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ASYMPTOTIC STABILITY AND OSCILLATORY BEHAVIOR OF A DIFFERENCE EQUATION

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Abstract. The aim of this paper is the study of the boundedness, asymptotic stability and oscillatory behavior of the following rational difference equation:

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + dx_{n-1}}$$

Our results extend the results of [10] . We also obtain the closed form of solutions of the above rational difference equation in general case.

1. Introduction

Discrete dynamical systems or difference equations is a assorted field which manipulate roughly every offshoot of pure and applied mathematics. Every dynamical system $S_{n+1} = f(S_n)$ locates a difference equation and vise versa. Lately, there has been considerable interest in studying difference equations. One of the purposes for this is a exigency for some techniques whose can be used in investigating equations arising in mathematical models describing real life situations in population biology, economic, probability theory, genetics, psychology, ... etc. Recently there has been a lot of interest in studying the boundedness character, stability and the periodic nature of non-linear difference equations. For some results in this area, see for example [[17]–[21]]. Difference equations have been studied in various branches of mathematics for a long time. First results in qualitative theory of such systems were obtained by Poincar and Perron in the end of nineteenth and the beginning of twentieth centuries. Many researchers have investigated the behavior of the solution of difference equations for example: Camouzis et al. [3] investigated the behaviour of solutions of the rational recursive sequence

$$x_{n+1} = \frac{\beta x_n^2}{1 + x_{n-1}^2}.$$

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Elabbasy et al. [7] investigated the global stability, boundedness, periodicity character and gave the solution of some special cases of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-k}}{\beta + \gamma \prod_{i=0}^{k} x_{n-i}}.$$

Grove, Kulenovic and Ladas [11] presented a summary of a recent work and a large of open problems and conjectures on the third order rational recursive sequence of the form

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1} + \delta x_{n-2}}{A + B x_n + C x_{n-1} + D x_{n-2}}.$$

In [22] Kulenovic, G. Ladas and W. Sizer studied the global stability character and the periodic nature of the recursive sequence

$$x_{n+1} = \frac{\alpha x_n + \beta x_{n-1}}{\gamma x_n + \delta x_{n-1}}.$$

Kulenovic and Ladas [21] studied the second-order rational difference equation

$$x_{n+1} = \frac{\alpha + \beta x_n + \gamma x_{n-1}}{A + B x_n + C x_{n-1}}.$$

For other important references, we refer the reader to ([1],[2],[4],[5],[6],[8],[9],[12], [13]–[16],[22]–[26]).

In this paper we study the following second-order rational difference equation

$$x_{n+1} = a x_n + \frac{b x_n x_{n-1}}{c x_n + d x_{n-1}} \tag{1.0.1}$$

where a, b, c, d are positive real constants. This equation has been first considered by Elsayed [10]. He proved sufficient conditions for boundedness and global behavior of this equation as well as he obtain the closed form of solution for a special case of this equation. In this paper we prove more general sufficient conditions for boundedness and asymptotic behavior of solutions. We also obtain the closed form of solutions in general form.

Here, we recall some notations and results which will be useful in our investigation.

Let I be some interval of real numbers and let

$$F : I^{k+1} \to I$$

be a continuously differentiable function. Then for every set of initial conditions $x_0, x_{-1} ..., x_{-k} \in I$, the difference equation

$$x_{n+1} = F(x_n, x_{n-1}, ..., x_{n-k}), n = 0, 1, ... \tag{1.0.2}$$

has a unique solution $x_{n}{}_{n=-k}^{\infty}$.[18]

Definition 1.0.1. A point $\overline{x} \in I$ is called ***an equilibrium point*** of equation(1.0.2) if

$$\overline{x} = F(\overline{x}, \overline{x}, ..., \overline{x})$$

That is, $x_n = \overline{x}$ for $n \geq 0$, is a solution of equation(1.0.2), or equivalently, $\overline{x}$ is a fixed point of F.

Definition 1.0.2. Let I be some interval of real numbers.

(i) The equilibrium point $\overline{x}$ of equation (1.0.2) is ***locally stable*** if for every > 0, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}?, x_0 \in I$ with

$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \delta$,

we have

$|x_n - \overline{x}| < \epsilon$ for all $n \geq -k$.

(ii) The equilibrium point $\overline{x}$ of equation(1.0.2) is ***locally asymptotically stable*** if $\overline{x}$ is locally stable solution of equation(1.0.2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, ..., x_{-1}?, x_0 \in I$ with

$|x_{-k} - \overline{x}| + |x_{-k+1} - \overline{x}| + ... + |x_0 - \overline{x}| < \gamma$,

we have

$\lim_{n \longrightarrow \infty} x_n = \overline{x}$.

(iii) The equilibrium point $\overline{x}$ of equation(1.0.2) is ***global attractor*** if for all $x_{-k}, x_{-k+1}, ..., x_{-1}?, x_0 \in I$, we have $\lim_{n \longrightarrow \infty} x_n = \overline{x}$.

