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AN EXTENSION ABOUT WIJSMAN $\mathcal{I}-$ASYMPTOTICALLY $\lambda-$STATISTICAL EQUIVALENCE

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Abstract. In this study we extend the notions Wijsman $\mathcal{I}-$asymptotically $\lambda-$statistical equivalent sequences, Wijsman strongly $\mathcal{I}-$asymptotically $\lambda-$equivalent sequences and Wijsman strongly Cesáro $\mathcal{I}-$asymptotically equivalent sequences by using the sequence $p = (p_k)$ which is the sequence of positive real numbers and $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$.

1. Introduction

Convergence is one of the most important notions in Mathematics. We know that statistical convergence extends the notion. We can easily show that any convergent sequence is statistically convergent, but not conversely. Let E be a subset of $\mathbb{N}$, the set of all natural numbers. $d(E) := \lim_n \frac{1}{n}{}_{j=1}^{n}\chi_E(j)$ is said to be natural density of E whenever the limit exists, where χ_E is the characteristic function of E. A number sequence (x_k) is statistically convergent to x provided that for every $\varepsilon > 0$, $d\{k \in \mathbb{N} : |x_k - x| \geq \varepsilon\} = 0$ or equivalently there exists a subset $K \subseteq \mathbb{N}$ with $d(E) = 1$ and $n_0(\varepsilon)$ such that $k > n_0(\varepsilon)$ and $k \in K$ imply that $|x_k - x| < \varepsilon$. In this case we write $st - \lim x_k = x$. Statistical convergence was given by Zygmund in the first edition of his monograph published in Warsaw in 1935. It was formally introduced by Fast ([5]) and Steinhaus ([18])and later was reintroduced by Schoenberg ([17]). It has become an active area of research in recent years. This concept has applications in different fields of mathematics such as number theory by Erdös and Tenenbaum ([4]), measure theory by Miller ([12]), trigonometric series by Zygmund ([21]), summability theory by Freedman and Sember ([6]), etc. The idea of $\mathcal{I}-$convergence for single sequences was introduced by Kostyrko, Salat and Wilczynski ([10]). We can say that this concept is a generalization of statistical convergence which is based on the structure of the ideal $\mathcal{I}$ of subsets of the set of natural numbers. Recently, it has become one of the most active areas of research in classical analysis. $\mathcal{I}-$convergence of real sequences coincides with

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the ordinary convergence if $\mathcal{I}$ is the ideal of all finite subsets of $\mathbb{N}$ and with the statistical convergence if $\mathcal{I}$ is the ideal of subsets of $\mathbb{N}$ of natural density zero.

In recent years, the concept of convergence of sequences of numbers has been extended by several authors to convergence of sequences of sets. The one of these such extensions considered in this paper is the concept of Wijsman convergence. ([1]-[2]-[9]-[14]-[20]) Before, Mursaleen defined $\lambda-$statistical convergence and he denoted this new method by S_λ and found its relation to statistical convergence, $[C,1]-$summability and $[V,\lambda]-$summability ([13]) where $\lambda = (\lambda_n)$ is a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. In this situation Λ will be the set of all non-decreasing sequences and $I_n = [n - \lambda_n + 1, n]$. Gumus and Savas gave the definition of $S_\lambda^L(\mathcal{I})-$asymptotically statistical equivalence and interested in some relations with $V_\lambda^L(\mathcal{I})$ and $C_1^L(\mathcal{I})$ spaces ([7]) In 2012, Nuray and Rhodes extended Mursaleen's study and gave the definition of statistical convergence for sequences of sets ([14]). Kisi and Nuray used $\mathcal{I}-$convergence for the similar concepts and they introduced new convergence notions for sequences of sets, which are called by Wijsman $\mathcal{I}-$convergence and Wijsman $\lambda-$statistical convergence ([8]-[9]).

Some authors like Savas and Gumus, generalized these concepts by using $p = (p_k)$ positive real number sequence ([16]).

