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NONLINEAR DYNAMICS OF COURNOT DUOPOLY GAME WITH SOCIAL WELFARE

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Abstract. This paper explores the idea of cournot duopoly with homogenous expectations in a context of bounded rationality, where two semipublic firms endeavor to maximize the weighted average of social welfare and their own profit. The stability analysis of the fixed points are analyzed and complex dynamic features including period doubling bifurcations of the unique Nash equilibrium is also investigated. Numerical simulations are carried out to show the complex behavior of the models' parameters. We discuss the possibility of partial privatization of the two firms. We investigate the effects of the government own on the profits and social welfare.

1. Introduction

Duopoly game is the most basic form of oligopoly, a market dominated by a small number of companies. There are often several duopolistic firms in economic market where competition among them is controlled by the amount of commodities they produce, the demand scheme they adopt and the profit of each firm wants to maximize. In the competition, firms produce the same or homogenous goods and they must focus not only on the market size, but also on the actions their competitors do. Cournot [8] and Bertrand [6] oligopoly are the two most notable models in oligopoly theory. In the first one firms control their output level which influences the market price, while in the second one, firms change the price to affect the market demand .

Currently, under the assumption of bounded rationality, the research result of the duopoly game model theory has been widely used in realistic problems of market price and quantity competition see, [13], [18] and [23]. Expectations play an important role in modelling economic phenomena, a producer can choose his expectations rules of many available techniques to adjust his production outputs. Some authors considered duopolies with homogeneous expectations and found a variety of complex dynamics in their games [2],

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[3]. On the other hand, each firm adjusts his outputs towards the profit maximizing amount as target by using a rule with different strategies [11], [20].
When bounded rational and adaptive expectations are chosen, the nonlinear models become complicated and no analytical tool are available. For these reasons in [1], [17] a detailed numerical analysis of the equilibria, complex dynamical behavior and some interesting feature through Neimark-Sacker and period-doubling bifurcations. Basic properties of a delayed Cournot duopoly game with heterogeneous bounded rationality have been analyzed by meaning of bifurcation diagrams and stability regions in [12]. Global bifurcations in a piecewise-smooth Cournot duopoly game with isoelastic demand function and unit costs have been presented in [22]. The conventional model of dynamic quantity-setting game where players are assumed to hold a uniform and accurate belief towards market is replaced by a more realistic model with subjective demand errors has been investigated in [19]. Fast adjustment cause a chaotic behavior of a nonlinear real estate model based on cobweb theory, where the demand function and supply function are quadratic have been analyzed in [15].
All related literature analyze firms' dynamic behavior by assuming a private oligopoly where they are merely keen on their individual profits. That is to say, all firms in the literature are assumed to be private. However, there are many firms with different ownership structures for example, (publicly-owned firms tend to maximize the social welfare, but partially publicly-owned firms tend to maximize the weighted average of the social welfare and its own profit), see in [4], [24].
The main purpose of this paper is to investigate the dynamic behavior of Cournot oligopoly game incorporating semipublic firms where the bounded rational players update their production strategies at discrete time periods by an adjustment mechanism based on maximize their individual profits and the social welfare.
The structure of this paper is as follows: In Section 2 we formulate a Cournot duopoly game based on a generalized inverse demand function and social welfare. In Section 3 the system modelled by a two dimensional map of two semipublic firms, the existence and local stability of fixed points are analyzed. Complex dynamics behavior occur under some changes of control parameters of the model which are shown by numerical experiments in Section 4 . Firms' decisions as to role of s_i in section 5 . Finally, conclusions are drawn in Section 6 .

2. The social welfare model

Consider an economy model with two types of agents: firms and consumers. There exists a competitive sector that produces the numeraire good. Duopolistic sector with two firms, namely firm 1 and firm2 each of which produces a differentiated good denoted by $p_i > 0$ and $q_i > 0$, $i = 1, 2$ the price and quantity of product of firm i, respectively. Assume that there exists a continuum of identical consumers that have preferences towards $q = (q_1, q_2)$ whose utility function is linear and separable in the numeraire good. The representative consumer maximizes $U(q_1, q_2) - p_1q_1 - p_2q_2$. where the function $U(q) : R^2_+ \to R_+$ is asumed to be quadratic and twice differentiable function. Following J. C. B-Ruiz ([4],[5]) and Singh & Vives [21], we assume that the utility function is quadratic (leading to linear demand functions) and given by:

$$U\left(q_1, q_2\right) = a\left(q_1 + q_2\right) - \frac{1}{2}\left(q_1^2 + 2bq_1q_2 + q_2^2\right), \tag{2.1}$$

where $a > 0$ is a parameter that captures the size of the market demand and the parameter $b \geq 0$ measures the degree to which goods are substitutes. The representative consumer's optimization problem is given by:

$$\max_{q_1, q_2} U(q_1, q_2) - p_1q_1 - p_2q_2 + M,$$

where $M \geq 0$ denotes the consumer's income. By solving this problem, one yields the following inverse demand functions $p_i = \frac{\partial U}{\partial q_i}, \;\; i = 1, 2.$
From Eq. (2.1) it follows that:

$$\frac{\partial U}{\partial q_i} = a - q_i - bq_j \qquad i,\; j = 1, 2, \quad i \neq j.$$

Then the inverse demand functions are given by:

$$p_i\left(q_i, q_j\right) = a - q_i - bq_j \qquad i,\; j = 1, 2, \quad i \neq j. \tag{2.2}$$

Therefore, firm $i\prime s$ cost function is quadratic and given by:

$$C_i\left(q_i\right) = F + q_i^2 \qquad i = 1, 2, \tag{2.3}$$

where marginal cost $F = 0$ with no loss of generality. Hence profits of firm i can be written as follows:

$$\begin{aligned} \pi_i &= p_iq_i - C_i\left(q_i\right), \qquad i = 1, 2 \\ \pi_i &= \left(a - q_i - bq_j\right)\; q_i - q_i^2 \quad i,\; j = 1, 2, \quad i \neq j. \end{aligned} \tag{2.4}$$

We measure social welfare at time t as the sum of consumer surplus (denoted by CS) and the total industry profits. Therefore, social welfare is given by:

$$W\left(t\right) = CS + \pi_1 + \pi_2, \tag{2.5}$$

where consumer surplus is expressed as the representative consumer's utility as follows:

$$\begin{aligned} CS &= U(q_1, q_2) - p_1q_1 - p_2q_2 \qquad (2.6) \\ &= \frac{1}{2}\left(q_1^2 + q_2^2\right) + bq_1q_2. \end{aligned}$$

