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SOME INTEGRALS FOR THE GENERALIZED BESSEL MAITLAND FUNCTIONS

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Abstract. The present paper is the investigation of some integrals for the generalized Bessel-Maitland functions $J_{\nu,q}^{\mu,\gamma}(z)$, which are expressed in terms of generalized (Wright) hypergeometric functions. Some interesting special cases involving Bessel functions, generalized Bessel functions, generalized Mitagg-Leffler functions, Struv's functions are deduced. These results are also established in terms of generalized Wright hypergeometric functions.

1. Introduction

In recent years, many integral formulas involving a variety of special functions have been developed by many authors (see [1],[2],([3]),[4],[16], for example). Several integral formulas involving product of Bessel functions have been developed and play an important role in several physical problems. In fact, Bessel functions are associated with a wide range of problems in diverse areas of mathematical physics. Here, we aim at presenting two generalized integral formulas involving the generalized Bessel-Maitland function, which are expressed in terms of generalized (Wright) hypergeometric functions. Some interesting special cases of our main results are also considered.

Definition 1 The special function of the form defined by the series representation as

$$J_\nu^\mu(z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{m!\ \Gamma(\nu+\mu m+1)}\ (\mu > 0;\ z \in C) \tag{1}$$

is known as Bessel-Maitland function, or the Wright generalized function (see[7]). An interesting generalization of the Bessel function $J_{\nu,\sigma}^\mu(z)$ defined by [10] as follow

$$J_{\nu,\sigma}^\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\ (z/2)^{\nu+2\sigma+2m}}{\Gamma(\sigma+m+1)\ \Gamma(\nu+\sigma+\mu m+1)}, \tag{2}$$

where$z \in C\backslash(-\infty,\ 0];\ \mu > 0,\ \nu,\ \sigma \in C$.

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Further, another generalization of the generalized Bessel-Maitland function $J_{\nu,q}^{\mu,\gamma}(z)$ defined by [14] as follows

$$J_{\nu,q}^{\mu,\gamma}(z) = \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}(-z)^m}{m!\ \Gamma(\nu+\mu m+1)}, \tag{3}$$

where $\mu, \nu, \gamma \in C, Re(\mu) \geq 0, Re(\nu) \geq -1, Re(\gamma) \geq 0$ and $q \in (0,1) \bigcup N$ and $(\gamma)_0 = 1, (\gamma)_{qm} = \frac{\Gamma(\gamma+qm)}{\Gamma(\gamma)}$, denotes the generalized pochhammer symbol (see [12]).

We investigate some special cases of the generalized Bessel-Maitland function (3) by giving particular values to the parameters μ, ν, γ, q.
If $q = 1, \gamma = 1$ and ν is replaced by $\nu + \sigma$ and z by $\left(\frac{z^2}{4}\right)$ in (3), we get

$$J_{\nu+\sigma,1}^{\mu,1}\left(\frac{z^2}{4}\right) = \Gamma(\sigma+m+1)\left(\frac{z}{2}\right)^{-\nu-2\sigma} J_{\nu,\sigma}^{\mu}(z), \tag{4}$$

where $J_{\nu,\sigma}^{\mu}(z)$ denotes Bessel-Maitland function defined by (3).
Also, if $\mu = 1$ and $\sigma = \frac{1}{2}$ in (4), we get

$$J_{\nu+\frac{1}{2},1}^{1,1}\left(\frac{z^2}{4}\right) = \Gamma\left(m+\frac{3}{2}\right)\left(\frac{z}{2}\right)^{-\nu-1} H_{\nu}(z), \tag{5}$$

where $H_\nu(z)$ denotes Struve's function (see [8]) defined as follows

$$H_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\ (z/2)^{\nu+2m+1}}{\Gamma(m+\frac{3}{2})\ \Gamma(\nu+m+\frac{3}{2})}. \tag{6}$$

