Electronic Journal of Mathematical Analysis and Applications
Vol. 4(2) July 2016, pp. 234-253.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

NONLINEAR DEGENERATED PARABOLIC EQUATIONS WITH LOWER ORDER TERMS

J. BENNOUNA , M. HAMMOUMI AND A. ABERQI

Abstract. We prove an existence result of a renormalized solution for a class of nonlinear degenerated parabolic problems with $L^1(\Omega \times (0, T))$-data.

1. Introduction

In this paper we study the existence of solutions for the following class of nonlinear parabolic problems

$$\begin{cases} \dfrac{\partial b(u)}{\partial t} - \text{div } a(x,t,u,\nabla u) + \text{div}(\phi(x,t,u)) = f & \text{in} \quad Q_T \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ b(u(x,0)) = b(u_0(x)) & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 3$, $Q_T = \Omega \times (0,T)$, $T > 0$, b is a strictly increasing C^1-function and div $a(x,t,u,\nabla u)$ is a Leray-Lions type operator defined on the weighted Sobolev space $W_0^{1,p}(\Omega,\nu)$ (see assumptions (15)-(17) of Section 2). The function $\phi(x,t,u)$ is a Carathéodory function with suitable assumptions (see assumptions (18)-(20)). The right-hand side belongs to $L^1(Q_T)$. Let us point out, the difficulties that arise in problem (1) are due to the following facts: the data f and $b(u_0)$ only belong to $L^1(Q_T)$ and $L^1(\Omega)$ respectively, the function $\phi(x,t,u)$ is just satisfies the following codition $|\phi(x,t,s)| \leq c(x,t)|s|^\gamma \nu(x)$, and the presence of the weighted function ν (see assumptions (4)-(5).

Under our assumptions, problem (1) does not admit, in general, a weak solution since the term $\phi(x,t,u)$ may not belong $(L^1_{loc}(Q))^N$. In order to overcome this difficulty, we work with the framework of renormalized solutions (see definition (3.1)). This notion was introduced by R.-J. DiPerna and P.-L. Lions [21] in their study of the Boltzmann equation. This notion was then adapted to an elliptic version of (1) by L. Boccardo and al (see [14]) when the right hand side is in $W^{-1,p'}(\Omega)$ and by J.-M. Rakotoson (see [26]) when the right hand side is in $L^1(\Omega)$, and finally by G. Dal Maso, F. Murat, L. Orsina and A. Prignet [17]

The existence and uniqueness of a renormalized solution for parabolic problems in the classical space has been proved by D. Blanchard and F. Murat [10] in the

---

2010 *Mathematics Subject Classification.* Primary 47A15; Secondary 46A32, 47D20.
Key words and phrases. Parabolic problems, Sobolev space, Renormalized Solutions.
Submitted Sep. 15, 2015.

case where $a(x,t,s,\xi)$ is independent of s, and with $\phi = 0$ and by D. Blanchard, F. Murat and H. Redwane with the nonstricte monotonicity on a with $\nu = 1$ (see [11] condition (7)).
For the degenerated parabolic equations the existence of weak solutions have been proved by L. Aharouch and al [1] in the case where a is strictly monotone, $\phi = 0$ and $f \in L^{p'}(0,T,W^{-1,p'}(\Omega,\nu^{1-p'}))$. See also the existence of renormalized solution proved by Y. Akdim and al [6] in the case where $a(x,t,s,\xi)$ is independent of s and $\phi = 0$.

In the case where $b(u) = u$ and $\nu = 1$ the existence of renormalized solutions for (1) has been established by R.-Di Nardo (see [19]). For the degenerated parabolic equation with $b(u) = u$, $div(\phi(x,t,u)) = H(x,t,u,\nabla u)$ and $f \in L^1(Q)$, the existence of renormalized solution has been proved by Y. Akdim and al (see [3]).

The case where $b(u) = b(x,u)$, $div(\phi(x,t,u)) = H(x,t,u,\nabla u)$ and $f \in L^1(Q)$, the existence of renormalized solutions has been established by H. Redwane (see [28]) in the classical Sobolev space ($\nu = 1$) and by Y. Akdim and al (see [2]) in the degenerate Sobolev space.

Note that, this paper can be seen as a generalization of ([4], [19]) in weighted case, and we prove the existence of a renormalized solution of (1)

The plan of the paper is as follows: In Section 2 we give some preliminaries and we make precise all the assumptions on a, ϕ, f, and $b(u_0)$. In Section 3 we give the definition of a renormalized solution of (1), and we establish (Theorem 3.1) the existence of such a solution.

2. Preliminaries and auxiliary results

We recall here some standard notations, properties and results which will be used through the paper.
Let Ω be a bounded open set of $\mathbb{R}^N$ and $Q_T = \Omega \times (0,T)$, T is a positive real number. Let $\nu(x)$ be a nonnegative function on Ω such that $\nu(x) \in L^r(\Omega)$, $r \geq 1$, $\nu(x)^{-1} \in L^t(\Omega)$, $p \geq 1 + 1/t$. We denote by $L^p(\Omega,\nu)$, or simply $L^p(\nu)$ if there is no confusion, $p \geq 1$, the space of measurable functions u on Ω such that

$$\|u\|_{L^p(\nu)} = \Big(\int_\Omega |u|^p \nu(x)dx\Big)^{\frac{1}{p}} < +\infty, \tag{2}$$

and by $W^{1,p}(\nu)$ the completion of the space $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_{W^{1,p}(\nu)} = \|u\|_{L^p(\nu)} + \|\nabla u\|_{L^p(\nu)}. \tag{3}$$

Moreover we denote by $W_0^{1,p}(\nu)$ the closure of $C_0^1(\overline{\Omega})$ in $W^{1,p}(\nu)$, provided with the induced topology defined by the induced norm, and by $W^{-1,p'}(\nu^{1-p'})$, $p' = \frac{p}{p-1}$, its dual space. $W^{1,p}(\nu)$ and $W_0^{1,p}(\nu)$ are reflexive Banach spaces if $1 < p < \infty$, (see [25]).

Denote $V = W_0^{1,p}(\nu)$, $H = L^2(\nu)$ and $V^* = W_0^{-1,p'}(\nu^{1-p'})$, with $p \geq 2$. The dual space of $X := L^p(0,T;W_0^{1,p}(\nu))$ denoted X^* is identified to $L^{p'}(0,T;V^*)$. Define $W_p^1(0,T,V,H) = \{v \in X : v' \in X^*\}$. Endowed with the norm

$$\|u\|_{W_p^1} = \|u\|_X + \|u'\|_{X^*},$$

$W_p^1(0,T,V,H)$ is a Banach space. Here u' stands for the generalized time derivative of u, that is,

$$\int_0^T u'(t)\varphi(t)dt = -\int_0^T u(t)\varphi'(t)dt \text{ for all } \varphi \in C_0^\infty(0,T).$$

Lemma 2.1. [30]

1. *The evolution triple* $V \hookrightarrow H \hookrightarrow V^*$ *is verified.*
2. *The imbedding* $W_p^1(0,T,V,H) \hookrightarrow C(0,T,H)$ *is continuous.*
3. *The imbedding* $W_p^1(0,T,V,H) \hookrightarrow L^p(Q_T,\nu)$ *is compact.*

Lemma 2.2. [1]
Let $g \in L^r(Q,\nu)$ *and let* $g_n \in L^r(Q,\nu)$, *with* $\|g_n\|_{L^r(Q,\nu)} \le C$, *with* $1 < r < +\infty$. *If* $g_n(x) \to g(x)$ *a.e in* Q *then* $g_n \rightharpoonup g$ *in* $L^r(Q,\nu)$

Lemma 2.3. [1] *Let* $\{v_n\}$ *be a bounded sequence in* $L^p(0,T;V)$ *such that*

$$\frac{\partial v_n}{\partial t} = \alpha_n + \beta_n \text{ in } \mathcal{D}'(Q),$$

with $\{\alpha_n\}$ *and* $\{\beta_n\}$ *two bounded sequences respectively in* X^* *and in* $L^1(Q)$. *Then* $v_n \to v$ *in* $L^p_{loc}(Q,\nu)$. *Furthermore,* $v_n \to v$ *strongly in* $L^1(Q)$.

From now on, we assume that the following assumptions hold true

$$\nu(x)^{-1} \in L^t(\Omega),\ t \ge \frac{N}{p},\ 1+\frac{1}{t} < p < N(1+\frac{1}{t}), \tag{4}$$

$$\nu(x) \in L^r(\Omega),\ r > \frac{Nt}{pt-N}, \tag{5}$$

An important tool that we will use here, is the following weighted version of the Sobolev inequality (see Theorem 3.1 and Corollary 3.5 in [25]).

Proposition 2.1. [25] *Assume that* (4) *and* (5) *hold true. Let* $\tilde{p}$ *denote the number associated to* p *defined by*

$$\frac{1}{\tilde{p}} = r'\Big(\frac{1}{p}(1+\frac{1}{t}) - \frac{1}{N}\Big).$$

Then the imbedding of $W_0^{1,p}(\nu)$ *into* $L^{\tilde{p}}(\nu)$ *is continuous. moreover, there exists a constant* $C_0 > 0$ *depending on* N, p, ν, t, *such that*

$$\|u\|_{L^{\tilde{p}}(\nu)} \le C_0 \||\nabla u|\|_{L^p(\nu)}, \forall u \in W_0^{1,p}(\nu). \tag{6}$$

Using this proposition, we can prove the following interpolation result.

