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BEHAVIOUR OF THE NEW CLASS OF THE RATIONAL DIFFERENCE EQUATIONS

O. MOAAZ, MAHMOUD A.E. ABDELRAHMAN

ABSTRACT. In this paper we study the behavior of the solution of the new class of the rational Difference Equations. Namely, we study the stability, boundedness, periodicity and the oscillation of the solution. Moreover some interesting counter examples are given in order to verify our strong results.

1. INTRODUCTION

The study of Difference Equations has been great interest in various branches of mathematics. These Difference Equations describing real life situations in probability theory, queuing theory, statistical problems, stochastic time series, combinatorial analysis, number theory, electrical network, quanta in radiation, genetics in biology, economics, psychology, sociology, etc. So our study of the Difference Equations is so interesting.

There is a set of nonlinear difference equations, known as the rational difference equations, all of which consists of the ratio of two polynomials in the sequence terms in the same from .there has been many work about the global asympototic of solutions of rational difference equations [1]-[13].

In this paper, we are concerned with analytical investigation of the solution of the following difference equation

$$x_{n+1} = a + b\frac{x_{n-l}}{x_{n-k}} + c\frac{x_{n-l}}{x_{n-s}},\ n = 0, 1, ..., \tag{1}$$

where the initial conditions $x_{-r}, x_{-r+1}, ..., x_{-1}, x_0,\ r = \max\{l, k, s\}$ are arbitrary positive real numbers and a, b, c, d are positive constants.

In this section we present the basic definitions and theorems of the our model, namely equilibrium points, local and global stability, boundedness, periodicity and the oscillation of the solution.

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Definition 1 (**Equilibrium point**)
Consider a difference equation in the form

$$x_{n+1} = F(x_{n-l}, x_{n-k}, x_{n-s}), \quad n = 0, 1, 2, ... \tag{2}$$

where F is a continuous function, while l, k and s are positive integers. A point $\overline{x}$ is said to be an equilibrium point of the equation (4) if it is a fixed point of F, i.e., $\overline{x} = F(\overline{x}, \overline{x}\ , \overline{x})$.

Definition 2 (**stability**)
Let $\overline{x} \in (0, \infty)$ be an equilibrium point of equation (4). Then we have

(a) (**local stability**)
An equilibrium point $\overline{x}$ of equation (4) is said to be locally stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $x_{-\nu} \in (0, \infty)$ for $\nu = 0, 1, ..., r$ with

$$\sum_{i=0}^{r} |x_{-i} - \overline{x}| < \delta,$$

then $\mid x_n - \overline{x} \mid < \varepsilon$ for all $n \geq -r$.

(b) (**local asymptotic stability**)
An equilibrium point $\overline{x}$ of equation (4) is said to be locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that, if $x_{-\nu} \in (0, \infty)$ for $\nu = 0, 1, ..., r$ with

$$\sum_{i=0}^{r} |x_{-i} - \overline{x}| < \gamma,$$

then

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(c) (**global stability**)
An equilibrium point $\overline{x}$ of equation (4) is said to be a global attractor if for every $x_{-\nu} \in (0, \infty)$ for $\nu = 0, 1, ..., r$ we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(d) (**global asymptotic stability**)
An equilibrium point $\overline{x}$ of equation (4) is said to be globally asymptotically stable if it is locally stable and a global attractor.

(v) (**unstability**)
An equilibrium point $\overline{x}$ of equation (4) is said to be unstable if it is not locally stable.

Definition 3 (**periodicity**)
A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with period t if $x_{n+t} = x_n$ for all $n \geq -r$. A sequence $\{x_n\}_{n=-r}^{\infty}$ is said to be periodic with prime period t if t is the smallest positive integer having this property

Definition 4 (**boundedness**)
Equation (4) is called permanent and bounded if there exists numbers m and M with $0 < m < M < \infty$ such that for any initial conditions $x_{-\nu} \in (0, \infty)$ for $\nu = 0, 1, ..., r$ there exists a positive integer N which depends on these initial conditions such that $0 < m < M < \infty$ for all $n \geq N$.