(iv) The equilibrium point $\overline{x}$ of equation(1.0.2) is ***globally asymptotically stable*** if $\overline{x}$ is locally stable, and $\overline{x}$ is also a global attractor of equation (1.0.2).

(v) The equilibrium point $\overline{x}$ of equation(1.0.2) is ***unstable*** if $\overline{x}$ is not locally stable.

2. BOUNDEDNESS OF SOLUTIONS

In this section we study the boundedness of the solutions of equation 1.0.1. The following theorem extends [10].

THEOREM 2.0.3. *Let x_n the sequence generated by equation 1.0.1. If $a + \frac{b}{c+d} \leq 1$, then x_n is bounded.*

Proof. 1) If $x_n \leq x_{n-1}$, then

$$x_{n+1} = ax_n + \frac{bx_n x_{n-1}}{cx_n + dx_{n-1}} \leq ax_n + \frac{b}{c+d}x_{n-1} \leq (a + \frac{b}{c+d})x_{n-1}$$

If $x_{n-1} \leq x_n$, then

$$x_{n+1} \leq ax_n + \frac{bx_n x_{n-1}}{(c+d)x_{n-1}} = (a + \frac{b}{c+d})x_n$$

Therefore

$$x_{n+1} \leq \max\{(a+\frac{b}{c+d})x_n, (a+\frac{b}{c+d})x_{n-1}\} = (a+\frac{b}{c+d}) \max\{x_n, x_{n-1}\} \quad (2.0.3)$$

Since $a + \frac{b}{c+d} \leq 1$, we get

$$x_{n+1} \leq \max\{x_n, x_{n-1}\} \leq ... \leq \max\{x_1, x_0\}$$

which implies the boundedness of x_n. □

3. CONVERGENCE

In this section we prove sufficient condition for global attractivity of the sequence 1.0.1. This result extends [[10], Theorems 1 and 2]. We obtain the result without using linearization and Theorems A and B of [10].

THEOREM 3.0.4. *1) If $a + \frac{b}{c+d} < 1$ then 0 is the global attractor of x_n and $x_n = O((a + \frac{b}{c+d})^n)$.*
2) If $a + \frac{b}{c+d} = 1$, then x_n has a global attractor $p > 0$.

Proof. First suppose $a + \frac{b}{c+d} < 1$, we prove 0 is a global attractor of x_n. (2.0.3) implies

$$0 \leq x_{n+1} \leq (a + \frac{b}{c+d}) \max\{x_n, x_{n-1}\} \leq (a + \frac{b}{c+d})^2 \max\{x_{n-1}, x_{n-2}\}$$
$$\leq ... \leq (a + \frac{b}{c+d})^n \max\{x_1, x_0\} \to 0$$

as $n \to +\infty$.

Now suppose that $a + \frac{b}{c+d} = 1$ then

$$x_{n+1} = ax_n + (1-a)\frac{(c+d)x_n x_{n-1}}{cx_n + dx_{n-1}} \leq ax_n + (1-a)\max\{x_n, x_{n-1}\} \tag{3.0.4}$$
$$\leq a \max\{x_n, x_{n-1}\} + (1-a)\max\{x_n, x_{n-1}\}$$

(first above inequality because of $f(x,y) = \frac{xy}{cx+dy}$ is non-decreasing with respect to x and y.) Take $y_n = \max\{x_n, x_{n-1}\}$ then $y_{n+1} \leq y_n$ therefore $y_n \to p \geq 0$ and

$$x_n \leq y_n \Rightarrow \limsup x_n \leq \limsup y_n = p \tag{3.0.5}$$

From (3.0.4), we have

$$y_{n+1} \leq ax_n + (1-a)y_n$$

Taking liminf of the above inequality we get $p \leq a \liminf x_n + (1-a)p$ which implies that

$$\liminf x_n \geq p \tag{3.0.6}$$

The equations 3.0.5 and 3.0.6 imply that $x_n \to p$. □

4. OSCILLATION

Definition 4.0.5. (**Oscillation**)
(a) A sequence $\{x_n\}_{n=0}^{\infty}$ is said to have ***eventually*** some property P, if there exists an integer $N \geq k$ such that every term of $\{x_n\}_{n=N}^{\infty}$ has this property.
(b) A sequence $\{x_n\}_{n=0}^{\infty}$ is said to ***oscillate about zero*** or simply to ***oscillate*** if the terms x_n are neither eventually all positive nor eventually all negative. Otherwise the sequence is called ***nonoscillatory***. A sequence $\{x_n\}_{n=0}^{\infty}$ is called ***strictly oscillatory*** if for every $n_0 \geq 0$, there exist $n_1, n_2 \geq n_0$ such that $x_{n_1}x_{n_2} < 0$.
(c) A sequence $\{x_n\}_{n=0}^{\infty}$ is said to ***oscillate about*** $\overline{x}$ if the sequence $x_n - \overline{x}$ oscillates. The sequence $\{x_n\}_{n=0}^{\infty}$ is called ***strictly oscillatory*** about $\overline{x}$ if the sequence $x_n - \overline{x}$ is strictly oscillatory.