We have also some important special cases of the generalized Bessel-Maitland function $J_{\nu,q}^{\mu,\gamma}(z)$, as follows.
If $q = 0$, equation (3) reduces to

$$J_{\nu,0}^{\mu,\gamma}(z) = J_{\nu}^{\mu}(z), \tag{7}$$

where $J_\nu^\mu(z)$ is generalized Bessel function defined by (1).
if $q = 0$ and ν is replaced by $\nu - 1$ and z by $-z$, (3) reduces to

$$J_{\nu-1,0}^{\mu,\gamma}(-z) = \Phi(\mu, \nu; z), \tag{8}$$

known as Wright function (see, [5] was introduced by Wright [21].
if $q = 0, \mu = 1$ and z is replaced by $\left(\frac{z^2}{4}\right)$, (3) reduces to

$$J_{\nu,0}^{1,\gamma}\left(\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^{-\nu} J_\nu(z), \tag{9}$$

an ordinary Bessel function [16].
If ν is replaced by $\nu - 1$ and z by $-z$, (3) reduces to

$$J_{\nu-1,q}^{\mu,\gamma}(-z) = E_{\mu,\nu}^{\gamma,q}(z), \tag{10}$$

where $\mu,\ \nu, \gamma \in C,\ Re(\mu) > 0,\ , Re(\nu) > 0,\ Re(\gamma) > 0,\ q \in (0,1) \bigcup N$, and $E_{\mu,\nu}^{\gamma;q}(z)$ denotes generalized Mittag-Leffler function, was given by Shukla and Prajapati [18].
If $q = 1$ and ν is replaced by $\nu - 1$ and z by $-z$, (3) reduces to

$$J_{\nu-1,1}^{\mu,\gamma}(-z) = E_{\mu,\nu}^{\gamma}(z), \tag{11}$$

was introduced by Prabhakar [15].
If $\gamma = 1, q = 1$ and ν is replaced by $\nu - 1$ z by $-z$, (3) reduces to

$$J_{\nu-1,1}^{\mu,1}(-z) = E_{\mu,\nu}(z), \tag{12}$$

where $\mu \in C, Re(\mu) > 0,\ Re(\nu) > 0$, was studied by Wiman [23].
If $\nu = 0, q = 1, \gamma = 1$ and z is replaced by $-z$,(3) reduces to

$$J_{0,1}^{\mu,1}(-z) = E_\mu(z), \tag{13}$$

where $\mu \in C, Re(\mu) > 0$, was introduced by Ghosta Mittag-Leffler [12].
If $\mu = k \in N$ and $q \in N$ in (3), we get

$$J_{\nu,q}^{k,\gamma}(z) = \frac{1}{\Gamma(\nu+1)}\, {}_qF_k \left[\begin{array}{ccc} \Delta(q;\gamma) & ; & \frac{-q^q z}{k^k} \\ \Delta(k;\nu+1) & ; & \end{array} \right]. \tag{14}$$

where $J_{\nu,q}^{k;\gamma}(z)$ is another representation of the generalized Bessel-Maitland function defined by (see [19]), ${}_qF_k(.)$ is the generalized hypergeometric function and the symbol $\Delta(q;\gamma)$ is a q-tuple $\frac{\gamma}{q},\ \frac{\gamma+1}{q}, ...,\ \frac{\gamma+q-1}{q};\ \Delta(k;\nu+1)$ is a k-tuple $\frac{\nu+1}{k}, \frac{\nu+2}{k}, ..., \frac{\nu+k}{k}$. The generalization of the generalized hypergeometric series ${}_pF_q$ (10) is due to Fox[9] and Wright ([24],[25],[26], for example) who studied the asymptotic expansion of the generalized Wright hypergeometric function defined by (see, also [21]).

$${}_p\Psi_q \left[\begin{array}{cccc} (\alpha_1\ ,\ A_1), & ....., & (\alpha_p\ ,\ A_p); & \\ & & & z \\ (\beta_1\ ,\ B_1), & ....., & (\beta_q\ ,\ B_q); & \end{array} \right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j k)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j k)} \frac{z^k}{k!} \tag{15}$$

where the coefficients $A_1, \cdots, A_p$ *and* $B_1, \cdots, B_q$ are positive real numbers such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0. \tag{16}$$

A special case of (15)

$${}_p\Psi_q \left[\begin{array}{cccc} (\alpha_1\ ,\ 1), & ....., & (\alpha_p\ ,\ 1); & \\ & & & z \\ (\beta_1\ ,\ 1), & ....., & (\beta_q\ ,\ 1); & \end{array} \right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)}\, {}_pF_q \left[\begin{array}{cccc} \alpha_1, & ....., & \alpha_p\ ; & \\ & & & z \\ \beta_1, & ....., & \beta_q\ ; & \end{array} \right] \tag{17}$$

where ${}_pF_q$ is the generalized hypergeometric series defined by [15].