Definition 5 The linearized equation of equation (4) about the equilibrium point $\overline{x}$ is defined by the linear difference equation

$$z_{n+1} = \sum_{i=0}^{k} h_i z_{n-i} \tag{3}$$

where

$$h_i = \frac{\partial F(\overline{x}, \overline{x}, ..., \overline{x})}{\partial x_{n-i}}, \ kispositiveinteger$$

Theorem 1[10] Assume that $h_i, i = 0, 1, 2 \in R$. Then

$$\sum_{i=0}^{2} \mid h_i \mid < 1,$$

is a sufficient condition for the asymptotic stability of equation (4).

2. The stability of solutions

In this section we study the local stability character of the solutions of equation (1). The positive equilibrium point of equation (1) is given by

$$\overline{x} = a + b + c,$$

Now, we define the continuous function $f : (0, \infty)^3 \to (0, \infty)$, such that

$$f(u_1, u_2, u_3) = a + b\frac{u_2}{u_1} + c\frac{u_2}{u_3},$$

Therefore, it follows that

$$\begin{aligned} \frac{\partial f}{\partial u_1}(u_1, u_2, u_3) &= -\frac{bu_2}{u_1^2}, \\ \frac{\partial f}{\partial u_2}(u_1, u_2, u_3) &= \frac{b}{u_1} + \frac{c}{u_3}, \\ \frac{\partial f}{\partial u_3}(u_1, u_2, u_3) &= -\frac{cu_2}{u_3^2}. \end{aligned} \tag{4}$$

Theorem 2 The equilibrium point $\overline{x}$ of equation(1) is locally stable, if $a > b+c$.

Proof. The linearized equation of (1) about the equilibrium point $\overline{x}$ is the linear difference equation

$$z_{n+1} = \sum_{i=0}^{2} \frac{\partial f(\overline{x}, \overline{x}, \overline{x})}{\partial u_{n-i}} z_{n-i}$$

From (4), we obtain

$$\begin{aligned} \frac{\partial f}{\partial u_1}(\overline{x}, \overline{x}, \overline{x}) &= -\frac{b}{a+b+c} = -h_0, \\ \frac{\partial f}{\partial u_2}(\overline{x}, \overline{x}, \overline{x}) &= \frac{b+c}{a+b+c} = -h_1, \end{aligned}$$

and

$$\frac{\partial f}{\partial u_3}(\overline{x}, \overline{x}, \overline{x}) = -\frac{c}{a+b+c} = -h_{2.}$$

Then the linearized equation

$$z_{n+1} + h_0 z_n + h_1 z_{n-1} + h_2 z_{n-2} = 0 \tag{5}$$

It is follows by Theorem 1 that, equation (5) is locally stable if

$$\sum_{i=0}^{2} |h_i| < 1.$$

This implies that

$$\left|\frac{b}{a+b+c}\right| + \left|\frac{b+c}{a+b+c}\right| + \left|\frac{c}{a+b+c}\right| < 1.$$

and so,

$$b + c < a$$

Hence, the proof is completed. □

The following counter example shows the stability of solution of equation (1).

Example 1 We consider the following initial data: $x_{-2} = 2, x_{-1} = 3, x_0 = 2$ for equation (1) with $l = 2, k = 0, s = 1, a = 1, b = 0.5$ and $c = 0.3$, Figure 1 .