Proposition 2.2. *Assume that* (4) *and* (5) *hold true. Let* v *be a function in* $W_0^{1,p}(\nu) \cap L^s(\Omega)$ *with* $2 \le p < N$ *and* $s > r'$. *Then there exists a positive constante* C, *depending on* $N,\ p,\ \nu,\ t$ *and* q, *such that*

$$\|v\|_{L^\sigma(\nu)} \le C\|\nabla v\|^{1-\theta}_{L^p(\nu)} \|v\|^\theta_{L^s(\Omega)}$$

for every θ *and* σ *satisfying*

$$0 \le \theta \le 1,\ 1 \le \sigma \le +\infty,\ \frac{1}{\sigma} = \theta + r'(1-\theta)\Big((1+\frac{1}{t})\frac{1}{p} - \frac{1}{N}\Big),\ r > \frac{Nt}{pt-N}.$$

Proof. For every $1 \leq \sigma \leq \tilde{p}$, we can write $\frac{1}{\sigma} = \theta + \frac{1-\theta}{\tilde{p}}$ for some $0 \leq \theta \leq 1$. So that by the Hölder inequality and (6), one has

$$\begin{aligned} \|v\|_{L^\sigma(\nu)} &\leq C_0 \||\nabla v|\|^{1-\theta}_{L^p(\nu)} \|v\|^{\theta}_{L^1(\nu)} \\ &\leq C_0 \||\nabla v|\|^{1-\theta}_{L^p(\nu)} \|\nu\|^{\theta}_{L^{s'}(\Omega)} \|v\|^{\theta}_{L^s(\Omega)}, \end{aligned}$$

which gives the desired result. □

An immediate consequence of the previous result, we get

Corollary 2.1. *Let $v \in L^p((0,T), W_0^{1,p}(\nu)) \cap L^\infty((0,T), L^s(\Omega))$, with $2 \leq p < N$ and $s > r'$. Then $v \in L^\sigma(\nu)$ with $\sigma = \frac{p\tilde{p}+\tilde{p}-p}{\tilde{p}}$. Moreover,*

$$\int_{Q_T} \nu(x)|v|^\sigma dxdt \leq C \parallel v \parallel^{\frac{\tilde{p}-p}{\tilde{p}}}_{L^\infty(0,T,L^s(\Omega))} \int_{Q_T} \nu(x)|\nabla v|^p dxdt.$$

Proof. By virtue of Proposition 2.2, we can write

$$\int_\Omega \nu(x)|v|^\sigma dx \leq C \||\nabla v|\|^{(1-\theta)\sigma}_{L^p(\nu)} \parallel v \parallel^{\theta\sigma}_{L^s(\Omega)} .$$

Integrating between 0 and T, we get

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \int_0^T \||\nabla v|\|^{(1-\theta)\sigma}_{L^p(\nu)} \|v\|^{\theta\sigma}_{L^s(\Omega)} dt. \tag{7}$$

Since $v \in L^p((0,T), W_0^{1,p}(\nu)) \cap L^\infty((0,T), L^s(\Omega))$, we have

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \|v\|^{\theta\sigma}_{L^\infty(0,T,L^s(\Omega))} \int_0^T \||\nabla v(t)|\|^{(1-\theta)\sigma}_{L^p(\nu)} dt.$$

Now we choose θ such that

$$(1-\theta)\sigma = p \text{ and } \theta\sigma = \frac{\tilde{p}-p}{\tilde{p}}.$$

This choice yields

$$\theta = \frac{\tilde{p}-p}{p\tilde{p}+\tilde{p}-p} \text{ and } \sigma = \frac{p\tilde{p}+\tilde{p}-p}{\tilde{p}}$$

Then, (7) becomes

$$\int_0^T \int_\Omega \nu(x)|v|^\sigma dxdt \leq C \parallel v \parallel^{\frac{\tilde{p}-p}{\tilde{p}}}_{L^\infty(0,T,L^s(\Omega))} \int_0^T \||\nabla v(t)|\|^{p}_{L^p(\nu)} dt.$$

□

In order to prove our existence result, we shall prove a technical lemma, following the same method used in [7]) that yields two estimates for $|u_n|^{p-1}$ and $|\nabla u_n|^{p-1}$ in the Lorentz spaces $L^{\frac{2p\tilde{p}-\tilde{p}-p}{2\tilde{p}(p-1)},\infty}(Q_T)$ and $L^{\frac{p(2p\tilde{p}-\tilde{p}-p)}{(p-1)(2p\tilde{p}+\tilde{p}-p)},\infty}(Q_T)$ respectively. Moreover by imbedding theorems, these a priori bounds imply two estimates in the Lebesgue spaces $L^m(Q_T)$ and $L^s(Q_T)$ with $m < \frac{2p\tilde{p}-\tilde{p}-p}{2\tilde{p}(p-1)}$ and $s < \frac{p(2p\tilde{p}-\tilde{p}-p)}{(p-1)(2p\tilde{p}+\tilde{p}-p)}$. In what follows, we define

$$meas_\nu E = \int_E \nu(x)dx,$$

for any measurable set $E \subseteq \mathbb{R}^N$. Tus, we can define the weighted Lorentz spaces $L^{r,\infty}(\nu)$, $1 \leq r \leq +\infty$ as the set of measurable functions u defined on Ω such that

$$\|u\|_{L^{r,\infty}(\nu)} = \sup_{t>0} t\, meas_\nu\{x \in \Omega : |u| > t\}^{\frac{1}{r}} < +\infty.$$

Throughout the paper, T_k, $k > 0$, denotes the truncation function at level k defined on $\mathbb{R}$ by $T_k(r) = \max(-k, \min(k, r))$.

Lemma 2.4. *Assume that Ω is an open subset of $\mathbb{R}^N$ of finite measure, $2 \leq p < N$, and that (4) and 5 hold true. Let u be a measurable function satisfying $T_k(u) \in L^p(0, T, W_0^{1,p}(\nu)) \cap L^\infty(0, T, L^2(\Omega))$ for every $k > 0$ and such that:*

$$\sup_{t\in(0,T)} \int_\Omega |T_k(u)|^2 dx + \int_{Q_T} \nu(x)|\nabla T_k(u)|^p dxdt \leq Mk, \quad \forall k > 0, \tag{8}$$

where M is a positive constant. Then we get $|u|^{p-1} \in L^{\frac{2p\tilde{p}-\tilde{p}-p}{2\tilde{p}(p-1)},\infty}(Q_T)$, and $|\nabla u|^{p-1} \in L^{\frac{p(2p\tilde{p}-\tilde{p}-p)}{(p-1)(2p\tilde{p}+\tilde{p}-p)},\infty}(Q_T)$, Moreover, we have the following estimates

$$\||u|^{p-1}\|_{L^{\frac{2p\tilde{p}-\tilde{p}-p}{2\tilde{p}(p-1)},\infty}(Q_T)} \leq CM^{(\frac{\tilde{p}-p}{2\tilde{p}}+1)\frac{2\tilde{p}(p-1)}{2p\tilde{p}-\tilde{p}-p}}, \tag{9}$$

$$\||\nabla u|^{p-1}\|_{L^{\frac{p(2p\tilde{p}-\tilde{p}-p)}{(p-1)(2p\tilde{p}+\tilde{p}-p)},\infty}(Q_T)} \leq CM^{\frac{2p\tilde{p}+2\tilde{p}-2p}{2p\tilde{p}+\tilde{p}-p}} \tag{10}$$

where C is a constant depend only on N, p, ν, and t.

Proof. We first prove (9). For any $k_0 > 0$, we can write

$$\begin{aligned} \||u|^{p-1}\|_{L^{\frac{2p\tilde{p}-\tilde{p}-p}{2\tilde{p}(p-1)},\infty}(Q_T)} &\leq \sup_{0<k<k_0} k\Big[meas_\nu\{(x,t) \in Q_T : |u|^{p-1} > k\}\Big]^{\frac{2\tilde{p}(p-1)}{2p\tilde{p}-\tilde{p}-p}} \\ &+ \sup_{k\geq k_0} k\Big[meas_\nu\{(x,t) \in Q_T : |u|^{p-1} > k\}\Big]^{\frac{2\tilde{p}(p-1)}{2p\tilde{p}-\tilde{p}-p}} \\ &\leq k_0|Q_T|^{\frac{2\tilde{p}(p-1)}{2p\tilde{p}-\tilde{p}-p}} + \sup_{k\geq k_0} k\Big[meas_\nu\{(x,t) \in Q_T : |u|^{p-1} > k\}\Big]^{\frac{2\tilde{p}(p-1)}{2p\tilde{p}-\tilde{p}-p}}. \end{aligned} \tag{11}$$

By corollary (2.1) and (8) we have

$$\begin{aligned} & k^{\frac{p\tilde{p}+\tilde{p}-p}{\tilde{p}}} meas_\nu\{(x,t) \in Q_T : |u| > k\} \\ &\leq \int_{Q_T} \nu(x)|T_k(u)|^{\frac{p\tilde{p}+\tilde{p}-p}{\tilde{p}}} dxdt \\ &\leq C \sup_{t\in(0,T)} \Big(\int_\Omega |T_k(u)|^2 dx\Big)^{\frac{\tilde{p}-p}{2\tilde{p}}} \int_{Q_T} \nu(x)|\nabla T_k(u)|^p dxdt \\ &\leq C(Mk)^{\frac{\tilde{p}-p}{2\tilde{p}}+1}. \end{aligned}$$

Hence,

$$meas_\nu\{(x,t) \in Q_T : |u|^{p-1} > k\} \leq CM^{\frac{\tilde{p}-p}{2\tilde{p}}+1} k^{-\frac{2p\tilde{p}-\tilde{p}-p}{2\tilde{p}(p-1)}}. \tag{12}$$

By (12) we deduce that $|u|^{p-1} \in L^{\frac{2p\tilde{p}-\tilde{p}-p}{2\tilde{p}(p-1)},\infty}(Q_T)$. Furthermore, putting (12) in (11) and taking $k_0 = \frac{M^{(\frac{\tilde{p}-p}{2\tilde{p}}+1)\frac{2\tilde{p}(p-1)}{2p\tilde{p}-\tilde{p}-p}}}{|Q_T|^{\frac{2\tilde{p}(p-1)}{p(2p\tilde{p}-\tilde{p}-p)}}}$ we get (9). We now prove the estimate involving the gradient of u. For every $\lambda > 0$ and every $k > 0$, we have

$$\begin{aligned} meas_\nu\{(x,t) \in Q_T : |\nabla u| > \lambda\} &\leq meas_\nu\{(x,t) \in Q_T : |\nabla u| > \lambda \text{ and } |u| \leq k\} \\ &+ meas_\nu\{(x,t) \in Q_T : |\nabla u| > \lambda \text{ and } |u| > k\}. \end{aligned}$$

