$$

Combining (3.1), (3.2), (3.32) and (3.33), we obtain (3.3).
The proof of Theorem 3.1 is complete. □

References

[1] M. Aassila, M.M. Cavalcanti and V.N. Domingos Cavalcanti, *Existence and uniform decay of the wave equation with nonlinear boundary damping and boundary memory source term*, Cal. Variations and Part. Diff. Equations, **15** (2002), 155-180.

[2] M. Bergounioux et al., *Mathematical model for a shock problem involving a linear viscoelastic bar*, Nonlinear Anal., **43** (2001), 547-561.

[3] E. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, (1955), p. 43.

[4] A. Dang and D. Alain, *Mixed problem for some semilinear wave equation with a nonhomogeneous condition*, Nonlinear Anal., **12** (1988), 581-592.

[5] N.T. Le, H.K. Le and L.T. Nguyen, *On a nonlinear wave equation associated with the boundary conditions involving convolution*, Nonlinear Anal., **11** (2009), 3943-3965.

[6] T.X. Le, G. Giang Vo, *The shock of a rigid body and a nonlinear viscoelastic bar associated with a nonlinear boundary condition*, J. Science Natural Sciences, Uni. Peda. HCM, **8** (2006), 70-81.

[7] J.L. Lions, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod, Gauthier-Villars, Paris, (1969), p. 61, Section 5.

[8] L.T. Nguyen, N.T. Le, *On a nonlinear wave equation with boundary conditions of two-point type*, Math. Anal. Appl., **385** (2012), 1070-1093.

[9] L.T. Nguyen, T.M. Tran, *A semilinear wave equation associated with a nonlinear integral equation*, Demonstratio Math., **36** (2003), 915-938.

[10] L.T. Nguyen, G. Giang Vo, *A wave equation associated with mixed nonhomogeneous conditions: Global existence and asymptotic expansion of solutions*, Nonlinear Anal., **66** (2007), 1526-1546.

[11] L.T. Nguyen, G. Giang Vo, *Existence and asymptotic expansion for a nonlinear wave equation associated with nonlinear boundary conditions*, Nonlinear Anal., **67** (2007), 1791-1819.

[12] P. Radu, *Weak solutions to the Cauchy problem of a semilinear wave equation with damping and source terms*, Ad. Differential Equations, **10** (2005), 1261-1300.

[13] J. Rivera, D. Andrade, *Exponential decay of a nonlinear wave equation with a viscoelastic boundary condition*, Math. Meth. Appl. Sci. **23** (2000), 41-61.

[14] M.L. Santos, *Decay rates for solutions of semilinear wave equations with a memory condition at the boundary*, Electron. J. Qual. Theory Differ. Equations **7** (2002), 1-17.

[15] D. Takači, A. Takači, *On the approximate solution of mathematical model of a viscoelastic bar*, Nonlinear Anal., **67** (2007), 1560-1569.

Giai Giang Vo

Department of Mathematics, Faculty of Information Technology,
Huflit University, 155 SuVanHanh, HCMC, Vietnam

E-mail address: g.giangvo@gmail.com