Another basic topic for us is asymptotically equivalent sequences. Marouf presented definitions for asymptotically equivalent sequences and asymptotic reguler matrices ([11]). In 2003, Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices ([15]).Two nonnegative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent if

$$\lim_k \frac{x_k}{y_k} = 1$$

and it is denoted by $x \sim y$.

In this work, we investigate that which results we will have if we apply the $p = (p_k)$ positive real number sequence to previous results. We also investigate what will happen if $p = (p)$.

2. Definitons and notations

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathcal{I}$
(ii) For each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$
(iii) For each $A \in \mathcal{I}$ and each $B \subseteq A$ we have $B \in \mathcal{I}$

An ideal is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

A family of sets $F \subset 2^{\mathbb{N}}$ is a filter in $\mathbb{N}$ if and only if
(i) $\emptyset \notin F$
(ii) For each $A, B \in F$ we have $A \cap B \in F$
(iii) For each $A \in F$ and each $B \supseteq A$ we have $B \in F$
$\mathcal{I}$ is a non-trivial ideal in $\mathbb{N}$ if and only if

$$F = F(\mathcal{I}) = \{M = \mathbb{N} \backslash A : A \in \mathcal{I}\}$$

is a filter in $\mathbb{N}$.

A real sequence $x = (x_k)$ is said to be $\mathcal{I}-$convergent to $L \in \mathbb{R}$ if and only if for each $\varepsilon > 0$ the set

$$A_\varepsilon = \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\}$$

belongs to $\mathcal{I}$. The number L is called the $\mathcal{I}-$limit of the sequence x.

Take for $\mathcal{I}$ class the $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$ $-$convergence coincides with the usual convergence.

Let (X, d) is a metric space. For any point $x \in X$ and any non-empty subset A of X, we define the distance from x to A by

$$d(x, A) = \inf_{a \in A} d(x, A).$$

([1]) Let (X, d) is a metric space. For any non-empty closed subsets $A, A_k \subseteq X$ we say that the sequence $\{A_k\}$ is Wijsman convergent to A if

$$\lim_{k \to \infty} d(x, A_k) = d(x, A)$$

for each $x \in X$. In this case we write $W - \lim_{k \to \infty} A_k = A$.

Lets define the following sequence of circles in the $(x, y)-$plane.

$$A_k = \left\{(x, y) : x^2 + y^2 + 2kx = 0\right\}.$$

As $k \to \infty$ the sequence is Wijsman convergent to the $y-$axis $A = \{(x, y) : x = 0\}$.

([14]) Let (X, d) is a metric space. For any non-empty closed subsets $A, A_k \subseteq X$; one says that the sequence $\{A_k\}$ is Wijsman statistically convergent to A if for $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n \to \infty} |\{k \leq n : |d(x, A_k) - d(x, A)| \geq \varepsilon\}| = 0.$$

In this case we write $st - \lim_W A_k = A$ or $A_k \to A(WS)$ where WS denotes the set of Wijsman statistical convergent sequences.

([1]) Let (X, d) is a metric space. For any non-empty closed subset A_k of X, we say that the sequence $\{A_k\}$ is bounded if

$$\sup_k d(x, A_k) < \infty$$

for each $x \in X$. In this case we write $\{A_k\} \in L_\infty$.

([1]) Let (X, d) is a metric space. For any non-empty closed subsets $A, A_k \subseteq X$, we say that the sequence $\{A_k\}$ is Wijsman Cesáro summable to A if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} d(x, A_k) = d(x, A)$$

for each $x \in X$ and we say that $\{A_k\}$ is Wijsman Cesáro summable to A if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |d(x, A_k) - d(x, A)| = 0$$

for each $x \in X$.

([19]) Let (X, d) is a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences

$\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent (Wijsman sense) if for each for each $x \in X$,

$$\lim_{k\to\infty} \frac{d(x,A_k)}{d(x,B_k)} = 1$$

and this is denoted by $A_k \sim B_k$

Choose the following sequences of circles in the $(x,y)-$plane.

$$\begin{array}{rcl} A_k & = & \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2+2ky=0\right\}, \\ B_k & = & \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2-2ky=0\right\}. \end{array}$$

Since,

$$\lim_{k\to\infty} \frac{d(x,A_k)}{d(x,B_k)} = 1$$

then $\{A_k\}$ and $\{B_k\}$ are asymptotically equivalent (Wijsman sense).