By (8) we know that

$$\begin{aligned} Mk \geq \int_{Q_T} \nu(x)|\nabla T_k(u)|^p dxdt &\geq \int_{\{|u|\leq k\}\cap\{|\nabla u|>\lambda\}} \lambda^p \nu(x)dx \\ &\geq \lambda^p meas_\nu\{(x,t)\in Q_T : |\nabla u| > \lambda \text{ and } |u|\leq k\}, \end{aligned}$$

which implies

$$meas_\nu\{(x,t)\in Q_T : |\nabla u|^{(p-1)} > \lambda \quad \text{and} \quad |u|\leq k\} \leq \frac{Mk}{\lambda^{p'}}.$$

The above formula together with (12) allow us to obtain

$$meas_\nu\{(x,t)\in Q_T : |\nabla u|^{(p-1)} > \lambda\} \leq \frac{Mk}{\lambda^{p'}} + CM^{\frac{\tilde p - p}{2\tilde p}+1} k^{-\frac{2p\tilde p-\tilde p-p}{2\tilde p}}. \tag{13}$$

If we take $k = M^{\frac{\tilde p - p}{2p\tilde p+\tilde p-p}} \lambda^{\frac{2p\tilde p}{(p-1)(2p\tilde p+\tilde p-p)}}$, (13) becomes

$$meas_\nu\{(x,t)\in Q_T : |\nabla u|^{(p-1)} > \lambda\} \leq C\frac{M^{\frac{2(p\tilde p+\tilde p-p)}{2p\tilde p+\tilde p-p}}}{\lambda^{\frac{p(2p\tilde p-\tilde p-p)}{(p-1)(2p\tilde p+\tilde p-p)}}}, \tag{14}$$

which proves (10). □

2.1. **Assumptions and main result.** We now make precise assumptions on each part of problem (1). Let Ω be a bounded open subset of $\mathbb{R}^N$, $N\geq 2$, $Q_T = \Omega\times(0,T)$, $T>0$, and $2\leq p<+\infty$. Let $\nu(x)$ be a nonnegative function satisfying (4) and (5). Suppose that $b:\mathbb{R}\to\mathbb{R}$ is a strictly increasing C^1-function, such that $b(0)=0$ and $b' > \beta > 0$ for some $\beta>0$, and for almost every $(x,t)\in Q_T$, for every $s\in\mathbb{R}$ and every $\xi\in\mathbb{R}^N$

$$|a(x,t,s,\xi)| \leq \nu(x)\big(h(x,t)+|s|^{p-1}+|\xi|^{p-1}\big),\ h(x,t)\in L^{p'}(\nu), \tag{15}$$

$$a(x,t,s,\xi)\xi \geq \alpha\nu(x)|\xi|^p,\ \text{with } \alpha>0, \tag{16}$$

$$(a(x,t,s,\xi)-a(x,t,s,\eta)\cdot(\xi-\eta) > 0,\ \xi\neq\eta, \tag{17}$$

$$|\phi(x,t,s)| \leq c(x,t)|s|^\gamma \nu(x), \tag{18}$$

$$c(x,t)\in (L^\tau(Q_T,\nu))^N, \quad \tau = \frac{p(3\tilde p-p)}{(p-1)(\tilde p-p)}, \tag{19}$$

$$\gamma = \frac{2(p-1)(p\tilde p+\tilde p-p)}{p(3\tilde p-p)} \tag{20}$$

$$f\in L^1(Q_T) \tag{21}$$

and

$$u_0\in L^1(\Omega) \text{ such that } b(u_0)\in L^1(\Omega). \tag{22}$$

We have to seek for a solution to problem (1) in the following sense.

Definition 2.1. *A measurable function u is a renormalized solution to problem (1), if*

$$b(u)\in L^\infty((0,T),L^1(\Omega)). \tag{23}$$

$$T_k(u)\in L^p((0,T),W_0^{1,p}(\Omega)),\ \textit{for any } k>0, \tag{24}$$

$$\lim_{m\to+\infty}\frac{1}{m}\int_{\{(x,t)\in Q_T:|u(x,t)|\leq m\}} a(x,t,u,\nabla u)\nabla u\,dx\,dt = 0, \tag{25}$$

and if for every function S in $W^{2,\infty}(\mathbb{R})$ which is piecewise C^1 and such that S' has a compact support

$$\begin{aligned}&\frac{\partial B_S(u)}{\partial t} - div\Big(a(x,t,u,\nabla u)S'(u)\Big) + S^{''}(u)a(x,t,u,\nabla u)\nabla u\\&+ div\Big(\phi(x,t,u)S'(u)\Big) - S''(u)\phi(x,t,u)\nabla u = fS'(u) \quad in \quad D^{'}(\Omega)\end{aligned} \tag{26}$$

and

$$B_S(u)(t=0) = B_S(u_0) \quad in \quad \Omega, \tag{27}$$

where $B_S(z) = \int_0^z b^{'}(s)S^{'}(s)ds.$

Remark 2.1. *Equation (26) is formally obtained through multiplication of (1) by $S'(u)$. However while $a(x,t,u,\nabla u)$ and $\phi(x,t,u)$ does not in general make sense in (1), all the terms in (26) have a meaning in $D'(Q_T)$. Indeed, if M is such that $suppS^{'} \subset [-M,M]$, the following identifications are made in (26):*

- *$B_S(u)$ belongs to $L^\infty(Q_T)$ since S is a bounded function and*

$$DB_S(u) = S'(u)b'(T_M(u))DT_M(u).$$

- *$S'(u)a(x,t,u,\nabla u)$ identifies with $S'(u)a(x,t,T_M(u),\nabla T_M(u))$ a.e in Q_T. Since we have $|T_M(u)| \le M$ a.e in Q_T and $S'(u) \in L^\infty(Q_T)$, we obtain from (15) and (24) that*

$$S'(u)a(x,t,T_M(u),\nabla T_M(u)) \in (L^{p'}(Q_T,\nu^{1-p'}))^N$$

- *$S''(u)a(x,t,u,\nabla u)\nabla u$ identifies with $S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u)$ a.e. in Q_T and*

$$S''(u)a(x,t,T_M(u),\nabla T_M(u))\nabla T_M(u) \in L^1(Q_T).$$

- *$S''(u)\phi(x,t,u)\nabla u$ and $S'(u)\phi(x,t,u)$ are respectively identify with the two terms $S''(u)\phi(x,t,T_M(u))\nabla T_M(u)$ and $S'(u)\phi(x,t,T_M(u))$ a.e. in Q_T.*

The above consideration shows that equation (26) hold in $D'(\Omega)$, $\frac{\partial B_S(u)}{\partial t}$ belongs to $L^1(Q)+L^{p'}(0,T,W^{-1,p'}(Q_T,\nu^{1-p'}))$ and $B_S(u) \in L^p(0,T,W_0^{1,p}(\Omega,\nu))\cap L^\infty(Q)$. It follows that $B_S(u)$ belongs to $C^0([0,T],L^1(\Omega))$ so the initial condition (27) makes sense.

Theorem 2.1. *Assume that* (4), (5) *and* (15)-(22) *hold true. Then, there exists at least a renormalized solution of the problem* (1).

Remark 2.2. *The result of Theorem 2.1 extends to the weighted case the analogous in* [4] *(with $\nu = 1$), in* [5] *(with $\phi(x,t,u) = \phi(u)$) and in* [19] *(with $b(u) = u$, $\nu = 1$).*

Remark 2.3. *Similar result can be obtained if the datum is of the forme $f - divF$, with $f \in L^1(\Omega)$ and $F \in (L^{p'}(\Omega,\nu^{1-p'}))^N$.*

3. Proof of Theorem 2.1

We divide the proof is divided into six steps.

Step 1: Approximate problem and a priori estimates.

For each $n > 0$, let us define the following approximation of b, a, ϕ, f, and u_0;

$$b_n(r) = T_n(b(r)) + \frac{1}{n}r. \quad \forall\ r \in \mathbb{R}, \tag{28}$$

$$a_n(x,t,s,\xi) = a(x,t,T_n(s),\xi).a.e \quad in \quad Q \quad \forall\, s \in \mathbb{R},\ \forall\, \xi \in \mathbb{R}^N, \tag{29}$$

$$\phi_n(x,t,r) = \phi(x,t,T_n(r)) \text{ a.e. } (x,t) \in Q_T,\ \forall\, r \in \mathbb{R}. \tag{30}$$

$$f_n \in L^{p'}(Q_T) \text{ such that } f_n \to f \text{ strongly in } L^1(Q_T) \tag{31}$$

and

$$u_{0n} \in D(\Omega) \text{ such that } b_n(u_{0n}) \to b(u_0) \text{ a.e. } (x,t) \in \Omega \text{ strongly in } L^1(\Omega), \tag{32}$$

Let us consider the approximate problem :

$$\begin{cases} \dfrac{\partial b_n(u_n)}{\partial t} - div(a_n(x,t,u_n,\nabla u_n)) + div(\phi_n(x,t,u_n)) = f_n \quad in \quad D'(Q_T), \\ u_n(x,t) = 0 \quad on \quad \partial\Omega \times (0,T) \\ b_n(u_n(x,0)) = b_n(u_{0n}(x)) \quad in \quad \Omega. \end{cases} \tag{33}$$

As a consequence, proving existence of a weak solution $u_n \in L^p((0,T), W_0^{1,p}(\nu))$ of (33) is an easy task (See [1], [24] and [27]).