We assume that both firms are semipublic. Hence, each firm maximizes the following weighted average of the social welfare and its own profit

$$\begin{aligned} V_1\left(t\right) &= s_1W\left(t\right) + \left(1 - s_1\right)\pi_1 \\ &= s_1\left(CS + \pi_2\right) + \pi_1 \\ &= \frac{-1}{2}\left(4 - s_1\right)q_1^2 - \frac{3}{2}s_1q_2^2 + \left(a - bq_2\right)q_1 + as_1q_2. \end{aligned} \tag{2.7}$$

Similar,

$$\begin{aligned} V_2\left(t\right) &= s_2W\left(t\right) + \left(1 - s_2\right)\pi_2 \\ &= s_2\left(CS + \pi_1\right) + \pi_2 \\ &= \frac{-1}{2}\left(4 - s_2\right)q_2^2 - \frac{3}{2}s_2q_1^2 + \left(a - bq_1\right)q_2 + as_2q_1. \end{aligned} \tag{2.8}$$

Where $s_1, s_2 \in [0,1]$ are the degree of public ownership. As they decrease, semipublic firms are tending to focus on private profits $\pi_1(t)$ and $\pi_2(t)$. By setting $s_1 = s_2 = 0$, we can get two private firms, by setting $s_1 = s_2 = 1$, we can get two public firms, setting $s_1 \in (0,1)$ and $s_2 = 0$, one can get semipublic firm 1 and private firm 2; by setting $s_1 = 0$ and $s_2 \in (0,1)$, one can get private firm 1 and semipublic firm 2; and finally by setting $s_1, s_2 \in (0,1)$, one can get two semipublic firms.
The objective of each firm is to maximize the weighted average of the social welfare and its own profit by choosing its output. Differentiating Eqs. (2.7) and (2.8) with respect to outputs of each firms gives rise to the following:

$$\Phi_1(t) = \frac{\partial V_1(t)}{\partial q_1(t)} = a - q_1(4 - s_1) - bq_2, \tag{2.9}$$

and

$$\Phi_2(t) = \frac{\partial V_2(t)}{\partial q_2(t)} = a - q_2(4 - s_2) - bq_1. \tag{2.10}$$

Following Bischi and Naimzada [7], the adjustment mechanism of outputs over time of the ith firm is described by:

$$q_i(t+1) = q_i(t) + \alpha_i \; q_i(t) \;\; \Phi_i(t), \tag{2.11}$$

where $\alpha_i > 0$ is a coefficient of the speed at which firm i adjusts its outputs. Account Eq. (2.11), the two-dimensional system that characterizes the dynamics of Cournot duopoly game is the following,

$$T : \begin{cases} q_1(t+1) = q_1(t) + \alpha_1 q_1(t)\left[a - q_1(t)(4 - s_1) - bq_2(t)\right] \\ q_2(t+1) = q_2(t) + \alpha_2 q_2(t)\left[a - q_2(t)(4 - s_2) - bq_1(t)\right] \end{cases}. \tag{2.12}$$

Equilibria or fixed points of T are solutions of the following equation $T(q_1, q_2) = (q_1, q_2)$. Trivially, One can obtain the following four equilibria of dynamic system (2.12).

$$1)\ E_1(0,0) \qquad 2)\ E_2\left(0, \frac{a}{4 - s_2}\right) \qquad 3) E_3\left(\frac{a}{4 - s_1}, 0\right), \tag{2.13}$$

and the unique Nash equilibrium $E_4(q_1^*, q_2^*)$ where,

$$\begin{aligned} q_1^* &= \frac{a\ \left[b - (4 - s_2)\right]}{b^2 - (4 - s_1)(4 - s_2) - s_1 s_2}, \\ q_2^* &= \frac{a\ \left[b - (4 - s_1)\right]}{b^2 - (4 - s_1)(4 - s_2) - s_1 s_2}. \end{aligned} \tag{2.14}$$

Provided that by the following conditions:

$$\begin{aligned} b &> \max\{(4 - s_1), (4 - s_2)\} \\ b^2 &> (4 - s_1)(4 - s_2) - s_1 s_2 \end{aligned} \tag{2.15}$$

or,

$$\begin{aligned} b &< \min\{(4 - s_1), (4 - s_2)\} \\ b^2 &< (4 - s_1)(4 - s_2) - s_1 s_2 \end{aligned} \tag{2.16}$$

3. Dynamic behavior of semipublic firms

We will focus on the analysis dynamics, equilibria of the dynamical system (2.12) and the unique Nash equilibrium $E_4\left(q_1^*, q_2^*\right)$. In order to study the local stability of equilibrium points of system (2.12), we consider the Jacobian matrix along the variable strategy $\left(q_1, q_2\right)$.

$$J\left(q_1, q_2\right)=\left(\begin{array}{cc} J_1 & -\alpha_1 b q_1 \\ -\alpha_2 b\ q_2 & J_2 \end{array}\right), \tag{3.1}$$

where,

$$\begin{aligned} J_1 &= 1+\alpha_1\left[a-2 q_1\left(4-s_1\right)-q_2 b\right] \\ J_2 &= 1+\alpha_2\left[a-b q_1-2 q_2\left(4-s_2\right)\right]. \end{aligned}$$

Theorem 1. *The trivial equilibrium point E_1 is source unstable point of system (2.12).*

Proof. An equilibrium is stable if and only if all eigenvalues of the related Jacobian matrix are less than one in the absolute value. The Jacobian matrix at equilibrium E_1 takes the form of:

$$J_{E_1(0,0)}=\left(\begin{array}{cc} 1+\alpha_1 a & 0 \\ 0 & 1+\alpha_2 a \end{array}\right),$$

whose eigenvalues are given by the diagonal entries are,

$$\begin{aligned} \lambda_1 &= 1+\alpha_1 a \\ \lambda_2 &= 1+\alpha_2 a \end{aligned}$$

since $\alpha_1, \alpha_2>0$ and $a>0$. It is clear that $\left|\lambda_1\right|>1$ and $\left|\lambda_2\right|>1$. Then equilibrium point $E_1(0,0)$ is source unstable point of system (2.12). □

Theorem 2. *Boundary equilibrium point $E_2\left(0, \frac{a}{4-s_2}\right)$ of system (2.12) has at least four different topological types for all parameters values:*

- i: *E_2 is a sink locally asymptotic if $0<a\alpha_2<2$, $b>4-s_2$ and $a\alpha_1<\frac{4-s_2}{4-s_2-b}$,*
- ii: *E_2 is a source point if $a\alpha_2>2$, $b<4-s_2$ and $a\alpha_1>\frac{4-s_2}{4-s_2-b}$,*
- iii: *E_2 is a non-hyperbolic point if $a\alpha_2=2$ or $b=4-s_2$,*
- iv: *E_2 is a saddle point for the other values of parameters except the values in $(i)-(iii)$.*