$${}_pF_q \left[\begin{array}{cccc} \alpha_1, & ....., & \alpha_p\ ; & \\ & & & z \\ \beta_1, & ....., & \beta_q\ ; & \end{array} \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!}$$

$$= {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots \beta_q;\ z), \tag{18}$$

$(\lambda)_n$ is called the Pochhammer's symbol [16].
For our present investigation, we required equation (3) and also need to recall the following Oberhettinger's integral formula [13].

$$\int_0^\infty x^{\delta-1}(x + a + \sqrt{x^2 + 2ax})^{-\lambda} dx = 2\lambda a^{-\lambda} \left(\frac{a}{2}\right)^\delta \frac{\Gamma(2\delta)(\Gamma(\lambda-\delta)}{\Gamma(1+\lambda+\delta)}, \tag{19}$$

provided $0 < R(\delta) < R(\lambda)$.

2. Main Results

In this section, we established two generalized integral formulas, which are expressed in terms of generalized (Wright) hypergeometric functions, by inserting the generalized Bessel-Maitland function (3) with suitable arguments into the integrand of (19).

Theorem 1 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_{\nu,q}^{\mu,\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \frac{2^{1-\delta}a^{\delta-\lambda}\Gamma(2\delta)}{\Gamma(\gamma)}\ {}_3\psi_3\left[\begin{array}{ll} (\lambda,q),\ (\lambda-\delta,1),\ (1+\lambda,1); & \\ & \frac{-y}{a} \\ (\nu+1,\mu),\ (1+\lambda+\delta,1),(\lambda,1); & \end{array}\right], \quad (20)$$

where $x > 0$; $\delta,\lambda,\mu,\nu,\gamma \in C$; $\Re(\delta) > 0, \Re(\nu) > -1, \Re(\mu) > 0, \Re(\gamma) > 0, 0 < \Re(\delta) < \Re(\lambda)$, and $q \in (0,1)\bigcup N$.

Theorem 2 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_{\nu,q}^{\mu,\gamma}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \frac{2^{1-\delta}a^{\delta-\lambda}\Gamma(\lambda-\delta)}{\Gamma(\gamma)}\ {}_3\psi_3\left[\begin{array}{ll} (\lambda,q),\ (2\delta,2),\ (1+\lambda,1); & \\ & \frac{-y}{2} \\ (\nu+1,\mu),\ (1+\lambda+\delta,2),(\lambda,1); & \end{array}\right] \quad (21)$$

where $x > 0$; $\delta,\lambda,\mu,\nu,\gamma \in C$; $\Re(\delta) > 0, \Re(\nu) > -1, \Re(\mu) > 0, \Re(\gamma) > 0, 0 < \Re(\delta) < \Re(\lambda)$, and $q \in (0,1)\bigcup N$.

Proof. By applying (3) to the integrand of (20) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_{\nu,q}^{\mu,\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \sum_{m=0}^{\infty}\frac{(\gamma)_{qm}(-1)^m y^m}{m!\ \Gamma(\nu+\mu m+1)}\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda-m}dx \quad (22)$$

In view of the conditions given in theorem 1, since
$\Re(\nu) > -1,\ 0 < \Re(\delta) < \Re(\lambda)$, We can apply the integral formula(19) to the integral (22) and obtain the following expression:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_{\nu,q}^{\mu,\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \frac{2^{1-\delta}a^{\delta-\lambda}\Gamma(2\delta)}{\Gamma(\gamma)}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(\gamma+qm)\Gamma(\lambda+1+m)\Gamma(\lambda-\delta+m)}{m!\ \Gamma(\nu+\mu m+1)\Gamma(\lambda+1)\Gamma(\lambda+\delta+1+m)}\left(\frac{y}{2}\right)^m, \quad (23)$$

which, upon using (17), yields (20). This completes the proof of theorem (1).
It is easy to see that a similar arguments as in the proof of theorem (2) will establish the integral formula (21).
Next we consider other variations of theorem 1 and theorem 2. In fact, we establish some integral formulas for the generalized Bessel-Maitland function $J_{\nu,q}^{\mu,\gamma}(z)$

expressed in terms of the generalized hypergeometric function ${}_qF_k$. To do this, we recall the well-known Legendre duplication formula (see, [20], for example) as:

$$(\lambda)_{2n} = 2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda+1}{2}\right)_n \quad (n\varepsilon N_0) \tag{24}$$

now we are ready to state the following two corollaries.