Step 2: The estimates derived in this step rely on standard techniques for problems of type (33).
Using in (33), the test function $T_k(u_n)\chi_{(0,\tau_1)}$, we get, for every $\tau_1 \in [0,T]$, we integrate between $(0,\tau_1)$ and by the condition (30) we have

$$\int_\Omega B_k^n(u_n(\tau_1))dx + \int_{Q_{\tau_1}} a_n(x,t,u_n,\nabla u_n)\nabla T_k(u_n)\,dx\,dt$$

$$\leq \int_{Q_{\tau_1}} c(x,t)|u_n|^\gamma \nu(x)|\nabla T_k(u_n)|\,dx\,dt + \int_{Q_{\tau_1}} f_n T_k(u_n)\,dx\,dt + \int_\Omega B_k^n(u_{0n})dx, \tag{34}$$

where $B_k^n(r) = \displaystyle\int_0^r T_k(s)b_n'(s)ds$. Due to definition of B_k^n we have:

$$0 \leq \int_\Omega B_k^n(u_{0n})dx \leq k\int_\Omega |b_n(u_{0n})|dx \leq k||b(u_0)||_{L^1(\Omega)} \quad \forall k > 0 \tag{35}$$

Using (34) and (16) we obtain:

$$\int_\Omega B_k^n(u_n(\tau_1))dx + \alpha\int_{Q_{\tau_1}} \nu(x)|\nabla T_k(u_n)|^p\,dx\,dt$$

$$\leq \int_{Q_{\tau_1}} c(x,t)|u_n|^\gamma \nu(x)|\nabla T_k(u_n)|\,dx\,dt + k(||b(u_0)||_{L^1(\Omega)} + ||f_n||_{L^1(Q)}) \tag{36}$$

If we take the supremum for $t \in (0,\tau_1)$ and we define $M = sup(||f_n||_{L^1(Q)}) + ||b(u_0)||_{L^1(\Omega)}$, we deduce from that above inequality (34) and (35)

$$\frac{\beta}{2}\int_\Omega |T_k(u_n)|^2\,dx + \alpha\int_{Q_t} \nu(x)|\nabla T_k(u_n)|^p\,dx\,dt \leq Mk + \int_{Q_t} c(x,t)|u_n|^\gamma \nu(x)|\nabla T_k(u_n)|\,dx\,dt. \tag{37}$$

By Corollary 2.1 and Young inequality we have:

$$\begin{aligned}&\int_{Q_t} c(x,t)|u_n|^\gamma \nu(x)|\nabla T_k(u_n)|\,dx\,dt\\&\leq C\frac{\gamma(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_n)|^2\,dx\\&+C\frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\\&\times\Big(\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_n)|^p\,dx\,dt\Big)^{(\frac{1}{p}+\frac{\gamma\tilde{p}}{p\tilde{p}+\tilde{p}-p})\frac{2(p\tilde{p}+\tilde{p}-p)}{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}}.\end{aligned}\tag{38}$$

Using the value $\gamma=\frac{2(p-1)(p\tilde{p}+\tilde{p}-p)}{p(3\tilde{p}-p)}$, (37) and (38), we obtain

$$\frac{\beta}{2}\int_\Omega |T_k(u_n)|^2\,dx+\alpha\int_{Q_t}\nu(x)|\nabla T_k(u_n)|^p\,dx\,dt$$

$$\leq Mk+C\frac{\gamma(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_n)|^2\,dx$$

$$+C\frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_n)|^p\,dx\,dt$$

Which is equivalent to

$$\Big(\frac{\beta}{2}-C\frac{\gamma(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big)\sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_n)|^2\,dx+$$

$$\Big(\alpha-C\frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big)\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_n)|^p\,dx\,dt\leq Mk$$

If we choose τ_1 such that

$$\Big(\frac{\beta}{2}-C\frac{\gamma(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big)\geq 0,\tag{39}$$

and

$$\Big(\alpha-C\frac{2p\tilde{p}+(2-\gamma)(\tilde{p}-p)}{2(p\tilde{p}+\tilde{p}-p)}||c(x,t)||_{L^\tau(Q_{\tau_1},\nu)}\Big)\geq 0,\tag{40}$$

then, let us denote by C the minimum between (39) and (40), we obtain

$$\sup_{t\in(0,\tau_1)}\int_\Omega |T_k(u_n)|^2\,dx+\int_{Q_{\tau_1}}\nu(x)|\nabla T_k(u_n)|^p\,dx\,dt\leq CMk\tag{41}$$

By (41) it follows that

$$T_k(u_n)\text{ is bounded in }L^p(0,T;W_0^{1,p}(\nu))\tag{42}$$

and

$$T_k(u_n)\text{ is bounded in }L^\infty(0,T;L^2(\Omega))\tag{43}$$

Moreover, proceeding as in [10], [12] is possible to prove that for any $S\in W^{2,\infty}(\mathbb{R})$ with S' has a compact support, the term

$$\frac{\partial S(u_n)}{\partial t}\text{ is bounded in }L^1(Q_T)+L^{p'}(0,T;W_0^{-1,p'}(\nu^{1-p'})),\tag{44}$$

On the other hand, the boundedness of $T_k(u_n)$ (42), (44) and the apriori estimate of u_n, in the Lorentz spaces imply that there exists a subsequence, still denoted by u_n, such that

$$u_n \to u \text{ a.e. in } Q_T, \tag{45}$$

where u is a measurable function defined on Q_T (see [9], lemma 2 p. 224).

We turn now to prove the almost every convergence of $b_n(u_n)$. Let $g_k \in C^2(\mathbb{R})$ such that $g_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $g_k(s) = k$ for $|s| \geq k$. Multiplying the approximate equation (33) by $g'_k(b_n(u_n))$ we get

$$\frac{\partial g_k(b_n(u_n))}{\partial t} - div\Big(a_n(x,t,u_n,\nabla u_n)g'_k(b_n(u_n))\Big) \tag{46}$$

$$+a_n(x,t,u_n,\nabla u_n)g''_k(b_n(u_n))b'_n(u_n)\nabla u_n + div\Big(\phi_n(x,t,u_n)g'_k(b_n(u_n))\Big)$$

$$-g''_k(b_n(u_n))b'_n(u_n)\phi_n(x,t,u_n)\nabla u_n = f_n g'_k(b_n(u_n)) \text{ in } D'(\Omega)$$

Now each term in (46) is taking into account because of (15), (29)and $T_k(u_n)$ is bounded in $L^p(0,T,W_0^{1,p}(\nu))$, we deduce that:

$$-div\big(a_n(x,t,u_n,\nabla u_n)g'_k(b_n(u_n))\big)+a_n(x,t,u_n,\nabla u_n)g''_k(b_n(u_n))b'_n(u_n)\nabla u_n+f_n g'_k b_n(u_n)$$

is bounded in $L^1(Q_T) + L^{p'}(0,T,W^{-1,p'}(\nu^{1-p'}))$ independently of n as soon as $k < n$. Due to definition of b and b_n, it is clear that $\{|b_n(u_n)| \leq k\} \subset \{|u_n| \leq k^*\}$ where k^* is a constant independent of n. As a first consequence we have:

$$Dg_k(b_n(u_n)) = g'_k(b_n(u_n))b'_n(T_{k^*}(u_n))DT_{k^*}(u_n) \quad \text{a.e in } Q \tag{47}$$

as soon as $k < n$. Secondly the following estimate hold true:

$$||g'_k(b_n(u_n))b'_n(T_{k^*}(u_n))||_{L^\infty(Q)} \leq ||g'_k||_{L^\infty(Q)}(\max_{|r|\leq k^*}(b'(r)+1)).$$

As a consequence of (41), (47) , we then obtain:

$$g_k(b_n(u_n)) \text{ is bounded in } L^p(0,T,W_0^{1,p}(\nu)). \tag{48}$$

Since $supp(g'_k)$ and $supp(g''_k)$ are both included in [-k,k] by (30) it follows that for all $k < n$ we have

$$\begin{aligned}
&\Big|\int_{Q_T} \phi_n(x,t,u_n)^{p'} g'_k(b_n(u_n))^{p'} \nu^{1-p'}(x)\,dx\,dt\Big| \\
&\leq \int_{Q_T} c(x,t)^{p'}|T_n(u_n)|^{p'\gamma}|g'_k(b_n(u_n))|^{p'}\nu(x)\,dx\,dt \\
&= \int_{\{|u_n|\leq k^*\}} c(x,t)^{p'}|T_{k^*}(u_n)|^{p'\gamma}|g'_k(b_n(u_n))|^{p'}\nu(x)\,dx\,dt
\end{aligned}$$

Furthermore, by Hölder and corollary 2.1, it results

$$\begin{aligned}
&\int_{\{|u_n|\leq k^*\}} c(x,t)^{p'}|T_{k^*}(u_n)|^{p'\gamma}|g'_k(b_n(u_n))|^{p'}\nu(x)\,dx\,dt \\
&\leq \|g'_k\|_{L^\infty(\mathbb{R})}||c(x,t)||^{p'}_{L^\tau(Q_T,\nu)}\Big[\sup_{t\in(0,T)}\Big(\int_\Omega |T_{k^*}(u_n)|^2\,dx\Big)^{\frac{\tilde{p}-p}{2\tilde{p}}} \\
&+\int_{Q_T}\nu(x)n|\nabla T_{k^*}(u_n)|^p\,dx\,dt\Big] \leq c_{k^*}
\end{aligned}$$

where c_{k^*} is a constant independently of n which will vary from line to line.
In the same by (30) we have:

$$\Big| \int_{Q_T} \phi_n(x,t,u_n)^{p'} (g_k''(b_n(u_n) b_n'(u_n) \nabla u_n)^{p'} \nu^{1-p'}(x)\, dx\, dt \Big| \tag{49}$$

$$\leq \int_{Q_T} (g_k''(b_n(u_n))^{p'} b_n'(u_n)^{p'} |c(x,t)|^{p'} |T_n(u_n)|^{p'\gamma} \nu(x) |\nabla u_n|^{p'}\, dx\, dt.$$