Proof. In order to prove this results, we consider the Jacobian matrix J at E_2 which takes the form:

$$J_{E_2\left(0, \frac{a}{4-s_2}\right)}=\left(\begin{array}{cc} 1+\alpha_1 a\left[1-\frac{b}{4-s_2}\right] & 0 \\ \frac{-a\ b\ \alpha_2}{4-s_2} & 1-\alpha_2 a \end{array}\right),$$

which has two eigenvalues ,

$$\begin{aligned} \lambda_1 &= 1-\alpha_2 a \\ \lambda_2 &= 1+\alpha_1 a\left[1-\frac{b}{4-s_2}\right] \end{aligned}$$

since for values $\alpha_i, a>0$, $s_2 \in(0,1)$, $i=1,2$ and $b \geq 0$. If $0<a\alpha_2<2$,hence $\left|\lambda_1\right|<1$. If $b>4-s_2$ and $a\alpha_1<\frac{4-s_2}{4-s_2-b}$, then $\left|\lambda_2\right|<1$ this lead to E_2 is a sink point. E_2 is a source point if $a\alpha_2 > 2$, $b < 4 - s_2$ and $a\alpha_1 > \frac{4-s_2}{4-s_2-b}$. E_2 is a non-hyperbolic point if $|\lambda_1| = 1$ or $|\lambda_2| = 1$and this is holds if $a\alpha_2 = 2$ or $b = 4 - s_2$. For the other values of parameters E_2 become saddle point. From condition (iii), it is easy to see that one of these eigenvalues of $E_2\left(0, \frac{a}{4-s_2}\right)$ is equal to -1 and the other other is neither 1 nor -1. Then all parameters locate in the following region:

$$\Omega_{E_2} = \left\{ \begin{array}{c} (a, b, s_2, \alpha_1, \alpha_2) : a\alpha_2 = 2, a\alpha_1 \neq \frac{4-s_2}{4-s_2-b} \text{ and } b \neq 4 - s_2, \\ \text{where } \alpha_i, a > 0, s_2 \in (0, 1) \end{array} \right\}.$$

Then $E_2\left(0, \frac{a}{4-s_2}\right)$ can pass through flip bifurcation when the parameters varying in the small neighborhood of Ω_{E_2}. □

Theorem 3. *Boundary equilibrium point $E_3\left(\frac{a}{4-s_1}, 0\right)$ of system (2.12) has at least four different topological types for all parameters values:*

i: *E_2 is a sink locally asymptotic if $0 < a\alpha_1 < 2$, $b > 4 - s_1$ and $a\alpha_2 < \frac{4-s_1}{4-s_1-b}$,*
ii: *E_2 is a source point if $a\alpha_1 > 2$, $b < 4 - s_1$ and $a\alpha_2 > \frac{4-s_1}{4-s_1-b}$,*
iii: *E_2 is a non-hyperbolic point if $a\alpha_1 = 2$ or $b = 4 - s_1$,*
iv: *E_2 is a saddle point for the other values of parameters except the values in $(i) - (iii)$.*

Proof. We consider the Jacobian matrix J at E_3 which takes the form:

$$J_{E_3\left(\frac{a}{4-s_1}, 0\right)} = \left(\begin{array}{cc} 1 - \alpha_1 a & \frac{-a\ b\ \alpha_1}{4-s_1} \\ 0 & 1 + \alpha_2 a \left[1 - \frac{b}{4-s_1}\right] \end{array} \right),$$

we derive two related eigenvalues of such $J_{E_3\left(\frac{a}{4-s_1}, 0\right)}$ matrix

$$\begin{aligned} \lambda_1 &= 1 - \alpha_1 a \\ \lambda_2 &= 1 + \alpha_2 a \left[1 - \frac{b}{4 - s_1}\right] \end{aligned}$$

since for values $\alpha_i, a > 0$, $s_1 \in (0, 1)$, $i = 1, 2$ and $b \geq 0$. If $0 < a\alpha_1 < 2$,hence $|\lambda_1| < 1$. If $b > 4 - s_1$ and $a\alpha_2 < \frac{4-s_1}{4-s_1-b}$, then $|\lambda_2| < 1$ this lead to E_3 is a sink point. E_3 is a source point if $a\alpha_1 > 2$, $b < 4 - s_1$ and $a\alpha_2 > \frac{4-s_1}{4-s_1-b}$. E_3 is a non-hyperbolic point if $|\lambda_1| = 1$ or $|\lambda_2| = 1$and this is holds if $a\alpha_1 = 2$ or $b = 4 - s_1$. For the other values of parameters E_3 become saddle point. From condition (iii), it is easy to see that one of these eigenvalues of $E_2\left(0, \frac{a}{4-s_1}\right)$ is equal to -1 and the other other is neither 1 nor -1. Then all parameters locate in the following region:

$$\Omega_{E_3} = \left\{ \begin{array}{c} (a, b, s_1, \alpha_1, \alpha_2) : a\alpha_1 = 2, a\alpha_2 \neq \frac{4-s_1}{4-s_1-b} \text{ and } b \neq 4 - s_1, \\ \text{where } \alpha_i, a > 0, s_1 \in (0, 1) \end{array} \right\}.$$

Then $E_2\left(0, \frac{a}{4-s_1}\right)$ can pass through flip bifurcation when the parameters varying in the small neighborhood of Ω_{E_3}. □

We now investigate the local stability and bifurcations of a unique Nash equilibrium $E_4(q_1^*, q_2^*)$ where,

$$\begin{aligned} q_1^* &= \frac{a\ [b-(4-s_2)]}{b^2-(4-s_1)(4-s_2)-s_1s_2}, \\ q_2^* &= \frac{a\ [b-(4-s_1)]}{b^2-(4-s_1)(4-s_2)-s_1s_2}. \end{aligned}$$

Then the Jacobian matrix at $E_4(q_1^*, q_2^*)$ has the form:

$$J_{E_4(q_1^*,q_2^*)} = \begin{pmatrix} J_{1,E_4(q_1^*,q_2^*)} & -\alpha_1 b q_1^* \\ -\alpha_2 b\ q_2^* & J_{2,E_4(q_1^*,q_2^*)} \end{pmatrix}. \tag{3.2}$$

Where

$$\begin{aligned} J_{1,E_4(q_1^*,q_2^*)} &= 1+\alpha_1[a-2q_1^*(4-s_1)-q_2^*b] \\ J_{2,E_4(q_1^*,q_2^*)} &= 1+\alpha_2[a-bq_1^*-2q_2^*(4-s_2)]. \end{aligned}$$