Corollary 1 Let the condition of theorem 1 be satisfied and replacing μ by k in the generalized Bessel-Maitland function $J^{\mu,\gamma}_{\nu,q}(z)$ in (20) and using(14). Then the following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J^{k,\gamma}_{\nu,q}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \frac{y\ 2^{-\delta}a^{\delta-\lambda}\Gamma(2\delta)\Gamma(\lambda+2)\Gamma(\lambda+1-\delta)}{\Gamma(\nu+1)\Gamma(\lambda+1)\Gamma(\lambda+\delta+2)}\ {}_qF_k\left[\begin{array}{ccc}\Delta(q;\gamma) & ; & \\ & & \frac{-q^q}{k^k} \\ \Delta(k;\nu+1) & ; & \end{array}\right], \tag{25}$$

where $\Re(\delta)>0, \Re(\lambda)>0$, and $\Delta(q;\gamma)$ is a q-tuple $\frac{\gamma}{q},\ \frac{\gamma+1}{q}, ..., \frac{\gamma+q-1}{q}$; $\Delta(k;\nu+1)$ is a k-tuple $\frac{\nu+1}{k}, \frac{\nu+2}{k}, ..., \frac{\nu+k}{k}$.

Corollary 2 Let the condition of theorem 2 be satisfied and replacing μ by k in the generalized Bessel-Maitland function $J^{\mu,\gamma}_{\nu,q}(z)$ in (21) and using the relation (14). Then the following integral formula holds true:

$$\int_0^\infty x^{\delta}(x+a+\sqrt{x^2+2ax})^{-\lambda} J^{k,\gamma}_{\nu,q}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= \frac{y\ 2^{-\delta}a^{\delta-\lambda}\Gamma(2\delta)\Gamma(\lambda+2)\Gamma(\lambda+1-\delta)}{\Gamma(\nu+1)\Gamma(\lambda+1)\Gamma(\lambda+\delta+2)}\ {}_qF_k\left[\begin{array}{ccc}\Delta(q;\gamma) & ; & \\ & & \frac{-q^q}{k^k} \\ \Delta(k;\nu+1) & ; & \end{array}\right], \tag{26}$$

provided $\lambda, \delta \in C;\ \Re(\delta)>0, \Re(\lambda)>0, \Re(\delta)<\Re(\lambda$ and$x>0$.

Proof. By writing the the right-hand side of (20) in the original summation formula, after a little simplification, we find that, when the last resulting summation is expressed in terms of ${}_qF_k$ in the relation (14), this completes the proof of corollary 1. Similarly, it is easy to see that a similar argument as in proof of corollary 1 will established the integral formula (26). Therefore we omit the details of the proof of the corollary 2.

3. Special Cases

On setting $q=0$ in theorem 1 and theorem 2 and making use of the relation (7), then the generalized Bessel-Maitland function $J^{\mu,\gamma}_{\nu,q}(z)$ will have following relation with Bessel-Maitland function $J^{\mu}_{\nu}(z)$ as follows:

1 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J^{\mu}_{\nu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta}a^{\delta-\lambda}\Gamma(2\delta)\ {}_2\psi_3\left[\begin{array}{cc}(\lambda-\delta,1),\ (1+\lambda,1); & \\ & \frac{-y}{a} \\ (\nu+1,\mu),\ (1+\lambda+\delta,1),(\lambda,1); & \end{array}\right], \tag{27}$$

provided $\lambda, \delta \in C; 0<\Re(\delta)<\Re(\lambda)$ and$x>0$.

2 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu^\mu\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\delta}a^{\delta-\lambda}\Gamma(\lambda-\delta)\ {}_2\psi_3\left[\begin{array}{c}(2\delta,2),\ (1+\lambda,1);\\ \\ (\nu+1,\mu),\ (1+\lambda+\delta,2),(\lambda,1);\end{array}\ \frac{-y}{a}\right], \qquad (28)$$

provided $\lambda, \delta \in C; 0 < \Re(\delta) < \Re(\lambda)$ and$x > 0$.

On setting $q = 0$ and ν is replaced by $\nu - 1$ and z is by $-z$ in theorem 1 and theorem 2 and making use of the relation (8), we obtain the following integral formulas involving the Wright function as follows:

3 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\Phi\left(\mu;\nu;\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\delta}a^{\delta-\lambda}\Gamma(2\delta)\ {}_2\psi_3\left[\begin{array}{c}(\lambda-\delta,1),\ (1+\lambda,1);\\ \\ (\nu,\mu),\ (1+\lambda+\delta,1),(\lambda,1);\end{array}\ \frac{y}{a}\right], \qquad (29)$$

provided $\lambda, \delta \in C; 0 < \Re(\delta) < \Re(\lambda)$ and$x > 0$.