Furthermore, by Hölder and corollary 2.1 ,we obtain for $k^* < n$:

$$\int_{Q_T} (g_k''(b_n(u_n))^{p'} b_n'(u_n)^{p'} |c(x,t)|^{p'} |T_n(u_n)|^{p'\gamma} \nu(x) |\nabla u_n|^{p'}\, dx\, dt$$

$$= \int_{Q_T} (g_k''(b_n(u_n))^{p'} b_n'(u_n)^{p'} |c(x,t)|^{p'} |T_k(u_n)|^{p'\gamma} \nu(x) |\nabla T_{k^*}(u_n)|^{p'}\, dx\, dt$$

$$\leq \|g_k''\|_{L^\infty(\mathbb{R})} \times \sup_{|r|\leq k^*} |b'(r)| \int_{Q_T} |c(x,t)|^{p'} |T_{k^*}(u_n)|^{p'\gamma} \nu(x) |\nabla T_k(u_n)|^{p'}\, dx\, dt \leq c_{k^*}$$

We conclude by (46) that

$$\frac{\partial g_k(b_n(u_n))}{\partial t} \text{ is bounded in } L^1(Q) + L^{p'}(0,T,W^{-1,p'}(\nu^{1-p'})). \tag{50}$$

As mentioned above, from ((48)) and ((50)), we deduce that for a subsequence, still indexed by n, $b_n(u_n)$ converges almost everywhere, as n goes to in infinity, to a measurable function χ defined on Q. Now since b^{-1} is continuous on $\mathbb{R}$, b_n^{-1} converges everywhere to b^{-1} when n goes to in infinity, so that :

$$u_n \to u = b^{-1}(\chi) \text{ a.e. } Q_T, \tag{51}$$

$$b_n(u_n) \to b(u) \text{ a.e. } Q_T, \tag{52}$$

and with the help of((44))

$$T_k(u_n) \rightharpoonup T_k(u) \quad in \quad L^p(0,T,W_0^{1,p}(\nu)) \tag{53}$$

for any $k \geq 0$ as n tends to infinity
Which implies, by using ((15)) , for all $k > 0$ that there exists a function $\sigma_k \in (L^{p'}(\nu^{1-p'}))^N$, such that

$$a(x,t,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup \sigma_k \text{ in } (L^{p'}(\nu^{1-p'}))^N \tag{54}$$

Actually $b(u)$ belongs to $L^\infty((0,T),L^1(\Omega))$. Indeed using $T_k(b_n(u_n))$ as test function in ((33)), by ((30)) we have

$$\int_\Omega B_k^n(u_n)dx + \int_{Q_T} a_n(x,t,u_n,\nabla u_n) \nabla T_k(b_n(u_n))\, dx\, dt \tag{55}$$

$$\leq \int_{Q_T} |c(x,t)||T_n(u_n)|^\gamma \nu(x) |\nabla T_k(b_n(u_n))|\, dx\, dt + k\ (||f_n||_{L^1(Q_T)} + ||b(u_0)||_{L^1(\Omega)}).$$

with $B_k(r) = \displaystyle\int_0^{b(r)} T_k(s)ds$. On the other hand, we have

$$\int_{Q_T} a_n(x,t,u_n,\nabla u_n) \nabla T_k(b_n(u_n))\, dx\, dt \tag{56}$$

$$= \int_{\{|b_n(u_n)|\leq k\}} a_n(x,t,u_n,\nabla u_n) T_k'(b_n(u_n)) b_n'(u_n) \nabla u_n\, dx\, dt \geq 0.$$

Since $b'(s) \geq \beta$, then for $k < n$ and for almost $t \in (0,T)$, we have

$$\int_{Q_T} |c(x,t)||T_n(u_n)|^\gamma \nu(x)|\nabla T_k(b_n(u_n))|dxdt \leq \max_{|s|\leq \frac{k}{\beta}} b'(s)||c(x,t)||_{L^\tau(Q_T,\nu)}$$
$$\times \sup_{t\in(0,T)} \Big(\int_\Omega |T_{\frac{k}{\beta}}(u_n)|^2\,dx\Big)^{\frac{(p-1)(\tilde{p}-p)}{p(3\tilde{p}-p)}} \times ||\nabla T_{\frac{k}{\beta}}(u_n)||_{L^p(Q_T,\nu)}^{\frac{2p\tilde{p}+\tilde{p}-p}{3\tilde{p}-p}} \leq c_k. \tag{57}$$

Using ((35)), ((57)) and ((55)) in ((56)), we have

$$\int_\Omega B_k^n(u_n(t)) \leq c_k + k\Big(||f_n||_{L^1(Q_T)} + ||b(u_0)||_{L^1(\Omega)}\Big)$$

Passing to limit-inf as $n \to +\infty$, we obtain that:

$$\int_\Omega B_k(u(t))\,dx \leq c_k + k\Big(||f_n||_{L^1(Q_T)}) + ||b(u_0)||_{L^1(\Omega)}\Big) \text{ for almost } t\in(0,T).$$

Due to definition of B_k, we have

$$k\int_\Omega |b(u(x,t))|\,dx \leq \int_\Omega B_k(u(t))\,dx + \frac{3}{2}k^2 meas(\Omega)$$
$$\leq k\Big(||f_n||_{L^1(Q_T)}) + ||b(u_0)||_{L^1(\Omega)}\Big) + c_k + \frac{3}{2}k^2 meas(\Omega).$$

shows that $b(u)$ belong to $L^\infty((0,T),L^1(\Omega))$

Lemma 3.1. *The subsequence of u_n defined in Step 1 satisfies*

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{m}\int_{\{|u_n|\leq m\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt = 0. \tag{58}$$

Proof. Using $\psi_m(u_n) = \frac{T_m(u_n)}{m}$ as a test function in ((33)), by ((30)) we get

$$\int_0^T <\frac{\partial b_n(u_n)}{\partial t},\psi_m(u_n)>\,dt + \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla\psi_m(u_n)\,dx\,dt \tag{59}$$
$$\leq \int_{Q_T} c(x,t)|T_n(u_n)|^\gamma\nu(x)|\nabla\psi_m(u_n)|\,dx\,dt + \int_{Q_T} f_n\psi_m(u_n)\,dx\,dt$$

hence

$$\int_\Omega B_m(u_n)(T)dx + \int_{Q_T} a_n(x,t,u_n,\nabla u_n)\nabla\psi_m(u_n)\,dx\,dt$$
$$\leq \int_{Q_T} c(x,t)|T_n(u_n)|^\gamma\nu(x)|\nabla\psi_m(u_n)|\,dx\,dt + \int_\Omega B_m(u_0)_n dx + \int_{Q_T} f_n\psi_m(u_n)\,dx\,dt,$$

where $B_m(r) = \displaystyle\int_0^r b_n'(s)\psi_m(s)ds$. Since $B_m(u_n)(T)\geq 0$, then for every $m<n$, we have

$$a_n(x,t,u_n,\nabla u_n)\nabla\psi_m(u_n) = \frac{1}{m}a(x,t,u_n,\nabla u_n)\nabla u_n \text{ a.e. in } Q$$

As a consequence

$$\frac{1}{m}\int_{\{|u_n|<m\}} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,dt \leq \frac{1}{m}\int_{Q_T} c(x,t)|T_m(u_n)|^\gamma\nu(x)|\nabla T_m(u_n)|\,dx\,dt$$
$$+\int_\Omega B_m(u_{0n})dx + \frac{1}{m}\int_{Q_T} f_n T_m(u_n)\,dx\,dt. \tag{60}$$

Proceeding as in ([11], [20]), using Young inequality and Corollary (2.1) we obtain for all $R < m$:

$$\frac{1}{m}\int_{\{|u_n|<m\}} a(x,t,u_n,\nabla u_n)\nabla u_n \, dx\, dt \tag{61}$$

$$\leq \frac{c_1}{m}\|c(x,t)\chi_{\{|u_n|\geq R\}}\|_{L^\tau(\nu)}\Big(\sup_{t\in(0,T)}\int_\Omega |T_m(u_n)|^2 dx\Big)^{\frac{1}{\tau}}\Big(\int_{Q_T}\nu(x)|\nabla T_m(u_n)|^p dxdt\Big)^{\frac{2p\tilde{p}+\tilde{p}-p}{p(3\tilde{p}-p)}}$$
$$+\frac{1}{m}\int_{\{|u_n|\leq R\}} c(x,t)|T_R(u_n)|^\gamma \nu(x)|\nabla T_R(u_n)| \, dx\, dt$$
$$+\int_\Omega B_m(u_{0n})dx + \frac{1}{m}\int_{Q_T} f_n T_m(u_n)\, dx\, dt.$$

Recalling that u_n is bounded in $L^\infty((0,T);L^1(\Omega))$, we obtain

$$\frac{1}{m}\int_{\{|u_n|<m\}} a(x,t,u_n,\nabla u_n)\nabla u_n \, dx\, dt \tag{62}$$

$$\leq c_2\|c(x,t)\chi_{\{|u_n|\geq R\}}\|^\tau_{L^\tau(\nu)} + \frac{\alpha}{2m}\int_{Q_T}\nu(x)|\nabla T_m(u_n)|^p dxdt$$
$$+\frac{1}{m}\int_{\{|u_n|\leq R\}} c(x,t)|T_R(u_n)|^\gamma \nu(x)|\nabla T_R(u_n)| \, dx\, dt$$
$$+\int_\Omega B_m(u_{0n})dx + \frac{1}{m}\int_{Q_T} f_n T_m(u_n)\, dx\, dt.$$

where c_2 is independent on m and R. Note that $T_m(u_n)$ converges to $T_m(u)$ in $L^\infty(Q_T)$ weak-$*$, and u is finit almost everywhere in Q_T, then $\frac{1}{m}T_m(u)$ converges to zero almost everywhere in Q_T. Using the elliptic condition on a and in view of (62), we deduce that

$$\frac{1}{2m}\int_{\{|u_n|<m\}} a(x,t,u_n,\nabla u_n)\nabla u_n \, dx\, dt \tag{63}$$

$$\leq c_2\|c(x,t)\chi_{\{|u_n|\geq R\}}\|^\tau_{L^\tau(\nu)} + \frac{1}{m}\int_{\{|u_n|\leq R\}} c(x,t)|T_R(u_n)|^\gamma \nu(x)|\nabla T_R(u_n)| \, dx\, dt$$
$$+\int_\Omega B_m(u_{0n})dx + \frac{1}{m}\int_{Q_T} f_n T_m(u_n)\, dx\, dt.$$