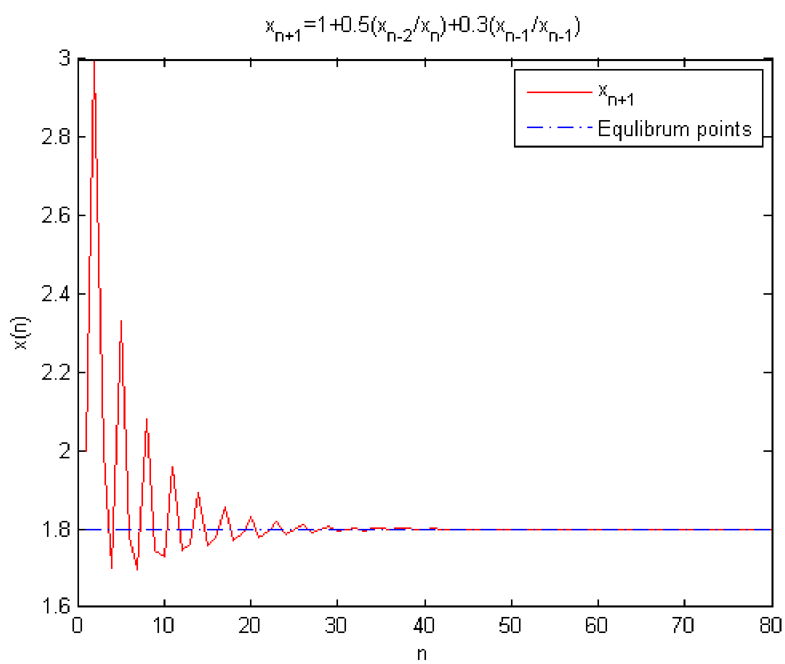

FIGURE 1. Stability of solution.

Now we introduce the following theorem, which will be useful for the investigation of the global attractivity of solution of equation (1).

Theorem 3 Let $[a, b]$ be an interval of real numbers and assume that

$$F : [a, b]^3 \to [a, b]$$

is a continuous function satissfying the following properties:

(i) *$F(u_1, u_2, u_3)$ is non-increeasing in u_1 and u_3 for each u_2 in $[a, b]$ and non-decreeasing in u_2 for each u_1, u_3 in $[a, b]$.*

(ii) *If* $(m, M) \in [a, b] \times [a, b]$ *is a solution of the system*

$$m = F(M, m, M) \quad \text{and} \quad M = F(m, M, m) \tag{6}$$

implies

$$m = M.$$

Then equation (1) has a unique equilibrium point $\overline{x} \in [a, b]$ *and every solution of equation (1) converges to* $\overline{x}$.

Proof. We define the sequances $\{m_i\}_{i=0}$ and $\{M_i\}_{i=0}$ as the following

$$m_0 = a, \; m_i = F(M_{i-1}, m_{i-1}, M_{i-1})$$

and

$$M_0 = b, \; M_i = F(m_{i-1}, M_{i-1}, m_{i-1})$$

for $i = 1, 2, ...$

From propertiy (i), we note that

$$m_{i+1} = F(M_i, m_i, M_i) \geq F(M_{i-1}, m_{i-1}, M_{i-1}) = m_i$$

and

$$M_{i+1} = F(m_i, M_i, m_i) \leq F(m_{i-1}, M_{i-1}, m_{i-1}) = M_i$$

Thus, we have the sequances $\{m_i\}_{i=0}$ non-decreeasing and $\{M_i\}_{i=0}$ non-increeasing and hence

$$m_0 \leq m_1 \leq ... \leq m_i \leq ... \leq M_i \leq ... \leq M_1 \leq M_0 = b$$

and so,

$$m_i \leq x_j \leq M_i \quad \text{for} \quad j \geq 3i + 1. \tag{7}$$

Now, we set

$$m = \lim_{i \to \infty} m_i$$

and

$$M = \lim_{i \to \infty} M_i$$

From (7), we obtain

$$M \geq \limsup_{i \to \infty} x_i \geq \liminf_{i \to \infty} x_i \geq m$$

By the continuity of F, we have that (m, M) is a solution of the system (6), then

$$m = \overline{x} = M.$$

□

Theorem 4 If $a \neq b + c$, then the positive equilibrium point $\overline{x}$ of Eq. (1) is global attractor.

Proof. From (4), we can easily see that the continuous function f defined by

$$f(u_1, u_2, u_3) = a + b\frac{u_2}{u_1} + c\frac{u_2}{u_3},$$

is non-increeasing in u_1 and u_3 and non-decreeasing in u_2.