Its characteristic equation is,

$$P(\lambda) = \lambda^2 - Tr\ \lambda + Det = 0,$$

where Tr is the trace and Det is the determinant of Jacobian matrix at E_4,

$$\begin{aligned} Tr &= 2+\alpha_1[a-2q_1^*(4-s_1)-q_2^*b]+\alpha_2[a-bq_1^*-2q_2^*(4-s_2)] \\ Det &= 1+\alpha_1[a-2q_1^*(4-s_1)-q_2^*b]+\alpha_2[a-bq_1^*-2q_2^*(4-s_2)] \\ &\quad +\alpha_1\alpha_2[(a-2q_1^*(4-s_1)-q_2^*b)(a-bq_1^*-2q_2^*(4-s_2))]-\alpha_1\alpha_2 b^2 q_1^* q_2^*. \end{aligned} \tag{3.3}$$

Since,

$$(Tr)^2 - 4\ Det = \left\{ \begin{pmatrix} \alpha_1[a-2q_1^*(4-s_1)-q_2^*b]- \\ \alpha_2[a-bq_1^*-2q_2^*(4-s_2)] \end{pmatrix}^2 + 4\alpha_1\alpha_2 b^2 q_1^* q_2^* \right. .$$

Its clear that $(Tr)^2 - 4Det > 0$, then we deduce that the eigenvalues of the unique Nash equilibrium $E_4(q_1^*, q_2^*)$ are real. Using Jury's conditions [10] , we have necessary and sufficient condition $|\lambda_i| < 1,\ i = 1, 2$, for local stability of Nash equilibrium E_4.

$$\begin{aligned} (i) &: P(1) = 1 - Tr + Det > 0 \\ (ii) &: P(-1) = 1 + Tr + Det > 0 \\ (iii) &: P(0) = 1 - Det > 0. \end{aligned} \tag{3.4}$$

The first condition is always satisfied for,

$$\alpha_1\alpha_2\left[(a-2q_1^*(4-s_1)-q_2^*b)(a-bq_1^*-2q_2^*(4-s_2))-b^2q_1^*q_2^*\right] > 0, \tag{3.5}$$

whereas, the second and the third conditions are the conditions of the local stability of Nash equilibrium in the parameters space (α_1, α_2) which become,

$$P(-1) = \begin{cases} 4+2\alpha_1\left[a-2q_1^*\left(4-s_1\right)-q_2^*b\right]+2\alpha_2\left[a-bq_1^*-2q_2^*\left(4-s_2\right)\right] \\ +\alpha_1\alpha_2\left[\left(a-2q_1^*\left(4-s_1\right)-q_2^*b\right)\left(a-bq_1^*-2q_2^*\left(4-s_2\right)\right)\right] \\ -\alpha_1\alpha_2b^2q_1^*q_2^*>0 \end{cases},$$

$$P(0) = \begin{cases} \alpha_1\left[a-2q_1^*\left(4-s_1\right)-q_2^*b\right]+\alpha_2\left[a-bq_1^*-2q_2^*\left(4-s_2\right)\right] \\ +\alpha_1\alpha_2\left[\left(a-2q_1^*\left(4-s_1\right)-q_2^*b\right)\left(a-bq_1^*-2q_2^*\left(4-s_2\right)\right)\right] \\ -\alpha_1\alpha_2b^2q_1^*q_2^*<0 \end{cases}. \tag{3.6}$$

The stability region is bounded by Eq. (3.6) with positive parameters (α_1, α_2) whose inequality given by:

$$\begin{cases} -4-\alpha_1\left[a-2q_1^*\left(4-s_1\right)-q_2^*b\right]-\alpha_2\left[a-bq_1^*-2q_2^*\left(4-s_2\right)\right] \\ <\alpha_1\left[a-2q_1^*\left(4-s_1\right)-q_2^*b\right]+\alpha_2\left[a-bq_1^*-2q_2^*\left(4-s_2\right)\right] \\ +\alpha_1\alpha_2\left[\left(a-2q_1^*\left(4-s_1\right)-q_2^*b\right)\left(a-bq_1^*-2q_2^*\left(4-s_2\right)\right)\right] \\ -\alpha_1\alpha_2b^2q_1^*q_2^*\left(2-s_1\right)<0 \end{cases}. \tag{3.7}$$

The unique Nash equilibrium $E_4(q_1^*, q_2^*)$ is stable node according to inequality (3.7) and looses its stability through a period doubling bifurcation whose curve intersect the axes α_1 and α_2 respectively as:

$$\begin{aligned} A_1(0,\alpha_2) &= \left(0\ ,\ \frac{2}{bq_1^*+2q_2^*\left(4-s_2\right)-a}\right), \\ A_2(\alpha_1,0) &= \left(\frac{2}{2q_1^*\left(4-s_1\right)+q_2^*b-a},\ 0\right). \end{aligned}$$

We summarize this result as follows:

Theorem 4. *The unique Nash equilibrium $E_4(q_1^*, q_2^*)$ is stable if and only if condition (3.7) holds and looses its stability through a period doubling bifurcation whose curve intersect with points A_1 & A_2.*

4. Numerical simulations

We present in this section some various numerical results of model (2.12) including its bifurcations diagrams, strange attractors, stable trajectories, Lyapunov exponents and the fractal dimension. We just focus on the Nash equilibrium $E_4(q_1^*, q_2^*)$ which is stable on some conditions and more importantly endowed with economic implications. It can be found that the stability region of the Nash equilibrium $E_4(q_1^*, q_2^*)$ depends on the model parameters. Assume that $a = 3$, $b = 2$, $s_1 = 0.8$ and $s_2 = 0.9$, the stability region of adjustment speeds for the Nash equilibrium $E_4(0.4969879518, 0.5421686747)$ can be described by Figure (1). When the adjustment speeds become larger, the stability property of the Nash equilibrium will disappear, as shown in Figures (2-7) show that a 2-cycle, a 4-cycle and a strange attractor may occur at some larger adjustment speeds.
As parameters (α_1, α_2) increase, the equilibrium points become unstable, infinitely many period doubling bifurcation of the phase quantity behavior becomes chaotic. This means that for a larger values of adjustment speeds of boundedly rational game, the dynamical model (2.12) converge to complex dynamics. This follow in Figures (8-11) show the

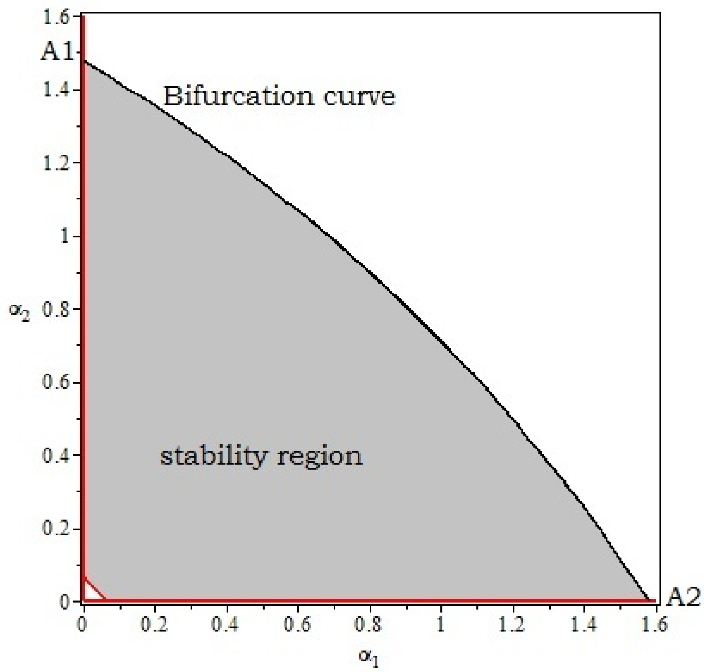

FIGURE 1. Stability region of Nash equilibrium point.