Since $T_R(u_n) \in L^p((0,T);W_0^{1,p}(\Omega))$ it follows that

$$\lim_{m\to+\infty}\limsup_{n\to+\infty}\frac{1}{m}\int_{\{|u_n|\leq R\}} c(x,t)|T_R(u_n)|^\gamma \nu(x)|\nabla T_R(u_n)| \, dx\, dt = 0, \forall R > 0. \tag{64}$$

In view of (21), (31), (32), (45), (53), using Lebesgue's convergence theorem and passing to limit in (63) as n tends to $+\infty$, then m tends to $+\infty$ and then R tends to $+\infty$, is an easy task and it allows us to obtain (58)

□

Step 4: In this step we introduce a time regularization of the $T_k(u)$ for $k > 0$ in order to perform the monotonicity method. This kind regularization has been introduced at the first time by R. Landes in [23]. Let v_0^κ be a sequence of function in $L^\infty(\Omega)\cap W_0^{1,p}(\Omega)$ such that $\|v_0^\kappa\|_{L^\infty(\Omega)} \leq k$ for all $\kappa > 0$ and v_0^κ converges to $T_k(u_0)$

a.e. in Ω and $\frac{1}{\kappa}\|v_0^\kappa\|_{L^p(\Omega)}$ converges to 0. For $k \geq 0$ and $\kappa > 0$, let us consider the unique solution $(T_k(u))_\kappa \in L^\infty(Q) \cap L^p(0,T:W_0^{1,p}(\Omega))$ of the monotone problem:

$$\frac{\partial (T_k(u))_\kappa}{\partial t} + \kappa((T_k(u))_\kappa - T_k(u)) = 0 \text{ in } D'(\Omega),$$

$$(T_k(u))_\kappa(t=0) = v_0^\kappa \text{ in } \Omega.$$

Remark that $(T_k(u))_\kappa \to T_k(u)$ a.e. in Q_T, weakly-$*$ in $L^\infty(Q)$ and strongly in $L^p((0,T),W_0^p(\Omega))$ as $\kappa \to +\infty$

$$||(T_k(u))_\kappa||_{L^\infty(Q)} \leq max(||(T_k(u))||_{L^\infty(Q)}, ||v_0^\kappa||_{L^\infty(\Omega)}) \leq k, \ \ \forall \ \kappa > 0 \ \ , \forall \ \ k > 0$$

Lemma 3.2. *Let $k \geq 0$ be fixed. Let S be an increasing $C^\infty(\mathbb{R})-$function such that $S(r) = r$ for $|r| \leq k$, and $suppS'$ is compact. Then*

$$\liminf_{\kappa\to+\infty}\lim_{n\to 0}\int_0^T\int_0^t < \frac{\partial b_n(u_n)}{\partial t}, S'(u_n)(T_k(u_n) - (T_k(u))_\kappa) >\geq 0.$$

where $< .,. >$ denotes the duality pairing between $L^1(\Omega) + W^{-1,p'}(\nu^{1-p'})$ and $L^\infty(\Omega) \cap W^{1,p}(\nu)$.

Proof. see H. Redwane [13] □

Step 5: We prove the following lemma which is the critical point in the development of the monotonicity method.

Lemma 3.3. *The subsequence of u_n satisfies for any $k \geq 0$*

$$\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a(x,t,u_n,\nabla T_k(u_n))\nabla T_k(u_n) \leq \int_0^T\int_0^t\int_\Omega \sigma_k \nabla T_k(u).$$

where σ_k is defined in ((54)).

Proof. Let S_m be a sequence of increasing C^∞-function such that $S_m(r) = r$ for $|r| \leq m$, $supp(S_m') \subset [-2m, 2m]$ and $\|S_m''\|_{L^\infty(\mathbb{R})} \leq \frac{3}{m}$ for any $m \geq 1$. We use the sequence $(T_k(u))_\kappa$ of approximation of $T_k(u)$, and plug the test function $S_m'(u_n)(T_k(u_n) - (T_k(u))_\kappa)$ for $m > 0$ and $\kappa > 0$. For fixed $k \geq 0$, let $W_\kappa^n = T_k(u_n) - (T_k(u))_\kappa$ we obtain upon integration over $(0,t)$ and then over $(0,T)$:

$$\begin{aligned}\int_0^T\int_0^t < \frac{\partial b_n(u_n)}{\partial t}, S_m'(u_n)W_\kappa^n > \,dt\,ds + \int_0^T\int_0^t\int_\Omega a_n(x,t,u_n,\nabla u_n)S_m'(u_n)\nabla W_\kappa^n\,dx\,ds\,dt \\ + \int_0^T\int_0^t\int_\Omega a_n(x,t,u_n,\nabla u_n)S_m''(u_n)\nabla u_n W_\kappa^n\,dx\,ds\,dt \qquad (65)\\ - \int_0^T\int_0^t\int_\Omega \phi_n(x,t,u_n)S_m'(u_n)\nabla W_\kappa^n\,dx\,ds\,dt \\ - \int_0^T\int_0^t\int_\Omega S_m''(u_n)\phi_n(x,t,u_n)\nabla u_n W_\kappa^n\,dx\,ds\,dt = \int_0^T\int_0^t\int_\Omega f_n S_m'(u_n)W_\kappa^n\,dx\,ds\,dt.\end{aligned}$$

Now we pass to the limit in ((65)) as $n \to +\infty$, $\kappa \to +\infty$ and then $m \to +\infty$ for k real number fixed. In order to perform this task we prove below the following results for any fixed $k \geq 0$

$$\liminf_{\kappa\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t < \frac{\partial b_n(u_n)}{\partial t}, W_\kappa^n > \,ds\,dt \geq 0 \qquad \text{for any } m \geq k, \qquad (66)$$

$$\lim_{\kappa\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega \phi_n(x,t,u_n)S'_m(u_n)\nabla W_\kappa^n\,dx\,ds\,dt=0 \quad \text{for any } m\geq 1, \tag{67}$$

$$\lim_{\kappa\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S''_m(u_n)\phi_n(x,t,u_n)\nabla u_n W_\kappa^n\,dx\,ds\,dt=0 \qquad \text{for any } m\geq 1, \tag{68}$$

$$\lim_{m\to+\infty}\limsup_{\kappa\to+\infty}\limsup_{n\to+\infty}\Big|\int_0^T\int_0^t\int_\Omega a_n(x,t,u_n,\nabla u_n)S''_m(u_n)\nabla u_n W_\kappa^n\,dx\,ds\,dt\Big|=0 \tag{69}$$

$$\lim_{\kappa\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega f_n S'_m(u_n)W_\kappa^n \quad dx\,ds\,dt=0. \tag{70}$$

Proof of ((66)): The function S_m belongs $C^\infty(\mathbb{R})$ and is increasing. we have $m\geq k$, $S_m(r)=r$ for $|r|\leq k$ while $suppS'_m$ is compact. In view of the definition of W_κ^n and lemma (3.2) applies with $S=S_m$ for fixed $m\geq k$. As a consequence ((66)) holds true.
Proof of ((67)): Let us recall the main properties of W_κ^n. For fixed $\kappa>0$: W_κ^n converges to $T_k(u)-(T_k(u))_\kappa$ weakly in $L^p(0,T,W_0^{1,p}(\nu))$ as $n\to+\infty$. Remark that

$$||W_\kappa^n||_{L^\infty(Q_T)}\leq 2k \qquad \text{for any } n>0,\ \kappa>0, \tag{71}$$

then we e deduce that

$$W_\kappa^n \rightharpoonup T_k(u)-(T_k(u))_\kappa \quad \text{a.e in } Q_T \text{ and } L^\infty(Q_T) \tag{72}$$

weakly-$*$ when $n\to+\infty$. one has $suppS''_m\subset[-2m,-m]\cup[m,2m]$ for any fixed $m\geq 1$ and $n>2m$.

$$\phi_n(x,t,u_n)S'_m(u_n)\nabla W_\kappa^n=\phi_n(x,t,T_{2m}(u_n))S'_m(u_n)\nabla W_\kappa^n \quad \text{a.e. in } Q_T$$

since $suppS'\subset[-2m,2m]$, on the other hand

$$\phi_n(x,t,T_{2m}(u_n))S'_m(u_n)\to\phi(x,t,T_{2m}(u))S'_m(u) \quad \text{a.e. in } Q_T$$

and

$$|\phi_n(x,t,T_{2m}(u_n))S'_m(u_n)|\leq\nu(x)c(x,t)(2m)^\gamma \quad \text{for } m\geq 1$$

by ((72)) and strongly convergence of $(T_k(u_n))_\kappa$ in $L^p(0,T,W_0^{1,p}(\nu))$ we obtain ((67)).
Proof of ((68)): For any fixed $m\geq 1$ and $n>2m$.

$$\phi_n(x,t,u_n)S''_m(u_n)\nabla u_n W_\kappa^n=\phi_n(x,t,T_{2m}(u_n))S''_m(u_n)\nabla T_{m+1}(u_n)W_\kappa^n \quad \text{a.e. in} Q_T$$

as in the previous step it is possible to pass to the limit for $n\to+\infty$ since by ((71)) and ((72))

$$\phi_n(x,t,T_{2m}(u_n))S''_m(u_n)W_\kappa^n\to\phi(x,t,T_{2m}(u))S''_m(u)W_\kappa \quad \text{a.e. in } Q_T.$$

Since $|\phi(x,t,T_{2m}(u))S''_m(u)W_\kappa|\leq 2k\nu(x)|c(x,t)|(2m)^\gamma$ a.e. in Q_T and $(T_k(u))_\kappa$ converges to 0 in $L^p(0,T;W_0^{1,p}(\nu))$, we obtain ((68)).
Proof of ((69)): In view of the definition of S_m we have $suppS''\subset[-2m,-m]\cup[m,2m]$ for any $m\geq 1$, as a consequence

$$\Big|\int_0^T\int_0^t\int_\Omega a_n(x,t,u_n,\nabla u_n)S''_m(u_n)\nabla u_n W_\kappa^n\,dx\,ds\,dt\Big|$$

$$\leq T\|S''_m(u_n)\|_{L^\infty(\mathbb{R})}\|W^n_\kappa\|_{L^\infty(Q_T)}\int_{m\leq|u_n|\leq 2m} a(x,t,u_n,\nabla u_n)\nabla u_n\,dx\,ds\,dt$$

for any $m \geq 1$, any $n > 2m$ any $\kappa > 0$. By ((58)) it is possible to establish ((69)).