4 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda}\Phi\left(\mu;\nu;\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\delta}a^{\delta-\lambda}\Gamma(\lambda-\delta)\ {}_2\psi_3\left[\begin{array}{c}(2\delta,2),\ (1+\lambda,1);\\ \\ (\nu,\mu),\ (1+\lambda+\delta,2),(\lambda,1);\end{array}\ \frac{-y}{2}\right], \qquad (30)$$

provided $\lambda, \delta \in C; 0 < \Re(\delta) < \Re(\lambda)$ and $x > 0$.

On setting $\mu = 1, q = 0$ and z is replaced by $\left(\frac{z^2}{4}\right)$ in theorem 1 and theorem 2 and making use of the relation (9), into accounts, yields the following integral formulas involving the ordinary Bessel function as follows:

5 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda-2\nu} J_\nu\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= \left(\frac{y}{2}\right)^\nu 2^{1-\delta}a^{\delta-\lambda-\nu}\Gamma(2\delta)\ {}_2\psi_3\left[\begin{array}{c}(\lambda+2\nu-\delta,2),\ (1+\lambda+2\nu,2);\\ \\ (\nu+1,1),\ (1+\lambda+2\nu+\delta,2),(\lambda+2\nu,2);\end{array}\ \frac{-y^2}{4a^2}\right], \qquad (31)$$

provided $\lambda, \delta \in C; 0 < \Re(\delta) < \Re(\lambda)$ and $x > 0$.

6 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda-2\nu} J_\nu\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$
$$= 2^{1-\delta}a^{\delta-\lambda-2\nu}\Gamma(\lambda+2\nu-\delta)\ {}_2\psi_3\left[\begin{array}{c}(2\delta,4),\ (1+\lambda+2\nu,2);\\ \\ (\nu+1,1),\ (1+\lambda+2\nu+\delta,4),(\lambda+2\nu,2);\end{array}\ \frac{-y^2}{16}\right], \qquad (32)$$

provided $\lambda, \delta \in C; 0 < \Re(\delta) < \Re(\lambda)$ and $x > 0$.

On replacing ν by $\nu-1$ and z by $-z$ in theorem 1 and theorem 2, and making use of the relation (10) , we get the following integral formulas involving the generalized Mittag-Leffler function as follows:

7 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu,\nu}^{\gamma,q}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta}a^{\delta-\lambda}\Gamma(2\delta)\ {}_3\psi_3\left[\begin{array}{cc} (\gamma,q),(\lambda-\delta,1),\ (1+\lambda,1); & \\ & \frac{y}{a} \\ (\nu,\mu),\ (1+\lambda+\delta,1),(\lambda,1); & \end{array}\right], \tag{33}$$

where $\delta,\lambda,\mu,\nu,\gamma \in C,\ \Re(\delta)>0,\Re(\lambda)>0,\Re(\mu)>0,\Re(\nu)\geq -1,\Re(\gamma)>0,q\in (0,1)\bigcup N$.

8 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu,\nu}^{\gamma,q}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta}a^{\delta-\lambda}\Gamma(\lambda-\delta)\ {}_3\psi_3\left[\begin{array}{cc} (\gamma,q),(2\delta,2),\ (1+\lambda,1); & \\ & \frac{y}{2} \\ (\nu,\mu),\ (1+\lambda+\delta,2),(\lambda,1); & \end{array}\right], \tag{34}$$

where $\delta,\lambda,\mu,\nu,\gamma \in C,\ \Re(\delta)>0,\Re(\lambda)>0,\Re(\mu)>0,\Re(\nu)\geq -1,\Re(\gamma)>0,q\in (0,1)\bigcup N$ and $x>0$.