Assume that (m, M) is a solution of the system

$$m = F(M, m, M) \quad \text{and} \quad M = F(m, M, m)$$

Thus, by equation (1), we find

$$m = a + b\frac{m}{M} + c\frac{m}{M} \quad \text{and} \quad M = a + b\frac{M}{m} + c\frac{M}{m}$$

and so,

$$Ma + (b + c)\, m = mM = ma + (b + c)\, M,$$

hence,

$$(M - m)\,(a - b - c) = 0.$$

If $a \neq b + c$, then we get $M = m$. It follows by Theorem 2 that $\overline{x}$ is a global attractor of equation (1) and then the proof is complete. □

3. Boundedness of the solutions

In this section, we investigate the boundedness of the positive solutions of equation (1).

Theorem 5 Assume that $\{x_n\}_{n=-r}$be a solution of equation (1).

(i) If $b + c < 1$ and the initial conditions $x_{\eta-r+1}$, ..., $x_{\eta-1}$, $x_\eta \in [b + c, 1]$, for some $\eta \geq 0$. Then we have the inequality

$$a + (b + c)^2 \leq x_n \leq a + 1, \quad \text{for } n \geq \eta. \tag{8}$$

(ii) If $b + c > 1$ and the initial conditions $x_{\eta-r+1}$, ..., $x_{\eta-1}$, $x_\eta \in [1, b + c]$, for some $\eta \geq 0$. Then we have the inequality

$$a + 1 \leq x_n \leq a + (b + c)^2, \quad \text{for } n \geq \eta. \tag{9}$$

Proof. For (i), let $x_{\eta-r+1}$, ..., $x_{\eta-1}$, $x_\eta \in [b + c, 1]$, for some $\eta \geq 0$. Thus, we get

$$\begin{aligned} x_{\eta+1} &= a + b\frac{x_{\eta-l}}{x_{\eta-k}} + c\frac{x_{\eta-l}}{x_{\eta-s}} \\ &\leq a + \frac{b}{b+c} + \frac{c}{b+c} \\ &= a + 1. \end{aligned} \tag{10}$$

Also, we obtain

$$\begin{aligned} x_{\eta+1} &\geq a + b\,(b + c) + c\,(b + c) \\ &= a + (b + c)^2. \end{aligned} \tag{11}$$

From (10) and (11), we deduce for all $n \geq \eta$ that the inequality (8) is valid. Hence, the proof of part (i) is completed. Similarly, we can prove part (ii). Thus, the proof is now completed. □

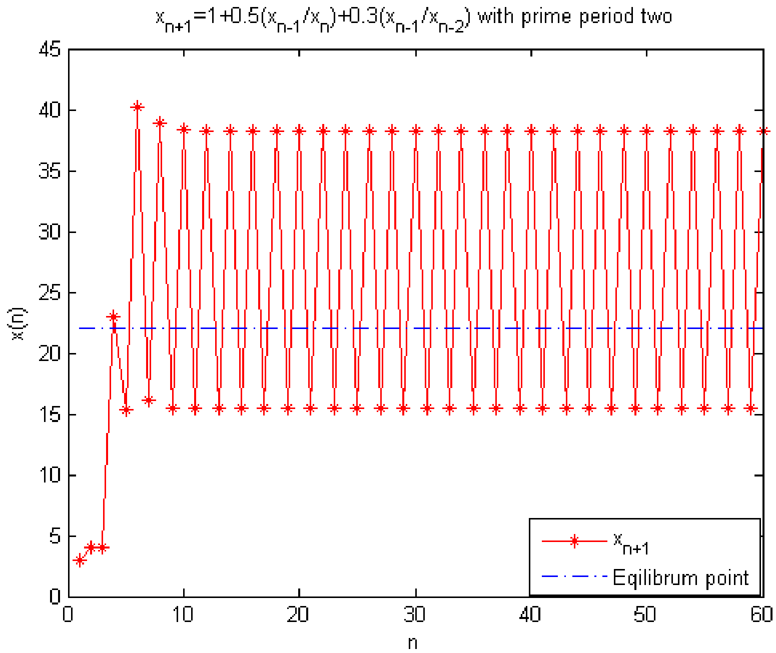

FIGURE 2. Prime period two.