Proof of ((70)): Lebesgue's convergence theorem implies that for any $\kappa > 0$ and any $m \geq 1$

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega f_n S'_m(u)(T_k(u)-(T_k(u))_\kappa)dxdsdt$$
$$=\int_0^T\int_0^t\int_\Omega f S'_m(u)(T_k(u)-(T_k(u))_\kappa).$$

Now for fixed $m \geq 1$, using that $||(T_k(u))_\kappa||_{L^\infty(Q)} \leq max(||(T_k(u))||_{L^\infty(Q)}, ||\upsilon_0^\kappa||_{L^\infty(\Omega)}) \leq k,\ \forall\kappa > 0,\ \forall k > 0$ (see[23]), it possible to pass to the limit as κ tends to $+\infty$ in the above inequality.

Now we turn back to the proof of lemma (3.3). Due to ((66))-((70)) we can to pass to the limit-sup when κ tends to $+\infty$ and to the limit as m tends to $+\infty$ in ((65)). using the definition of W^n_κ we deduce that for any $k \geq 0$

$$\lim_{m\to+\infty}\limsup_{\kappa\to+\infty}\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(x,t,u_n,\nabla u_n)(\nabla T_k(u_n)-\nabla(T_k(u)_\kappa)dxdsdt$$
$$\leq 0.$$

Since $S'_m(u_n)a_n(x,t,u_n,\nabla u_n)\nabla T_k(u_n) = a(x,t,u_n,\nabla u_n)\nabla T_k(u_n)$ for $k \leq n$ and $k \leq m$, using the properties of S'_m the above inequality implies that for $k \leq m$:

$$\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a_n(x,t,u_n,\nabla u_n)(\nabla T_k(u_n))\,dx\,ds\,dt \tag{73}$$

$$\leq \lim_{n\to+\infty}\limsup_{\kappa\to+\infty}\limsup_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(x,t,u_n,\nabla u_n)\nabla(T_k(u)_\kappa\,dx\,ds\,dt$$

On the other hand, for $2m < n$

$$S'_m(u_n)a_n(x,t,u_n,\nabla u_n) = S'_m(u_n)a(x,t,T_{2m}(u_n),\nabla T_{2m}(u_n)) \quad \text{a.e. in } Q_T.$$

Furthermore we have

$$a_n(x,t,u_n,\nabla u_n) \rightharpoonup \sigma_k \quad \text{weakly in } (L^{p'}(Q_T,\nu^{1-p'}))^N \tag{74}$$

it follows that for a fixed $m \geq 1$

$$S'_m(u_n)a_n(x,t,u_n,\nabla u_n) \to S'_m(u_n)\sigma_{m+1} \quad \text{weakly in} \quad (L^{p'}(Q_T,\nu^{1-p'}))^N$$

when n tends to $+\infty$. Finally, using the strong convergence of $(T_k(u)_\kappa)$ to $T_k(u)$ in $L^p(0,T,W_0^{1,p}(\nu))$ as κ tends to $+\infty$, we get

$$\lim_{\kappa\to+\infty}\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega S'_m(u_n)a_n(x,t,u_n,\nabla u_n)\nabla(T_k(u_n)_\kappa\,dx\,ds\,dt \tag{75}$$
$$=\int_0^T\int_0^t\int_\Omega S'_m(u_n)\sigma_{m+1}\nabla T_k(u)\,dx\,ds\,dt$$

as soon as $k \leq m$. Now for $k \leq m$ we have

$$a(x,t,T_{m+1}(u_n),\nabla T_{m+1}(u_n))\chi_{\{|u_n|\leq k\}} = a(x,t,T_k(u_n),\nabla T_k(u_n))\chi_{\{|u_n|\leq k\}} \quad \text{a.e. in } Q_T$$

which implies that, by ((51)), ((74)), and by passing to the limit when n tends to $+\infty$,

$$\sigma_{m+1}\chi_{|u|\leq k} = \sigma_k\chi_{\{|u|\leq k\}} \quad \text{a.e. in } Q_T - \{|u| = k\} \quad \text{for } k \leq m \tag{76}$$

Finally, by ((76)) and ((74)) we have for $k \leq m:$ $\sigma_{m+1}\nabla T_k(u) = \sigma_k \nabla T_k(u)$ a.e. in Q_T. Recalling ((73)), ((75)) the proof of the lemma is complete. □

Step 6: In this step we prove that the weak limit σ_k of $a(x,t,T_k(u_n),\nabla T_k(u_n))$ can be identified with $a(x,t,T_k(u),\nabla T_k(u))$. In order to prove this result we recall the following monotonicity estimates:

Lemma 3.4. *the subsequence of u_n defined in Step 1 satisfies for any $k \geq 0$*

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\Big) \cdot\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big) = 0 \tag{77}$$

Proof. Using ((17)) we have

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega \Big(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\nabla T_k(u))\Big) \cdot\Big(\nabla T_k(u_n) - \nabla T_k(u)\Big) \geq 0. \tag{78}$$

Furthermore, by ((15)), ((51)) we have

$$a(x,t,T_k(u_n),\nabla T_k(u))) \to a(x,t,T_k(u),\nabla T_k(u))) \quad \text{a.e. in } Q_T,$$

and

$$|a(x,t,T_k(u_n),\nabla T_k(u_n)))| \leq \nu(x)[h(x,t)+|T_k(u_n)|^{p-1}+|\nabla T_k(u_n)|^{p-1}] \quad \text{a.e. in } Q_T,$$

uniformly with respect to n. As a consequence

$$a(x,t,T_k(u_n),\nabla T_k(u))) \to a(x,t,T_k(u),\nabla T_k(u))) \text{ strongly in } (L^{p'}(Q_T,\nu^{1-p'}))^N. \tag{79}$$

Finally, using ((51)), ((74)) and ((79)) make it possible to pass to the limit-sup as n tends to $+\infty$ in ((78)) and to obtain the result. □

In this lemma we identify the weak limit σ_k and we prove the weak-L^1 convergence of the "truncated" energy $a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)$ as n tends to $+\infty$.

Lemma 3.5. *For fixed $k \geq 0$, we have*

$$\sigma_k = a(x,t,T_k(u),\nabla T_k(u))) \quad \text{a.e. in } Q_T, \tag{80}$$

and as n tends to $+\infty$

$$a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \rightharpoonup a(x,t,T_k(u),\nabla T_k(u)))\nabla T_k(u) \tag{81}$$

weakly in $L^1(Q_T)$.

Proof. We observe that for any $k > 0$, any $n > k$ and any $\xi \in \mathbb{R}^N$:

$$a_n(x,t,T_k(u_n),\xi) = a(x,t,T_k(u_n),\xi) \quad \text{a.e. in } Q_T.$$

Since

$$T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } L^p((0,T),W_0^p(\nu)), \tag{82}$$

and by ((77)) we obtain

$$\lim_{n\to+\infty}\int_0^T\int_0^t\int_\Omega a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx\,ds\,dt = \int_0^T\int_0^t\int_\Omega \sigma_k\nabla T_k(u)\,dx\,ds\,dt. \tag{83}$$

Since, for fixed $k > 0$, the function $a(x,t,s,\xi)$ is continuous and bounded with respect to s, the usual Minty's argument applies in view of ((82)), ((74)) and ((83)). It follows that ((80)) holds true. In order to prove ((83)), by ((16)), ((77)) and proceeding as in [11, 12] it's easy to show ((81)). $\square$

Taking the limit as n tends to $+\infty$ in ((58)) and using ((81)) show that u satisfies ((25)). Our aim is to prove that u satisfies ((26)) and ((27)). Now we want to prove that u satisfies the equation ((26)).
Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that $suppS' \subset [-k,k]$ where k is a real positive number. Pointwise multiplication of the approximate equation ((33)) by $S'(u_n)$ leads to

$$\frac{\partial B_S^n(u_n)}{\partial t} - div\Big(a_n(x,t,u_n,\nabla u_n)S'(u_n)\Big) + S''(u_n)a(x,t,u_n,\nabla u_n)\nabla u_n \tag{84}$$
$$+ div\Big(\phi_n(x,t,u_n)S'(u_n)\Big) - S''(u_n)\phi_n(x,t,u_n)\nabla u_n = f_nS'(u_n) \quad \text{in } D'(Q_T).$$

In what follows we pass to the limit as n tends to $+\infty$ in each term of ((84)).