([19]) Let (X,d) is a metric space. For any non-empty closed subsets $A_k, B_k \subseteq X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically statistically equivalent (Wijsman sense) provided that for each for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{n} \left| \left\{ k \le n : \left| \frac{d(x,A_k)}{d(x,B_k)} - L \right| \ge \varepsilon \right\} \right| = 0$$

and this is denoted by $A_k \overset{\mathcal{WS}_L}{\sim} B_k$.

([9]) Let (X,d) is a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an admissible ideal. For any non-empty closed subsets $A, A_k \subset X$ we say that the sequence $\{A_k\}$ is Wijsman $\mathcal{I}-$convergent to A if for each $\varepsilon > 0$ and for each $x \in X$ the set

$$A(x;\varepsilon) = \{k \in \mathbb{N} : |d(x,A_k) - d(x,A)| \ge \varepsilon\}$$

belongs to $\mathcal{I}$. This is denoted by $\mathcal{I}_W - \lim A_k = A$ or $A_k \to A(\mathcal{I}_W)$ where $\mathcal{I}_W$ is the set of Wijsman $\mathcal{I}-$convergent sequences.

Let $X = \mathbb{R}^2$ and let $\{A_k\}$ is the following sequence:

$$A_k = \begin{cases} \left\{(x,y) : x^2+y^2-2ky=0\right\}, & \text{if } k \ne n^2 \\ \{(x,y) : y=-1\}, & \text{if } k = n^2 \end{cases}$$

and $A = \{(x,y) : y = 0\}$. The sequence $\{A_k\}$ is not Wijsman convergent to A but if we take $\mathcal{I} = \mathcal{I}_d$, then $\{A_k\}$ is Wijsman $\mathcal{I}-$convergent to A where $\mathcal{I}_d$ is the ideal of sets that have zero density.

([8]) Let (X,d) is a metric space. For any non-empty closed subsets $A, A_k \subset X$, we say that the sequence $\{A_k\}$ is Wijsman $\lambda-$statistically convergent or $WS_\lambda-$convergent to A provided that for every $\varepsilon > 0$ and for each $x \in X$,

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\{k \in I_n : |d(x,A_k) - d(x,A)| \ge \varepsilon\}\right| = 0.$$

This is denoted by $S_\lambda - \lim_W A_k = A$ or $A_k \to A(WS_\lambda)$ where

$$WS_\lambda = \left\{ \{A_k\} : A \subseteq X,\ S_\lambda - \lim_W A_k = A \right\}.$$

If $\lambda_n = n$, Wijsman $\lambda-$statistical convergence is the same as Wijsman statistical convergence for the sequences of sets.

([9])Let (X,d) is a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an admissible ideal. For any non-empty closed subsets A_k, $B_k \subset X$ such that $d(x,A_k) > 0$ and $d(x,B_k) > 0$ for each $x \in X$ we say that the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically Wijsman $\mathcal{I}-$equivalent of multiple L if for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{k \in \mathbb{N} : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\} \in \mathcal{I}$$

and it is denoted by $A_k \overset{\mathcal{I}_W}{\sim} B_k$.

Let $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an non-trivial ideal, (X, d) is a metric space, A_k, $B_k \subset X$ are non-empty closed subsets. Let $X = \mathbb{R}, \{A_k\}$, $\{B_k\}$ are following sequences.

$$A_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq n,\ 0 \leq y \leq \frac{1}{n}x\right\}, & \text{if } k \neq n^2 \\ \{(0,0)\}, & \text{otherwise} \end{cases}$$

and

$$B_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq n,\ 0 \leq y \leq -\frac{1}{n}x\right\}, & \text{if } k \neq n^2 \\ \{(0,0)\}, & \text{otherwise} \end{cases}$$

If we take $\mathcal{I} = \mathcal{I}_d$, the sequences $\{A_k\}$ and $\{B_k\}$ are asymptotically Wijsman $\mathcal{I}-$equ-ivalent where $\mathcal{I}_d$ is the ideal of sets which have zero density.