4. Periodic and Oscillatory solutions

Theorem 6 Assume that $l-odd$ and $k, s-even$ positive integers. If $a \neq b+c$, then equation (1) has no prime period two solution.

Proof. Suppose that there exists a distinct prime period two solution

$$..., \beta, \gamma, \beta, \gamma, ...$$

of equation (1). Thus, we have $x_{n-l} = \beta$ and $x_{n-k} = x_{n-s} = \gamma$.

From equation (1), we find

$$\beta = a + b\frac{\beta}{\gamma} + c\frac{\beta}{\gamma} \tag{12}$$

and

$$\gamma = a + b\frac{\gamma}{\beta} + c\frac{\gamma}{\beta}. \tag{13}$$

Subtracting (12) from (13), we get

$$(\beta - \gamma)(a - b - c) = 0,$$

Since $a \neq b+c$, then $\beta = \gamma$. This is a contradiction. Thus, the proof is completed. □

Remark 1 In the previous theorem, we can get the prime period two if $a = b+c$. This nice result will be illustrated in the following counter example, Figure 2.

Example 2 We consider the following initial data: $x_{-2} = 3, x_{-1} = 4, x_0 = 4$ for equation (1) with $l = 1, k = 0, s = 2, a = 11, b = 8$ and $c = 3$, Figure 2 .

Theorem 7 Assume that $l-even$ and $k,s-odd$ positive integers. Then equation (1) has no prime period two solution.

Proof. Let there exists a distinct prime period two solution

$$...,\beta,\gamma,\beta,\gamma,...$$

of equation (1). Thus, we have $x_{n-l}=\gamma$ and $x_{n-k}=x_{n-s}=\beta$.

From equation (1), we find

$$\beta=a+b\frac{\gamma}{\beta}+c\frac{\gamma}{\beta}$$

and

$$\gamma=a+b\frac{\beta}{\gamma}+c\frac{\beta}{\gamma}.$$

Hence, we obtain

$$\beta^2=a\beta+(b+c)\gamma \tag{14}$$

and

$$\gamma^2=a\gamma+(b+c)\beta. \tag{15}$$

By subtracting (14) from (15), we get

$$\beta+\gamma=a-b-c \tag{16}$$

By adding (14) and (15), we have

$$\beta\gamma=-(a-b-c)(b+c) \tag{17}$$

But, from (16) and (17), we see that $(\beta+\gamma)(\beta\gamma)<0$,which contradicts β, γ are positives. This completes the proof. □

Remark 2 The all remaining cases for l,k,s matches with one of the above two cases given in Theorems 4 and 4 .

The following theorem shows the oscillation behaviour of solutions of equation (1).

Theorem 8 Assume that l is *odd* and k,s are *even*. Then, equation (1) has an oscillatory solution.

Proof. Initially, without loss of generality we may assume that $s=\max\{l,k,s\}$, and let the sequance $\{x_n\}_{n=-s}$ be a solution of the equation (1). Now, we suppose that

$$\textbf{(i)}\quad x_{-s+(2\mu-1)}>\overline{x} \quad\text{and}\quad x_{-s+2\lambda}<\overline{x}$$

for $\mu=1,2,...,\frac{s}{2}$ and $\lambda=0,1,...,\frac{s}{2}$. Hence, we obtain

$$\begin{aligned} x_1 &= a+b\frac{x_{-l}}{x_{-k}}+c\frac{x_{-l}}{x_{-s}}, \\ &> a+b\frac{\overline{x}}{\overline{x}}+c\frac{\overline{x}}{\overline{x}} \\ &= \overline{x} \end{aligned}$$

and

$$\begin{aligned} x_2 &= a+b\frac{x_{-l+1}}{x_{-k+1}}+c\frac{x_{-l+1}}{x_{-s+1}}, \\ &< \overline{x} \end{aligned}$$