Since S is bounded and continuous, u_n converges to u a.e. in Q_T implies that $B_S^n(u_n)$ converge to $B_S(u)$ a.e. in Q_T and $L^\infty(Q_T)$ weak-$*$, Then $\dfrac{\partial B_S^n}{\partial t}$ converges to $\dfrac{\partial B_S}{\partial t}$ in $D'(\Omega)$. We observe that the term $a_n(x,t,u_n,\nabla u_n)S'(u_n)$ can be identified with $a(x,t,T_k(u_n),\nabla T_k(u_n))S'(u_n)$ for $n \geq k$, so using the pointwise convergence of $u_n \to u$ in Q_T, the weakly convergence of $T_k(u_n) \rightharpoonup T_k(u)$ in $L^p((0,T),W_0^p(\nu))$, we get

$$a_n(x,t,u_n,\nabla u_n)S'(u_n) \rightharpoonup a(x,t,T_k(u_n),\nabla T_k(u))S'(u) \quad \text{in } L^{p'}(Q_T,\nu^{1-p'}),$$

and

$$S''(u_n)a_n(x,t,u_n,\nabla u_n)\nabla u_n \rightharpoonup S''(u)a(x,t,T_k(u_n),\nabla T_k(u))\nabla T_k(u) \quad \text{in } L^1(Q_T).$$

Furthermore, since $\phi_n(x,t,u_n)S'(u_n) = \phi_n(x,t,T_k(u_n))S'(u_n)$ a.e. in Q_T. By ((30)) we obtain $|\phi_n(x,t,T_k(u_n))S'(u_n)| \leq \nu(x)|c(x,t)|k^\gamma$, it follows that

$$\phi_n(x,t,T_k(u_n))S'(u_n) \to \phi_n(x,t,T_k(u))S'(u) \quad \text{strongly in } L^{p'}(Q_T,\nu^{1-p'}).$$

In a similar way, it results

$$S''(u_n)\phi_n(x,t,u_n)\nabla u_n = S''(T_k(u_n))\phi_n(x,t,T_k(u_n))\nabla T_k(u_n) \quad \text{a.e. in } Q_T.$$

Using the weakly convergence of $T_k(u_n)$ in $L^p((0,T);W_0^p(\nu))$ it is possible to prove that

$$S''(u_n)\phi_n(x,t,u_n)\nabla u_n \to S''(u)\phi(x,t,u)\nabla u \quad \text{in } L^1(Q_T).$$

Finally by ((31)) we deduce that $f_nS'(u_n)$ converges to $fS'(u)$ in $L^1(Q_T)$.

It remains to prove that $B_S(u)$ satisfies the initial condition $B_S(t=0) = B_S(u_0)$ in Ω. To this end, firstly remark that S being bounded, $B_S^n(u_n)$ is bounded in $L^\infty(Q)$. Secondly the above considerations of the behavior of the terms of this equation show that $\frac{\partial B_S^\epsilon(u_\epsilon)}{\partial t}$ is bounded in $L^1(Q_T) + L^{p'}(0,T;W^{-1,p'}(\nu^{1-p'}))$. As a consequence, an Aubin's type lemma (See e.g [29]) implies that $B_S^n(u_n)$ lies in a compact set of $C^0([0,T],L^1(\Omega))$. On the other hand, the smoothness of of S implies that $B_S(t=0) = B_S(u_0)$ in Ω.

References

[1] L. Aharouch, E. Azroul and M. Rhoudaf: Strongly nonlinear variational degenerated parabolic problems in weighted sobolev spaces. *The Australian journal of Mathematical Analysis and Applications*, Vol.5, Article 13, pp. 1-25, 2008.

[2] Y. Akdim, J. Bennouna, M. Mekkour: Renormalized solutions of nonlinear degenerated parabolic equations with natural growth terms and L^1 data. *International journal of evolution equations*, Volume 5 Number 4, pp. 421-446, 2011.

[3] Y. Akdim, J. Bennouna, M. Mekkour: Solvability of degenerate parabolic equations without sign condition and three unbounded nonlinearities. *Electronic Journal of Differetial Equations*, No. 03, pp.1-25, 2011.

[4] A. Aberqi, J. Bennouna, M. Mekkour, H. Redwane: Renormalized solutions for nonlinear parabolic equation with lower order terms *The Australian journal of Mathematical Analysis and Applications*, Vol.10, Article 15, pp. 1-36, 2013.

[5] Y. Akdim, J. Bennouna, M. Mekkour and H. Redwane: Existence of a Renormalised Solutions for a Class of Nonlinear Degenerated Parabolic Problems with L^1 Data. *Journal of partial differential equations*, Vol. 26, No. 1, pp. 82-104, March 2013.

[6] Y. Akdim, J. Bennouna, M. Mekkour and M. Rhoudaf: Renormalized solution of nonlinear degenerated parabolic problems with L^1-data: existence and uniqness. *Recent developments is nonlinear analysis, Proccedings of the conference in Mathematics and Mathematical Physics world scientific Publishing C.O Ltd.*

[7] Ph. Bénilan, L. Boccardo, Th. Gallouët, R. Gariepy, M.Pierre, J.L. Vázquez: An L^1 theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci* (4) 22 (1995), 241-273.

[8] D. Blanchard: Truncations and monotonicity methods for parabolic equations. *Nonlinear Anal.* 21, pp. 725-43, 1993.

[9] D. Blanchard, G. Francfort: A few results on a class of degenerate parabolic equations. Ann. Scu. Norm. Sup. Pisa, 8(2), 1991, pp. 213-249.

[10] D. Blanchard, F. Murat: Renormalized solutions of nonlinear parabolic with L^1 data: existence and uniqueness. *Proc. Roy. Soc. Edinburgh Sect*, A 127, pp. 1137-1152, 1997.

[11] D. Blanchard, F. Murat, H. Redwane: Existence and uniqueness of renormalized solution for fairly general class of non linear parabolic problems, *J.Differential Equations.*No. 177, 331-374, 2001.

[12] D. Blanchard, H. Redwane: Renormalized solutions for class of nonlinear evolution problems, *J. Math. Pure.* 77, pp. 117-151, 1998.

[13] D. Blanchard, H. Redwane, Renormalized solutions of nonlinear parabolic evolution problems, J. Math. Pure Appl. 77 (1998) 117151

[14] L. Boccardo, D. Giachetti, J.-I. Diaz, F. Murat: Existence and regularity of renormalized solutions of some elliptic problems involving derivatives of nonlinear terms. *Journal of differential equations* 106, pp. 215-237, 1993.

[15] L. Boccardo, T. Gollouët: Nonlinear elliptic and parabolic equations involving measure data, *J. Funct. Anal.* pp. 87, 149-169, 1989.

[16] A. Dall'aglio: Approximated solutions of equations with L^1 data. Application to the H-convergence of quasi-linear parabolic equations. *Ann. Mat. Pura Appl.* (4) 170, 207-240, 1996.

[17] G. Dal Maso, F. Murat, L. Orsina and A. Prignet: Renormalized solutions of elliptic equations with general measure data, *Ann. Scuala Norm. Sup. Pisa Cl.Sci.* (4), 28, 1999.

[18] R. Dautray, J.L. Lions, Mathematical analysis and a numerical methodsfor science and technology, Vol 5. Springer- Verlag (1992).

[19] R.-Di Nardo: Nonlinear parabolic equations with a lower order term, Prepint N.60-2008, Departimento di Matematica e Applicazioni "R.Caccippoli", Universita degli Studi di Napoli "FedericoII".

[20] R. Di Nardo, F. Filomena and O. Guib?: Existence result for nonlinear parabolic equations with lower order terms. Anal. Appl. Singap. 9, no. 2, 161-186, 2011.

[21] R.-J. Diperna, P.-L. Lions: On the Cauchy Problem for the Boltzmann Equations: Global existence and weak stability, *ann. of Math.* 130, pp. 285-366, 1989.

[22] A. Kufner, Weighted Sobolev Spaces, John Wiley and Sons, (1985)

[23] R. Landes: On the existence of weak solutions for quasilinear parabolic initial-boundary problems, *Proc. Roy. Soc. Edinburgh Sect.* A89, pp. 321-366, 1981.
[24] J.-L. Lions: Quelques méthodes de resolution des problémes aux limites non linéaires, Dunod et Gauthier-Villars, Paris, 1969.
[25] M.K.W Murthy, G. Stampacchia: Boundary value problems for some degenerate elliptic operators, *Ann. Mat. Pura Appl.* 80 (1968), 1-122.
[26] J.-M. Rakotoson: Resolution of the critical cases for the problems with L^1 data, Asymptotic Analysis when right hand side is in $L^1(\Omega)$. 6, pp. 285-293, 1993.
[27] H. Redwane: Solution renormalisèes de problèmes paraboliques et elleptique non linéaires. Ph. D. thesis, Rouen (1997).
[28] H. Redwane: Existence of Solution for a class of a parabolic equation with three unbounded nonlinearities, *Adv. Dyn. Syt. A2PL.* 2, pp. 241-264, 2007.
[29] J. Simon: Compact set in the space $L^p(0,T,B)$, *Ann. Mat. Pura, Appl.* 146, pp. 65-96, 1987.
[30] E.Zeidler : Nonlinear Functional Analysis and its Applications, Springer-Verlag(New York-Heidlberg, (1990)).

A. Aberqi
University Sidi Mohammed Ben Abdellah, Faculty of Sciences Dhar El Mahraz Laboratory LAMA, Department of Mathematics P.O. Box 1796 Atlas Fez,Morocco
E-mail address: aberqi_ahmed@yahoo.fr

J. Bennouna
University Sidi Mohammed Ben Abdellah, Faculty of Sciences Dhar El Mahraz Laboratory LAMA, Department of Mathematics P.O. Box 1796 Atlas Fez,Morocco
E-mail address: jbennouna@hotmail.com

M. Hammoumi
University Sidi Mohammed Ben Abdellah, Faculty of Sciences Dhar El Mahraz Laboratory LAMA, Department of Mathematics P.O. Box 1796 Atlas Fez,Morocco
E-mail address: hammoumi.mohamed09@gmail.com