On setting $q=1$ and ν is replaced by $\nu-1$ and z by $-z$ in theorem 1 and theorem 2 and making use of the relation (11), we get the following integrals formulas involving the Mittag-Leffler functions as follows:

9 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu,\nu}^{\gamma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta}a^{\delta-\lambda}\Gamma(2\delta)\ {}_3\psi_3\left[\begin{array}{cc} (\gamma,1),(\lambda-\delta,1),\ (1+\lambda,1); & \\ & \frac{y}{a} \\ (\nu,\mu),\ (1+\lambda+\delta,1),(\lambda,1); & \end{array}\right], \tag{35}$$

where $\delta,\lambda\in C;\ \Re(\delta)>0,\Re(\lambda)>0$.

10 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu,\nu}^{\gamma}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta}a^{\delta-\lambda}\Gamma(\lambda-\delta)\ {}_3\psi_3\left[\begin{array}{cc} (\gamma,1),(2\delta,2),\ (1+\lambda,1); & \\ & \frac{y}{2} \\ (\nu,\mu),\ (1+\lambda+\delta,2),(\lambda,1); & \end{array}\right] \tag{36}$$

where $\delta,\lambda\in C;\ \Re(\delta)>0,\Re(\lambda)>0,\ x>0$.

On setting $\gamma=1$ and ν is replaced by $\nu-1$ and z by $-z$ in theorem 1 and theorem 2 and making use of the relation (12), we get the following integral formulas involving the Mittag-Leffler function as given below.

11 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu,\nu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right)dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \Gamma(2\delta)\ {}_3\psi_3 \left[\begin{array}{c} (\lambda-\delta,1),\ (1+\lambda,1)),\ (1,1); \\ \\ (\nu,\mu),\ (1+\lambda+\delta,1), (\lambda,1); \end{array} \frac{y}{a} \right], \tag{37}$$

where $\delta, \lambda, \mu, \nu, \in C$; $\Re(\delta) > 0, \Re(\lambda) > 0,\ x > 0$.

12 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu,\nu}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \Gamma(\lambda-\delta)\ {}_3\psi_3 \left[\begin{array}{c} (2\delta,2),\ (1+\lambda,1),\ (1,1); \\ \\ (\nu,\mu),\ (1+\lambda+\delta,2),\ (\lambda,1); \end{array} \frac{y}{2} \right], \tag{38}$$

where $\delta, \lambda, \mu, \nu, \in C$; $\Re(\delta) > 0, \Re(\lambda) > 0,\ x > 0$.

On setting $\nu = 0, q = 1, \gamma = 1$ and z is replaced by $-z$ in theorem 1 and theorem 2 and making use of the relation(14), we get the following integral formulas as follows:

13 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \Gamma(2\delta)\ {}_3\psi_3 \left[\begin{array}{c} (\lambda-\delta,1),\ (1+\lambda,1),\ (1,1); \\ \\ (1,\mu),\ (1+\lambda+\delta,1),\ (\lambda,1); \end{array} \frac{y}{a} \right], \tag{39}$$

where $\delta, \lambda, \mu, \in C$; $\Re(\delta) > 0, \Re(\lambda) > 0$.

14 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} E_{\mu}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \Gamma(\lambda-\delta)\ {}_3\psi_3 \left[\begin{array}{c} (2\delta,2),\ (1+\lambda,1),\ (1,1); \\ \\ (1,\mu),\ (1+\lambda+\delta,2),\ (\lambda,1); \end{array} \frac{y}{2} \right], \tag{40}$$

where $\delta, \lambda, \mu, \in C$; $\Re(\delta) > 0, \Re(\lambda) > 0$.

On setting $q = 1, \gamma = 1, \nu = \nu + \sigma$ and z is replaced by $\left(\frac{z^2}{4}\right)$ in theorem 1 and theorem 2 and using the relation (4), then the generalized Bessel-Maitland function $J^{\mu,\gamma}_{\nu,q}(z)$ will have following relation with Bessel-Maitland function $J^{\mu}_{\nu,\sigma}(z)$ as follows.

15 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J^{\mu}_{\nu,\sigma}\left(\frac{y}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \Gamma(2\delta)\ {}_3\psi_3 \left[\begin{array}{c} (\lambda-\delta,2),\ (1+\lambda,2),\ (1,1); \\ \\ (\nu+\sigma+1,\mu),\ (1+\lambda+\delta,2),\ (\lambda,2); \end{array} \frac{-y^2}{4a^2} \right], \tag{41}$$

where $\delta, \lambda, \sigma, \nu \in C$; $\Re(\delta) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$.