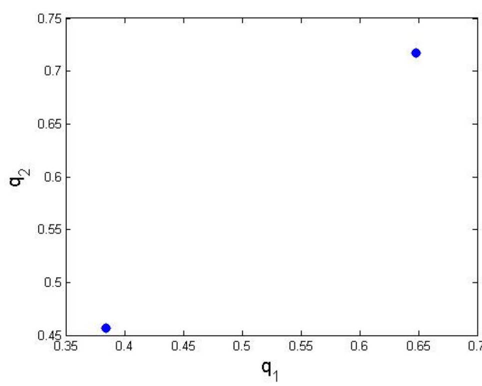

FIGURE 2. 2-cycle of model (2.12) for $\alpha_1 = 0.8$, $\alpha_1 = 0.7$

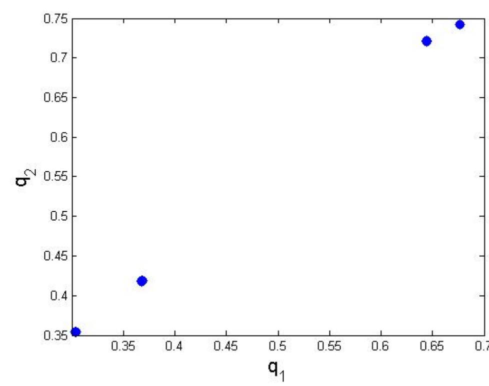

FIGURE 3. 4-cycle of model (2.12) for $\alpha_1 = 0.84$, $\alpha_1 = 0.8$

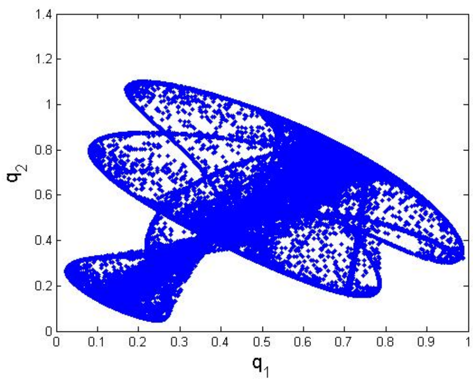

FIGURE 4. Strange attractor of model (2.12) for $\alpha_1 = 0.95$, $\alpha_1 = 0.94$

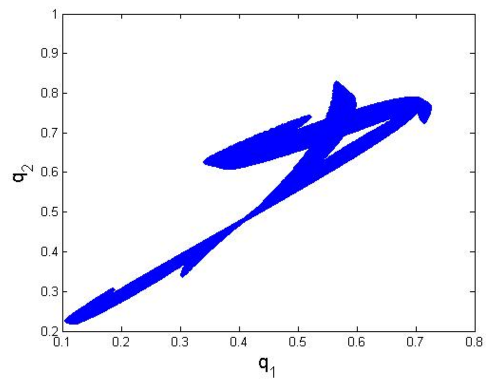

FIGURE 5. Strange attractor of model (2.12) for $\alpha_1 = 1$, $\alpha_1 = 0.85$

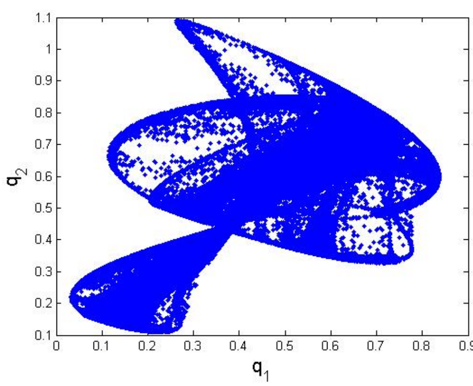

FIGURE 6. Strange attractor of model (2.12) for $\alpha_1 = 1$, $\alpha_1 = 0.9$

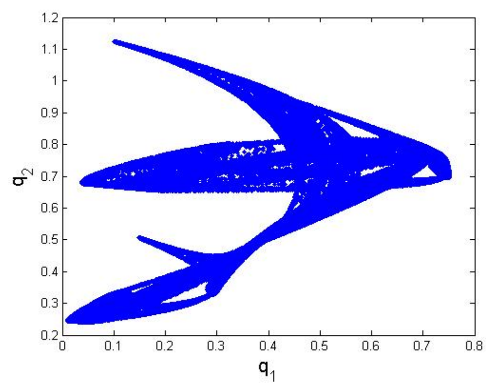

FIGURE 7. Strange attractor of model (2.12) for $\alpha_1 = 1.1$, $\alpha_1 = 0.8$

bifurcation diagram with respect to (adjustment speeds of boundedly rational players (α_1, α_2)). Let $\alpha_1 = \alpha_2 = 0.97$, one can get the bifurcation diagrams with respect to the other adjustment speeds in Figures (8 & 9). Assume that $a = 3$, $b = 2$, $s_1 = 0.8$, $\alpha_1 = 0.9$, we provide the bifurcation diagrams for α_2 when $s_2 = 0$ as a private firm in Figure (10). When $s_2 = 1$ as a public firm the bifurcation diagrams for α_2 shown in Figure (11). Assume that $\alpha_1 = 0.5$, $\alpha_2 = 0.75$, the Nash equilibrium is stable and the related gradient dynamic is shown in Figure (12).