([9])Let (X, d) is a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$ we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strong Cesáro $\mathcal{I}-$asymptotically equivalent (Wijsman sense) of multiple L provided that for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\} \in \mathcal{I}$$

and this is denoted by $A_k \overset{C_1^L(\mathcal{I}_W)}{\sim} B_k$.

([8]) Let (X, d) is a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$ we say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman $\mathcal{I}-$asymptotically statistical equivalent of multiple L provided that for every $\delta, \varepsilon > 0$ and for each $x \in X$,

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\left\{k \leq n : \left|\frac{d(x, A_k)}{d(x, B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \in \mathcal{I}$$

and this is denoted by $A_k \overset{S^L(\mathcal{I}_W)}{\sim} B_k$.

Let $\mathcal{I}$ is an admissible ideal in $\mathbb{N}$ .and $X = \mathbb{R}^2$. For any non-empty closed subsets A_k, $B_k \subseteq X$, $\{A_k\}$ and $\{B_k\}$ are following sequences:

$$A_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 = \frac{1}{k}\right\}, & \text{if } k \neq n^2 \\ \{(0,0)\}, & \text{otherwise} \end{cases}$$

$$B_k = \begin{cases} \left\{(x, y) \in \mathbb{R}^2 : x^2 + (y+1)^2 = \frac{1}{k}\right\}, & \text{if } k \neq n^2 \\ \{(0,0)\}, & \text{otherwise} \end{cases}$$

If we take $\mathcal{I} = \mathcal{I}_d$ we have,

$$\left\{k \in \mathbb{N} : \left|\frac{d(x, A_k)}{d(x, B_k)} - 1\right| \geq \varepsilon\right\} \in \mathcal{I}$$

thus $A_k \overset{\mathcal{I}_W^1}{\sim} B_k$.

([8]) Let (X, d) is a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are strongly $\lambda_{\mathcal{I}}-$asymptotically equivalent (Wijsman sense) of multiple L provided that for every $\varepsilon > 0$ and for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \in \mathcal{I}$$

and this is denoted by $A_k \overset{V_\lambda^L(\mathcal{I}_W)}{\sim} B_k$.

([9]) Let (X, d) is a metric space. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, we say that the sequences $\{A_k\}$ and $\{B_k\}$ are $\mathcal{I}-$asymptotically $\lambda-$statistically equivalent (Wijsman sense) of multiple L provided that for every $\varepsilon, \delta > 0$ and for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

and in this case we write $A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k$.

3. Main Results

Let (X, d) is a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is a non-trivial ideal, $p = (p_k)$ is a sequence of positive real numbers and $\lambda \in \Lambda$. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, the sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman strongly $\lambda_{\mathcal{I}}-$asymptotically $p-$equivalent or $V_\lambda^{L(p)}(\mathcal{I}_W)-$ asymptotically equivalent of multiple L provided that for every $\varepsilon > 0$ and every $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \sum_{k \in I_n} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right|^{p_k} \geq \varepsilon \right\} \in \mathcal{I}.$$

In this situation we write $A_k \overset{V_\lambda^{L(p)}(\mathcal{I}_W)}{\sim} B_k$.

If we take $p_k = p$ for all $k \in \mathbb{N}$ we write $A_k \overset{V_\lambda^{Lp}(\mathcal{I}_W)}{\sim} B_k$ instead of $A_k \overset{V_\lambda^{L(p)}(\mathcal{I}_W)}{\sim} B_k$.