16 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J^{\mu}_{\nu,\sigma}\left(\frac{xy}{x+a+\sqrt{x^2+2ax}}\right) dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \Gamma(\lambda-\delta)\ {}_3\psi_3 \left[\begin{array}{ll} (2\delta,4),\ (1+\lambda,2),\ (1,1); & \\ & \frac{-y^2}{16} \\ (\nu+\sigma+1,\mu),\ (1+\lambda+\delta,4),\ (\lambda,2); & \end{array} \right], \quad (42)$$

where $\delta, \lambda, \sigma, \nu \in C;\ \Re(\delta) > 0, \Re(\lambda) > 0, \Re(\mu) > 0$.

Also, setting $\mu = 1,\ \sigma = \frac{1}{2}$ in (41) and (42) and using the relation (5), then we get the integral formulas involving the Struv's function $H_\nu(z)$ as follows.

17 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} H_\nu \left(\frac{y}{x+a+\sqrt{x^2+2ax}} \right) dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \Gamma(2\delta)\ {}_3\psi_3 \left[\begin{array}{ll} (\lambda-\delta,2),\ (1+\lambda,2),\ (1,1); & \\ & \frac{-y^2}{4a^2} \\ (\nu+\frac{3}{2},1),\ (1+\lambda+\delta,2), (\lambda,2); & \end{array} \right], \quad (43)$$

where $\delta, \lambda, \nu \in C;\ \Re(\delta) > 0, \Re(\lambda) > 0$..

18 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} H_\nu \left(\frac{xy}{x+a+\sqrt{x^2+2ax}} \right) dx$$

$$= 2^{1-\delta} a^{\delta-\lambda} \Gamma(\lambda-\delta)\ {}_3\psi_3 \left[\begin{array}{ll} (2\delta,4),\ (1+\lambda,2),\ (1.1); & \\ & \frac{-y^2}{16} \\ (\nu+\frac{3}{2},1),\ (1+\lambda+\delta,4), (\lambda,2); & \end{array} \right], \quad (44)$$

where $\delta, \lambda, \nu \in C;\ \Re(\delta) > 0, \Re(\lambda) > 0$.

19 The following integral formula holds true:

$$\int_0^\infty x^{\delta-1}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu^k \left(\frac{y}{x+a+\sqrt{x^2+2ax}} \right) dx$$

$$= \frac{y\ 2^{-\delta} a^{\delta-\lambda} \Gamma(2\delta)\Gamma(\lambda+2)\Gamma(\lambda+1-\delta)}{\Gamma(\nu+1)\Gamma(\lambda+1)\Gamma(\lambda+\delta+2)}\ {}_0F_k \left[\begin{array}{ll} --- & ; \\ & \frac{-q^q}{k^k} \\ \Delta(k;\nu+1) & ; \end{array} \right], \quad (45)$$

where k is positive integer $\delta, \lambda, \nu \in C;\ \Re(\delta) > 0, \Re(\lambda) > 0$.

20 The following integral formula holds true:

$$\int_0^\infty x^{\delta}(x+a+\sqrt{x^2+2ax})^{-\lambda} J_\nu^k \left(\frac{xy}{x+a+\sqrt{x^2+2ax}} \right) dx$$

$$= \frac{2^{-\delta} a^{\delta-\lambda} \Gamma(\lambda-\delta)\Gamma(\lambda+2)\Gamma(2\delta+2)}{\Gamma(\nu+1)\Gamma(\lambda+1)\Gamma(\lambda+\delta+3)}\ {}_0F_k \left[\begin{array}{ll} --- & ; \\ & \frac{-q^q}{k^k} \\ \Delta(k;\nu+1) & ; \end{array} \right], \quad (46)$$

where k is positive integer $\delta, \lambda, \nu \in C;\ \Re(\delta) > 0, \Re(\lambda) > 0$.

4. Concluding Remark

In the present paper, we investigate new integrals involving the generalized Bessel-Maitland function $J_{\nu,q}^{\mu,\gamma}(z)$, in terms of the generalized (Wright) hypergeometric function. Certain special cases of integrals involving the generalized Bessel-Maitland function have been investigated in the literature by a number of authors with different arguments (see,[12],[15], [18], [19], for example). Also, the generalized Bessel-Maitland function $J_{\nu,q}^{\mu,\gamma}(z)$ are expressed in terms of Fox H-function [9], and the generalized hypergeometric function ${}_qF_k$ [14]. Therefore, the results presented in this paper are easily converted in terms of a similar type of new interesting integrals with different arguments after some suitable parametric replacements.

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