Electronic Journal of Mathematical Analysis and Applications
Vol. 4(2) July 2016, pp. 266-271.
ISSN: 2090-729(online)
http://fcag-egypt.com/Journals/EJMAA/

FIXED POINT THEOREMS ON HILBERT SPACES VIA WEAK EKELAND VARIATIONAL PRINCIPLE

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Abstract. In this paper, we give new fixed point theorems on Hilbert spaces by using weak Ekeland variational principle for potential operators.

1. Introduction

In this paper, we'll generalize the following result proved in [1].

Theorem 1.1. *Let H be a Hilbert space and $A : H \longrightarrow H$ a compact potential operator such that there exists a bounded linear operator B on H and $v^* \in H$ satisfying*

$$\Big(A(su),\, u\Big) \leq \Big(B(su),\, u\Big) + (v^*,\, u),\ \forall\, s \in [0,1],\ \forall\, u \in H \ \text{ with } \ \|B\| < 1.$$

Then, the operator A has a fixed point in H.

We start with the following preliminaries.

Theorem 1.2. *(Weak Ekeland variational principle) ([2], [3]). Let E be a complete metric space and let $\varphi : E \to \mathbb{R}$ a functional that is lower semi-continuous, bounded from below. Then, for each $\varepsilon > 0$, there exists $u_\varepsilon \in E$ with $\varphi(u_\varepsilon) \leq \inf_E \varphi + \varepsilon$ and whenever $v \in E$ with $v \neq u_\varepsilon$, then $\varphi(u_\varepsilon) < \varphi(v) + \varepsilon d(u_\varepsilon, v)$.*

2. Main results

We are now in a position to give our main results.

Theorem 2.1. *Let $T : \overline{U} \to H$ be a compact potential operator, where U is an open convex and bounded subset of a Hilbert space H with $0 \in U$. If there exists a constant $C > 0$ such that*

$$\int_0^1 (T(su), u)\, ds \leq \frac{1}{2}\|u\|^2 - C\|u\| \ \text{ for all } \ u \in \partial U, \tag{2.1}$$

then T has a fixed point in $\overline{U}$.

2010 *Mathematics Subject Classification.* 34B15, 47G40, 47H10, 58E30.
Key words and phrases. Fixed point theorems, Hilbert spaces, weak Ekeland variational principle, Potential operators.
Submitted March 6, 2016.

Proof. Consider the complete metric space $\overline{U}$ endowed with the distance induced by the norm of H and the functional φ defined on $\overline{U}$ by

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_0^1 (T(su), u)\, ds.$$

It is clear that φ is differentiable with $\varphi' = I - T$.

Claim 1: The functional φ is bounded from below.

Indeed, because $\overline{U}$ is a bounded and convex set with $0 \in U$ and the operator T is compact, there exists $M > 0$ such that $\|T(su)\| \leq M$ for all $u \in \overline{U}$ and $s \in (0, 1)$. By using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\|u\|^2 - \int_0^1 (T(su), u)\, ds \\ &\geq \frac{1}{2}\|u\|^2 - \int_0^1 \|T(su)\|\|u\|\, ds \\ &\geq \frac{1}{2}\|u\|^2 - M\|u\|.\end{aligned}$$

Then, φ is bounded from below.

Let $0 < \varepsilon \leq C$. Then, by the weak Ekeland variational principle, there exists $u_\varepsilon \in \overline{U}$ with $\varphi(u_\varepsilon) \leq \inf_{\overline{U}} \varphi + \varepsilon$ and whenever $v \in \overline{U}$ with $v \neq u_\varepsilon$, then $\varphi(u_\varepsilon) < \varphi(v) + \varepsilon\|u_\varepsilon - v\|$.

Claim 2: $u_\varepsilon \notin \partial U$.

Indeed, if $u_\varepsilon \in \partial U$, then, for $v = 0$, we have $\varphi(u_\varepsilon) < \varphi(0) + \varepsilon\|u_\varepsilon - 0\|$. Because $\varphi(0) = 0$, we obtain that $\varphi(u_\varepsilon) < \varepsilon\|u_\varepsilon\| \leq C\|u_\varepsilon\|$. i.e. $\frac{1}{2}\|u_\varepsilon\|^2 - \int_0^1 (T(su_\varepsilon), u_\varepsilon)\, ds < C\|u_\varepsilon\|$ and this is a contradiction with the hypotheses.

Claim 3: u_ε is an approximate fixed point of T.

Indeed, let $t > 0$ and $h \in H$. We put $v_\varepsilon = u_\varepsilon + th$. We remark that because $u_\varepsilon \in U$ and U is open then $u_\varepsilon + th \in U$ for t small enough. We have then

$$\frac{\varphi(u_\varepsilon) - \varphi(u_\varepsilon + th)}{t} \leq \varepsilon\|h\|.$$

By passing to the limit as $t \to 0^+$, we obtain that $- < \varphi'(u_\varepsilon), h > \leq \varepsilon\|h\|$. As $h \in H$ is arbitrary, we obtain $| < \varphi'(u_\varepsilon), h > | \leq \varepsilon\|h\|$ which means that $\|\varphi'(u_\varepsilon)\| \leq \varepsilon$. This means that u_ε is an approximate critical point of φ and then it is an approximate fixed point of T.