Let (X, d) is a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is a non-trivial ideal, $p = (p_k)$ is a sequence of positive real numbers and $\lambda \in \Lambda$. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$, the sequences $\{A_k\}$ and $\{B_k\}$ are said to be Wijsman strongly Cesáro $\mathcal{I}-$asymptotically $p-$equivalent of multiple L if every $\varepsilon > 0$ and every $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right|^{p_k} \geq \varepsilon \right\} \in \mathcal{I}.$$

In this case we write $A_k \overset{\sigma^{L(p)}(\mathcal{I}_W)}{\sim} B_k$.

Let (X, d) is a metric space and $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an admissible ideal and $\lambda \in \Lambda$. For any non-empty closed subsets A_k, $B_k \subseteq X$ such that $d(x, A_k) > 0$ and $d(x, B_k) > 0$ for each $x \in X$ we say that the sequences $\{A_k\}$ and $\{B_k\}$ are Wijsman $\mathcal{I}-$asymptotically λ statistical equivalent of multiple L if every $\varepsilon, \delta > 0$ and for each $x \in X$,

$$\left\{ n \in \mathbb{N} : \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \left| \frac{d(x, A_k)}{d(x, B_k)} - L \right| \geq \varepsilon \right\} \right| \geq \delta \right\} \in \mathcal{I}$$

and it is denoted by $A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k$.

Let (X,d) is a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an admissible ideal and $\lambda \in \Lambda$. If $A_k \overset{V_\lambda^{L_p}(\mathcal{I}_W)}{\sim} B_k$ then $A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k$.

Proof. (i) Let $A_k \overset{V_\lambda^{L_p}(\mathcal{I}_W)}{\sim} B_k$ and $\varepsilon > 0$. For each $x \in X$,

$$\begin{aligned}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p &\geq \sum_{k\in I_n \ \& \ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p \\ &\geq \varepsilon^p \left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|\end{aligned}$$

and so

$$\frac{1}{\varepsilon^p \lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right|.$$

Then for any $\delta > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \delta\right\} \\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_r}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon^p \delta\right\} \in \mathcal{I}.\end{aligned}$$

Therefore $A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k$. □

Let (X,d) is a metric space, $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is an admissible ideal and $\lambda \in \Lambda$. If $\{A_k\}$, $\{B_k\} \in L_\infty$ and $A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k$ then $A_k \overset{V_\lambda^{L_p}(\mathcal{I}_W)}{\sim} B_k$.

Proof. $\{A_k\}$, $\{B_k\} \in L_\infty$ and $A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k$. Then there is an M such that $\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \leq M$ for each $x \in X$ and all k. For each $\varepsilon > 0$

$$\begin{aligned}\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p &= \frac{1}{\lambda_n}\sum_{k\in I_n \ \& \ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p \\ &+ \frac{1}{\lambda_n}\sum_{k\in I_n \ \& \ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| < \varepsilon}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^p \\ &\leq \frac{1}{\lambda_n}M^p\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \\ &+ \frac{1}{\lambda_n}\varepsilon^p\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| < \varepsilon\right\}\right| \\ &\leq \frac{M^p}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| + \varepsilon^p.\end{aligned}$$

Then for any $\delta > 0$,

$$\begin{aligned}&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\} \\ &\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \varepsilon\right\}\right| \geq \frac{\varepsilon^p}{M^p}\right\} \in \mathcal{I}\end{aligned}$$

and we have the proof. □

$$A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k \cap L_\infty = A_k \overset{V_\lambda^{L_p}(\mathcal{I}_W)}{\sim} B_k \cap L_\infty.$$

Proof. This follows from Theorem 3.1. and Theorem 3.2. □

Let $\lambda \in \Lambda$ and $\inf p_k = h$ and $\sup p_k = H$. $A_k \overset{V_\lambda^{L(p)}(\mathcal{I}_W)}{\sim} B_k$ implies $A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k$.