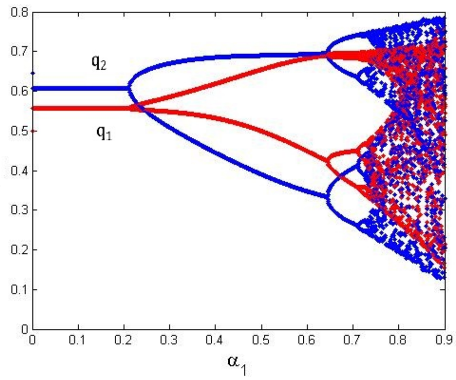

Figure 8. The bifurcation diagram for α_1 as $\alpha_2 = 0.97$

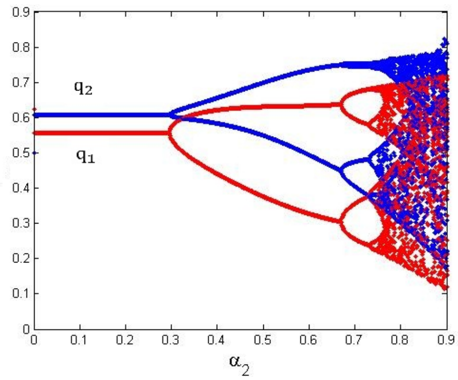

Figure 9. The bifurcation diagram for α_2 as $\alpha_1 = 0.97$

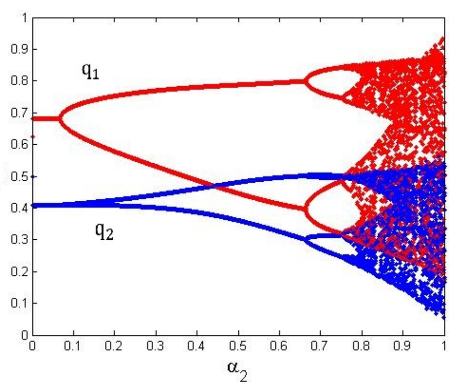

Figure 10. The bifurcation diagram for α_2 as $s_2 = 0$

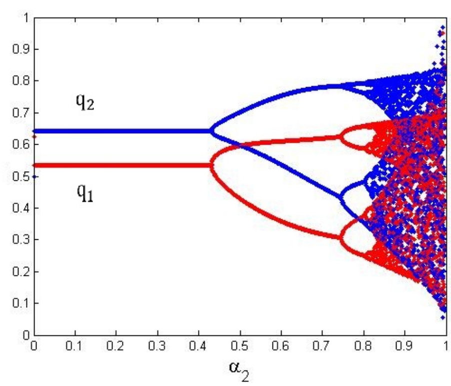

Figure 11. The bifurcation diagram for α_2 as $s_2 = 1$

The usual test for chaos is calculation of the maximal Lyapunov exponent. A positive maximal Lyapunov exponent indicates chaos. The maximal Lyapunov exponent can be negative (stable fixed point), zero (bifurcation point), and positive (chaos). Moreover the values of Lyapunov exponent for the values of the parameters gave the same indications about stable and chaotic regions. Figure (13) displays the related maximal Lyapunov exponents of model (2.12) as a function in α_1, corresponding to the chaotic attractor in Figure (4) with initial $(q_{1,0}, q_{2,0}) = (0.1, 0.6)$.
Kaplan and Yorke (1979) [14] introduced a quantity defined in terms of the Lyapunov exponents $\Lambda_i \;\; (i = 1, 2, ...n)$. The quantity Kaplan and Yorke introduced, is commonly

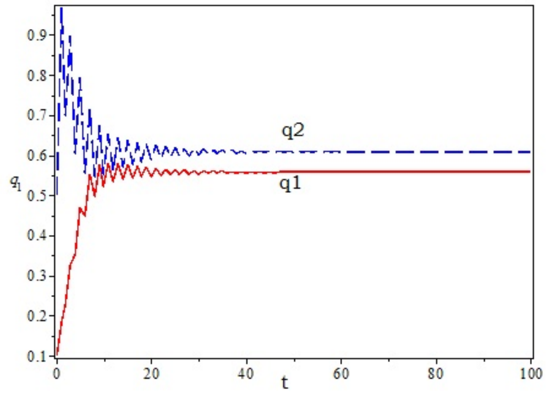

FIGURE 12. Stable trajectory of model (2.12)

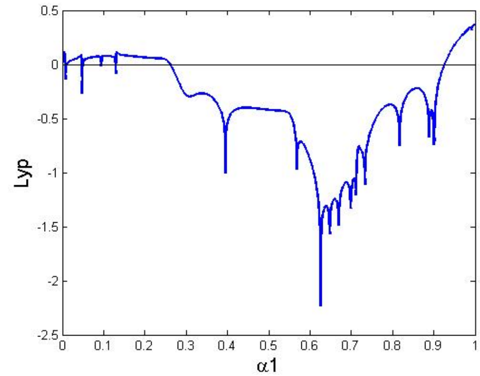

FIGURE 13. Maximal Lyapunov exponent versus α_2

called the 'Lyapunov dimension' and is given by:

$$d_L = j + \frac{\sum\limits_{i=1}^{i=j} \Lambda_i}{|\Lambda_i \ |},$$

where j is the largest integer such that $\sum\limits_{i=1}^{i=j} \Lambda_i > 0$, $\sum\limits_{i=1}^{i=j+1} \Lambda_i < 0$.
In our system of the two-dimensional map (2.12) has the Lyapunov dimension which given by:

$$d_L = 1 + \frac{\Lambda_1}{|\Lambda_2 \ |}, \Lambda_1 > 0 > \Lambda_2.$$

By the definition of Lyapunov dimension and simulation of the computer, we have the Lyapunov dimension of the strange attractor of system (2.12). At the parameter values $(a, b, s_1, s_2, \alpha_1, \alpha_2) = (3, 2, 0.8, 0.9, 1, 0.9)$, two different Lyapunov exponents exists and are $\Lambda_1 \approx 0.817517$ and $\Lambda_2 \approx -2.678498$. Therefor the map exhibits a fractal structure and its attractor has the fractal dimension $d_L \approx 1 + \frac{0.817517}{2.678498} \approx 1.305214$, which is chaotic behavior.

5. Firms' decisions as to role of s_i

It must be noted that the literature on public firms (see, De Fraja and Delbono1989, [9]) usually assumes that private firms maximize profits, public firms maximize social welfare and firms with a mixture of private and public ownership maximize the weighted average of social welfare and their own profit. Matsumura (1998) [16] investigated a quantity-setting duopoly involving a private firm and a privatized firm jointly owned by the public and private sectors. He showed that this type of firm is a reasonable choice for the government in the context of a mixed duopoly with single product firms. An

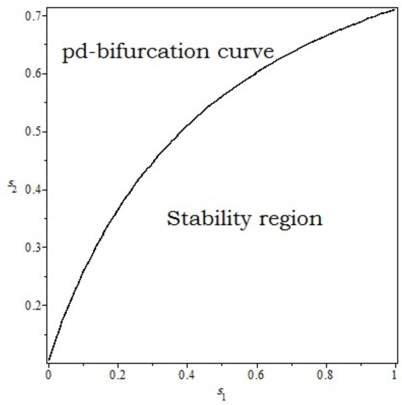

FIGURE 14. Stability region in (s_1, s_2) plane.