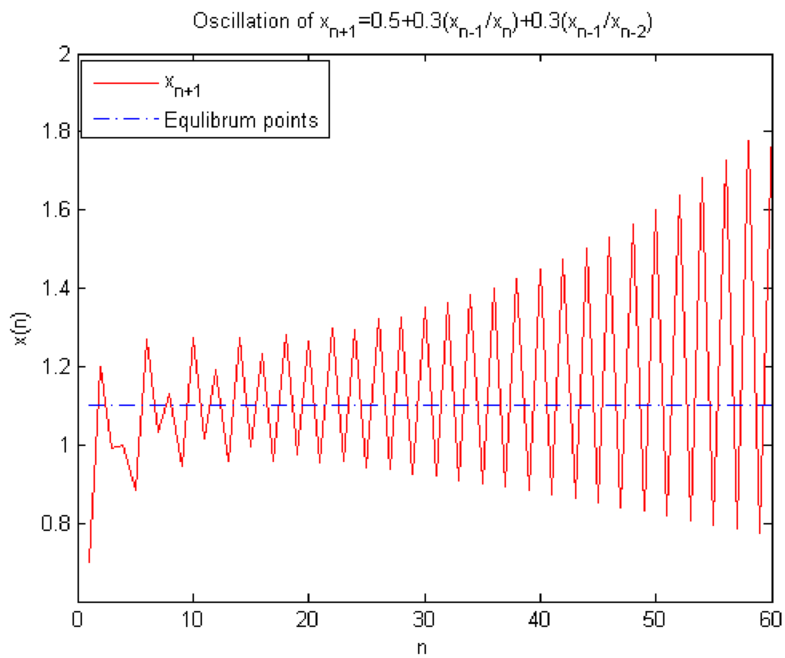

FIGURE 3. Oscillation of solution.

Next, let $x_{2\upsilon-1} > \overline{x}$ and $x_{2\upsilon} < \overline{x}$ for $\upsilon = 1, 2, ..., t$, and we shall prove that $x_{2t+1} > \overline{x}$ and $x_{2t+2} < \overline{x}$. From equation (1), we find

$$\begin{aligned} x_{2t+1} &= a + b\frac{x_{-l+2t}}{x_{-k+2t}} + c\frac{x_{-l+2t}}{x_{-s+2t}}, \\ &> \overline{x} \end{aligned}$$

and

$$\begin{aligned} x_{2t+2} &= a + b\frac{x_{-l+2t+1}}{x_{-k+2t+1}} + c\frac{x_{-l+2t+1}}{x_{-s+2t+1}}, \\ &< \overline{x} \end{aligned}$$

Similarly, if

$$\textbf{(ii)} \quad x_{-s+(2\mu-1)} < \overline{x} \quad \text{and} \quad x_{-s+2\lambda} > \overline{x}$$

for $\mu = 1, 2, ..., \frac{s}{2}$ and $\lambda = 0, 1, ..., \frac{s}{2}, ,$ we can prove that $x_{2\upsilon-1} < \overline{x}$ and $x_{2\upsilon} > \overline{x}$ for $\upsilon = 1, 2, ...$ Hence, the proof is completed. □

The following counter example show the oscillation behaviour of solution of equation (1).

Example 3 We consider the following initial data: $x_{-3} = 1, x_{-2} = 0.99, x_{-1} = 1.2, x_0 = 0.7$ for equation (1) with $l = 3, k = 0, s = 2, a = 0.5, b = 0.3$ and $c = 0.3$, Figure 3.

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O. Moaaz

Mansora University, Mansora, Egypt

E-mail address: o_moaaz@mans.edu.eg

Mahmoud A.E. Abdelrahman

Mansora University, Mansora, Egypt

E-mail address: mahmoud.abdelrahman@mans.edu.eg