Proof. Assume that $A_k \overset{V_\lambda^{L(p)}(\mathcal{I}_W)}{\sim} B_k$ and $\varepsilon > 0$. Then,

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|^{p_k} &= \frac{1}{\lambda_n}\sum_{k\in I_n\ \&\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|^{p_k} \\
&+ \frac{1}{\lambda_n}\sum_{k\in I_n\ \&\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|<\varepsilon}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_{k\in I_n\ \&\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_{k\in I_n\ \&\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon}(\varepsilon)^{p_k} \\
&\geq \frac{1}{\lambda_n}\sum_{k\in I_n\ \&\ \left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon}\min\left\{(\varepsilon)^h,(\varepsilon)^H\right\} \\
&\geq \frac{1}{\lambda_n}\min\left\{(\varepsilon)^h,(\varepsilon)^H\right\}\left|\left\{k\in I_n:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|
\end{aligned}$$

and

$$\begin{aligned}
&\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\left|\left\{k\in I_n:\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|\geq\varepsilon\right\}\right|\geq\delta\right\} \\
&\subseteq\left\{n\in\mathbb{N}:\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)}-L\right|^{p_k}\geq\delta\min\left\{(\varepsilon)^h,(\varepsilon)^H\right\}\right\}\in I.
\end{aligned}$$

Thus we have $x \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} y$. □

Let $\{A_k\}$ and $\{B_k\}$ are bounded sequences, $\inf_k p_k = h$ and $\sup_k p_k = H$. Then,

$$A_k \overset{S_\lambda^L(\mathcal{I}_W)}{\sim} B_k \text{ implies } A_k \overset{V_\lambda^{L(p)}(\mathcal{I}_W)}{\sim} B_k.$$

Proof. Suppose that $\{A_k\}$ and $\{B_k\}$ are bounded and $\varepsilon > 0$. Then there is an integer K such that $\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \leq K$ for all k.

$$\begin{aligned}
\frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} &= \frac{1}{\lambda_n}\sum_{k\in I_n \ \& \ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|\geq\varepsilon}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} \\
&+ \frac{1}{\lambda_n}\sum_{k\in I_n \ \& \ \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|<\varepsilon}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} \\
&\leq \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| \max\left\{K^h, K^H\right\} \\
&+ \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| < \frac{\varepsilon}{2}\right\}\right| \frac{\max(\varepsilon)^{p_k}}{2} \\
&\leq \max\left\{K^h, K^H\right\}\frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| + \frac{\max\left\{\varepsilon^h, \varepsilon^H\right\}}{2}
\end{aligned}$$

and

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} \geq \varepsilon\right\} \\
&\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\left|\left\{k \in I_n : \left|\frac{d(x,A_k)}{d(x,B_k)} - L\right| \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{2\varepsilon - \max\left\{\varepsilon^h, \varepsilon^H\right\}}{2\max\left\{K^h, K^H\right\}}\right\}
\end{aligned}$$

belongs to $\mathcal{I}$ so we have $A_k \overset{V_\lambda^{L(p)}(\ \mathcal{I}_w)}{\sim} B_k$. □

Let $\lambda \in \Lambda$ and I is an admissible ideal in $\mathbb{N}$. If $A_k \overset{V_\lambda^{L(p)}(\mathcal{I}_W)}{\sim} B_k$ then $A_k \overset{\sigma^{(p)}(\mathcal{I}_W)}{\sim} B_k$.

Proof. Assume that $A_k \overset{V_\lambda^{L(p)}(\mathcal{I}_W)}{\sim} B_k$ and $\varepsilon > 0$. Then,

$$\begin{aligned}
\frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} &= \frac{1}{n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} + \frac{1}{n}\sum_{k=1}^{n-\lambda n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} \\
&\leq \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} + \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} \\
&\leq \frac{2}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k}
\end{aligned}$$

and so,

$$\begin{aligned}
&\left\{n \in \mathbb{N} : \frac{1}{n}\sum_{k=1}^{n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} \geq \varepsilon\right\} \\
&\subseteq \left\{n \in \mathbb{N} : \frac{1}{\lambda_n}\sum_{k\in I_n}\left|\frac{d(x,A_k)}{d(x,B_k)} - L\right|^{p_k} \geq \frac{\varepsilon}{2}\right\}
\end{aligned}$$

and it completes the proof. □

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