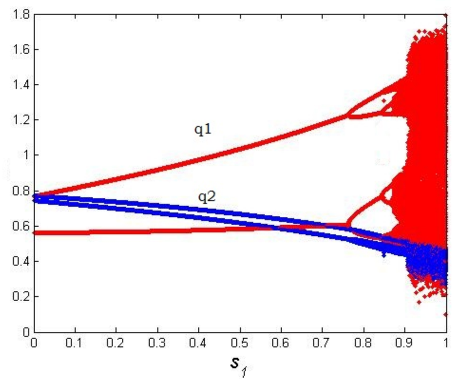

FIGURE 15. bifurcation diagram model (2.12) with respect s_1

endogenous order of moves is analyzed in a mixed market where a firm jointly owned by the public sector and private domestic shareholders (a semipublic firm) competes with n private firms (see, [5]).
In this section we study the firm's decisions as to s_i roles and the effects of theses decisions on the output levels of the two firms, profits, consumer surplus and social welfare.
We shall now analyze s_i effect on the dynamic properties of the Cournot duopoly map (2.12) (for instance, existence of output levels and stability of the market). To study the behavior of model (2.12) when the degree of public ownership s_i varied in interval $[0, 1]$,one can consider the initial condition $(q_{1,0}, q_{2,0})$ situated in the basin of attractor of the unique Nash equilibrium E_4. Conditions (3.5)-(3.6) define a region of stability market in the plane of public ownership (s_1, s_2) . So for a given value of parameters $a = 5, b = 4.1, \alpha_1 = 0.5, \alpha_2 = 0.2$. Figure (14) shows the stability region of the Nash equilibrium market for the above values of parameters. Figure(15) is about bifurcation diagram of system (2.12) with respect to s_1 while the other $a = 5, b = 3, \alpha_1 = 0.7, \alpha_2 = 0.1, s_2 = 0.04$, when the degree of the ownership of firm 1 about $0 < s_1 < 0.7$ the output levels of the two firms are stable. Period doubling bifurcation comes up at $s_1 = 0.776$, while s_1 increases more complicated dynamical behaviors can be observed to the firms. The observations from Figure (16) show that, the equilibrium market of the two firms is stable for s_2 taking its value from 0.354 up to 1. Then stability of the outputs level become unstable and bifurcates into periodic window leading the system to chaos when s_2 decreases about 0.265.
In this section, we discuss four firms' decisions: 1) if firm 1 is private and firm 2 is semipublic. 2) if firm 1 is public and firm 2 is semipublic. 3) if firm 1 is semipublic and firm 2 is private. 4) if firm 1 is semipublic and firms 2 is public.

5.1. **Case 1: firm 1 is private and firm 2 is semipublic.** In this subsection, we discuss the effect of government owns s_i on the market where private ($s_1 = 0$) firm

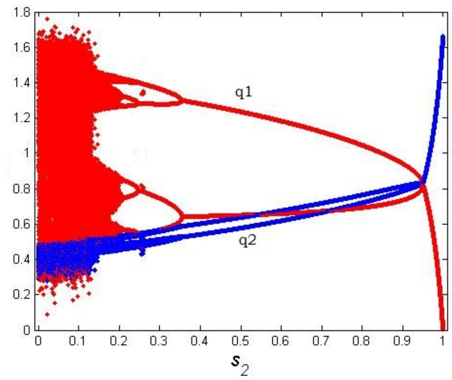

FIGURE 16. bifurcation diagram model (2.12) with respect s_2

1 maximizes its profit with respect to q_1 whereas semipublic ($s_2 = s = (0,1)$) firm 2 maximizes the social welfare with respect to q_2. In this case, there is a mixed duopoly denoted by superscript $0s$ their output levels given by:

$$\begin{aligned} q_1^{0s} &= \frac{a\ [b-(4-s)]}{b^2+4s-16} \\ q_2^{0s} &= \frac{a\ [b-4]}{b^2+4s-16}. \end{aligned} \tag{5.1}$$

From Eq. (5.1) in Eqs. (2.4, 2.5, 2.6) we get

$$\pi_1^{0s} = \frac{2a^2\ [b-(4-s)]^2}{(b^2+4s-16)^2}$$

$$\pi_2^{0s} = \frac{a^2\ (b-4)^2\,(2-s)}{(b^2+4s-16)^2}$$

$$CS^{os} = \frac{a^2(2b^3+2b^2s-14b^2-6bs+s^2+16b-8s+32)}{2\,(b^2+4s-16)^2}$$

$$W^{os} = \frac{a^2(2b^3-6b^2+18bs+5s^2-48b-72s+160)}{2\,(b^2+4s-16)^2}$$

Theorem 5. *When govermment owns s effects in mixed duopoly (case 1):*

(1) The profit of firm 1 increasing (decreasing) if and only if $s > (<)\ 4-b$.
(2) The profit of firm 2 increasing (decreasing) if and only if $s > (<) \frac{b^2}{4}$.
(3) Consumer surplus increasing (decreasing) if and only $s > (<) \frac{b^3-7b^2+8b+16}{3b}$.
(4) Social welfare increasing (decreasing) if and only $s > (<) \frac{b^2-8b+16}{16-5b}$.

Proof. For a given value of parameter b, the profit of the private firm in the mixed duopoly,

$$\frac{\partial \pi_1^{0s}}{\partial s} = \frac{4a^2 b(b-4)(b-4+s)}{\left(b^2+4s-16\right)^3} > (<)0 \text{ if and only if } s > (<)4-b\ .$$

The profit of firm 2

$$\frac{\partial \pi_2^{0s}}{\partial s} = \frac{4a^2 (b-4)^2(4s-b^2)}{\left(b^2+4s-16\right)^3} > (<)0 \text{ if and only if } s > (<)\frac{b^2}{4}.$$

Consumer surplus

$$\begin{aligned} \frac{\partial CS^{0s}}{\partial s} &= \frac{a^2 (b-4)(b^3-7b^2-3bs+8b+16)}{\left(b^2+4s-16\right)^3} > (<)0 \text{ if and only} \\ s &> (<)\frac{b^3-7b^2+8b+16}{3b}. \end{aligned}$$

Social welfare

$$\frac{\partial W^{0s}}{\partial s} = \frac{a^2 (b-4)(b^2+5bs-8b-16s+16)}{\left(b^2+4s-16\right)^3} > (<)0 \text{ if and only } s > (<)\frac{b^2-8b+16}{16-5b}.$$

□

5.2. **Case 2: firm 1 is public and firm 2 is semipublic.** In this case there is a mixed duopoly denoted by superscript $1s$, by setting $s_1 = 1$ and $s_2 = s = (0,1)$ their output levels given by:

$$\begin{aligned} q_1^{1s} &= \frac{a(b-4+s)}{b^2+2s-12} \\ q_2^{1s} &= \frac{a(b-3)}{b^2+2s-12} \end{aligned} \tag{5.2}$$