LEMMA 4.0.6. *Suppose that $a + \frac{b}{c+d} = 1$. If $x < y$, then $x < ax + \frac{bxy}{cx+dy} < y$ and if $y < x$, then $y < ax + \frac{bxy}{cx+dy} < x$.*

Proof. Let $f(x,y) = ax + \frac{bxy}{cx+dy}$. f is increasing respect to x and y. Hence, if $x < y$, then $f(x,x) < f(x,y) < f(y,y)$. The lemma is proved by assumption $a + \frac{b}{c+d} = 1$. □

Proposition 4.0.7. *Suppose $a + \frac{b}{c+d} = 1$ and $p = \lim x_n$, (the limit is exists by Theorem 2), then the following statements hold:*
1) If $x_0 = x_1 = x > 0$, then $x_n = x$, for all n.
2) If $x_0 < x_1$, then $x_{2n} \leq p \leq x_{2n+1}$, where . x_{2n} is increasing and x_{2n+1} is

decreasing.
3) If $x_0 > x_1$, then $x_{2n} \geq p \geq x_{2n+1}$, where $p = \lim x_n$. x_{2n+1} is increasing and x_{2n} is decreasing.
Consequently in cases 2 and 3 the sequence x_n oscillates about its limit point p.

Proof. (1) is trivial.
(2) and (3) is proved by induction and lemma 4.0.6 . □

5. CLOSED FORM OF THE SOLUTION

THEOREM 5.0.8. *The closed form solution of equation 1.0.1 is given by*

$$x_n = \frac{x_0}{c^n}\prod_{i=1}^{n}\left(\frac{C_1(\frac{ac+d+\sqrt{(ac+d)^2+4bc}}{2})^i + C_2(\frac{ac+d-\sqrt{(ac+d)^2+4bc}}{2})^i}{C_1(\frac{ac+d+\sqrt{(ac+d)^2+4bc}}{2})^{i-1} + C_2(\frac{ac+d-\sqrt{(ac+d)^2+4bc}}{2})^{i-1}} - d\right)$$

Proof. From equation 1.0.1, we get

$$\frac{x_{n+1}}{x_n} = a + \frac{b}{c\frac{x_n}{x_{n-1}} + d}.$$

We set $c\frac{x_n}{x_{n-1}} + d = \frac{z_n}{z_{n-1}}$, then we get

$$\frac{z_{n+1}}{cz_n} - \frac{d}{c} = a + \frac{bz_{n-1}}{z_n}.$$

Then

$$z_{n+1} = (ac+d)z_n + bcz_{n-1}.$$

That is a second order linear homogeneous difference equation. The solution of this linear equation is obtained easily as follows,

$$z_n = C_1\left(\frac{ac+d+\sqrt{(ac+d)^2+4bc}}{2}\right)^n + C_2\left(\frac{ac+d-\sqrt{(ac+d)^2+4bc}}{2}\right)^n$$

Hence the solution of 1.0.1 is obtained as follows,

$$x_n = \frac{x_0}{c^n}\prod_{i=1}^{n}\left(\frac{C_1(\frac{ac+d+\sqrt{(ac+d)^2+4bc}}{2})^i + C_2(\frac{ac+d-\sqrt{(ac+d)^2+4bc}}{2})^i}{C_1(\frac{ac+d+\sqrt{(ac+d)^2+4bc}}{2})^{i-1} + C_2(\frac{ac+d-\sqrt{(ac+d)^2+4bc}}{2})^{i-1}} - d\right)$$

□

6. PERIOD TWO SOLUTION

THEOREM 6.0.9. *Equation 1.0.1 has no positive solutions of prime period two for all $a, b, c, d \in (0, \infty)$.*

Proof. From equation 1.0.1 , we have

$$q = ap + \frac{bpq}{cp + dq}$$

$$p = aq + \frac{bqp}{cq + dp}$$

we try to solve these equations. .
By subtracting, we deduce that

$$p-q=a(q-p)+bpq(\frac{1}{cq+dp}-\frac{1}{cp+dq})$$

Hence

$$p-q=a(q-p)+bpq\Big(\frac{(cp+dq)-(cp+dq)}{(cp+dq)(cq+dp)}\Big)$$

$$p-q=-a(p-q)+bpq(p-q)\Big(\frac{c-d}{(cp+dq)(cq+dp)}\Big)$$

$$(p-q)(1+a+bpq\Big(\frac{d-c}{(cp+dq)(cq+dp)}\Big))=0$$

From the last equation we can see that if $d\geq c$,equation 1.0.1 has no positive solutions of prime period two for all $a,b,c,d\in(0,\infty)$.

□

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