From Eq. (5.2) in Eqs. (2.4, 2.5, 2.6) we get

$$\begin{aligned} \pi_1^{1s} &= \frac{a^2\ (b-4+s)\,(b-4)}{\left(b^2+2s-12\right)^2} \\ \pi_2^{1s} &= \frac{a^2\ (3-b)\,(bs-2b-2s+6)}{\left(b^2+2s-12\right)^2} \end{aligned}$$

$$CS^{1s} = \frac{a^2(2b^3+2b^2s-12b^2-4bs+s^2+10b-8s+25)}{2\left(b^2+2s-12\right)^2}$$

$$W^{1s} = \frac{a^2(2b^3-6b^2+8bs+s^2-30b-28s+93)}{2\left(b^2+2s-12\right)^2}$$

Theorem 6. *When govermment owns s effects in mixed duopoly (case 2):*

(1) The profit of firm 1 increasing (decreasing) if and only if $s > (<)\frac{b^2}{2} - 2b + 2$.
(2) The profit of firm 2 increasing (decreasing) if and only if $s > (<)\frac{b^3-2b^2-4b}{2(b-2)}$.
(3) Consumer surplus increasing (decreasing) if and only $s > (<)\frac{b^4-6b^3+8b^2+4b-2}{b^2-4b+4}$.

(4) Social welfare increasing (decreasing) if and only $s > (<) \frac{2(b^2-6b+9)}{b^2-8b+16}$.

Proof. For a given value of parameter b, the profit of the private firm in the mixed duopoly,

$$\frac{\partial \pi_1^{1s}}{\partial s} = \frac{a^2(b-4)(b^2-4b-2s+4)}{(b^2+2s-12)^3} > (<) 0 \text{ if and only if } s > (<) \frac{b^2}{2} - 2b + 2 \ .$$

The profit of firm 2

$$\begin{aligned} \frac{\partial \pi_2^{1s}}{\partial s} &= \frac{a^2(3-b)(b^3-2b^2-2bs-4b+4s)}{(b^2+2s-12)^3} > (<) 0 \text{ if and only if} \\ s &> (<) \frac{b^3-2b^2-4b}{2(b-2)}. \end{aligned}$$

Consumer surplus

$$\begin{aligned} \frac{\partial CS^{1s}}{\partial s} &= \frac{a^2(b^4-6b^3-b^2s+8b^2+4bs+4b-4s-2)}{(b^2+2s-12)^3} > (<) 0 \text{ if and only} \\ s &> (<) \frac{b^4-6b^3+8b^2+4b-2}{b^2-4b+4}. \end{aligned}$$

Social welfare

$$\begin{aligned} \frac{\partial W^{1s}}{\partial s} &= \frac{a^2(b^2s-2b^2-8bs+12b+16s-18)}{(b^2+2s-12)^3} > (<) 0 \text{ if and only} \\ s &> (<) \frac{2(b^2-6b+9)}{b^2-8b+16}. \end{aligned}$$

□

5.3. **Case 3: firm 1 is semipublic and firm 2 is private.** In this case there is a mixed duopoly denoted by superscript $s0$, where semipublic ($s_1 = s = (0,1)$) firm 1 maximizes its the social welfare with respect to q_1 whereas private ($s_2 = 0$) firm 2 maximizes the profit with respect to q_2. their output levels given by:

$$\begin{aligned} q_1^{s0} &= q_2^{0s} = \frac{a \ [b-4]}{b^2+4s-16} \\ q_2^{s0} &= q_1^{0s} = \frac{a \ [b-(4-s)]}{b^2+4s-16}. \end{aligned} \quad (5.3)$$

From Eq. (5.3) in Eq. (2.4) we get

$$\begin{aligned} \pi_1^{s0} &= \pi_2^{0s} = \frac{a^2 \ (b-4)^2 (2-s)}{(b^2+4s-16)^2} \\ \pi_2^{s0} &= \pi_1^{0s} = \frac{2a^2 \ [b-(4-s)]^2}{(b^2+4s-16)^2}. \end{aligned}$$

By comparing the output levels in this case by the output in case1 we get that $q_1^{s0} = q_2^{0s}$ and $q_2^{s0} = q_1^{0s}$, thus firms obtain profits as $\pi_1^{s0} = \pi_2^{0s}$and $\pi_2^{s0} = \pi_1^{0s}$. However, in this case the social welfare and consumer surplus as the same as in case1.

5.4. **Case 4: firm 1 is semipublic and firm 2 is public.** In this case there is a mixed duopoly denoted by superscript $s1$, by setting $s_1 = s = (0,1)$ and $s_2 = 1$ their output levels given by:

$$\begin{aligned} q_1^{s1} &= q_2^{1s} = \frac{a(b-3)}{b^2+2s-12} \\ q_2^{s1} &= q_1^{1s} = \frac{a(b-4+s)}{b^2+2s-12} \end{aligned} \tag{5.4}$$

From Eq. (5.4) in Eq. (2.4) we get:

$$\begin{aligned} \pi_1^{s1} &= \pi_2^{1s} = \frac{a^2\ (3-b)\,(bs-2b-2s+6)}{\left(b^2+2s-12\right)^2} \\ \pi_2^{s1} &= \pi_1^{1s} = \frac{a^2\ (b-4+s)\,(b-4)}{\left(b^2+2s-12\right)^2} \end{aligned}$$

By comparing the output levels in this case by the output in case2 we get that $q_1^{s1} = q_2^{1s}$ and $q_2^{s1} = q_1^{1s}$, thus firms obtain profits as $\pi_1^{s1} = \pi_2^{1s}$and $\pi_2^{s1} = \pi_1^{1s}$. However, in this case the social welfare and consumer surplus as the same as in case2.

6. Conclusion

In this paper, we investigated the complex dynamics of duopoly game with bounded rationality. Assuming that two semipublic firms update their outputs "i.e. weighted average of the social welfare and their own profit" using classical gradient adjustment dynamic. We show that the trivial and two boundary equilibria are unstable. We mainly address the problems of the locally asymptotical stability of the unique Nash equilibrium and some complex dynamic features such as period doubling bifurcations, strange attractors and chaotic phenomena arising in the instability parameter region. We discussed that if the behavior of the firm is characterized by relatively high speeds of adjustment, the Nash equilibrium becomes unstable through period doubling bifurcations, more complex attractors are created around it.

In the literature on mixed oligopoly there are numerous papers analyzing the partial privatization of publicly-owned firms. In these partially privatized firms there is a mixture of private and public ownership (semipublic firms). Therefore, We analyze the effects of government owns s_i in the semipublic firms on the profits and social welfare.

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