Claim 4: Existence of a fixed point.

Indeed, for $\varepsilon = \frac{1}{n}$, we remark that $\frac{1}{n} \leq C$ when $n \to +\infty$. We obtain that $\|\varphi'(u_n)\| \leq \frac{1}{n}$ which means that $\|\varphi'(u_n)\| \longrightarrow 0$ as $n \to +\infty$ and thus we have $\|u_n - T(u_n)\| \longrightarrow 0$ as $n \to +\infty$.

Since the operator T is compact, there exists a subsequence $(u_{n_k}) \subset (u_n)$ such that $T(u_{n_k}) \to w$ with $w \in \overline{U}$. Then $u_{n_k} \to w$. Indeed, we have

$\|u_{n_k} - w\| \leq \|u_{n_k} - T(u_{n_k})\| + \|T(u_{n_k}) - w\| \to 0$, which means that $u_{n_k} \to w$ and then $T(u_{n_k}) \to T(w)$. Thus $T(w) = w$ and w is a fixed point of T. □

As a direct consequence of the above theorem and by using Cauchy-Schwarz inequality, we obtain:

Corollary 2.2. *Let $T : \overline{U} \to H$ be a compact potential operator, where U is an open convex and bounded subset of a Hilbert space H with $0 \in U$. If there exists a* *constant* $C > 0$ *such that*

$$\int_0^1 \|T(su)\|\,ds \leq \frac{1}{2}\|u\| - C \ \textit{ for all } \ u \in \partial U,$$

then T *has a fixed point in* $\overline{U}$.

Corollary 2.3. *Let* $T : H \to H$ *be a compact potential operator. If there exist a bounded linear operator* B *on* H *with* $\|B\| < 1$ *and* $v^* \in H$ *satisfying*

$$(T(su), u) \leq (B(su), u) + (v^*, u) \ \ \forall s \in (0,1), \ \forall u \in \partial B(0,R) \textit{ for some } R > \frac{2\|v^*\|}{1-\|B\|}, \tag{2.2}$$

then T *has a fixed point in* $\overline{B}(0,R)$.

Proof. From the fact that $R > \frac{2\|v^*\|}{1-\|B\|}$, then there exists $C > 0$, such that $R \geq \frac{2(\|v^*\|+C)}{1-\|B\|}$. By using the hypothesis and Cauchy-Schwarz inequality, we obtain that

$$\int_0^1 (T(su), u)\,ds \leq \frac{1}{2}\|B\|\|u\|^2 + \|v^*\|\|u\|.$$

Let the open convex set $U = B(0,R)$. To apply Theorem 2.1, it is sufficient to have

$$\frac{1}{2}\|B\|\|u\|^2 + \|v^*\|\|u\| \leq \frac{1}{2}\|u\|^2 - C\|u\| \quad \forall u \in \partial B(0,R),$$

which is equivalent to

$$\|u\| \geq \frac{2(\|v^*\|+C)}{1-\|B\|} \quad \forall u \in \partial B(0,R),$$

that is

$$R \geq \frac{2(\|v^*\|+C)}{1-\|B\|},$$

and this is true. □

Remark 2.4. We can generalize the above results by replacing the open convex U with a star-shaped open set.

3. Application

Consider the Dirichlet boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{3.1}$$

where $f : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

Lemma 3.1. *If* u *is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,s) f(s, u(s))\,ds,$$

where

$$G(t,s) = \begin{cases} t(1-s), & t \leq s, \\ s(1-t), & s \leq t, \end{cases} \tag{3.2}$$

then u *is a solution of problem (3.1).*

Let T be the operator defined on $H_0^1(0,1)$ by

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))\,ds.$$

Then, T satisfies the problem

$$\begin{cases} -(Tu)''(t) = f(t,u(t)), & t \in [0,1], \\ (Tu)(0) = (Tu)(1) = 0, \end{cases} \tag{3.3}$$

and let φ be the functional defined on $H_0^1(0,1)$ by

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \int_0^1 \Big(\int_0^{u(t)} f(t,s) ds \Big) dt,$$

where $H_0^1(0,1)$ is the standard Sobolev space endowed with the norm $\|u\| = \Big(\int_0^1 u'^2(t) dt \Big)^{\frac{1}{2}}$.

Definition 3.2. We say that $u \in H_0^1(0,1)$ is a weak solution of (3.1) if

$$\int_0^1 \left[u'(t)v'(t) - f(t,u(t))v(t) \right] dt = 0, \quad \text{for all } v \in H_0^1(0,1).$$

Lemma 3.3. *([1]) The operator $T : H_0^1(0,1) \longrightarrow H_0^1(0,1)$ is compact.*

Theorem 3.4. *Assume that the following condition holds:*
(H) *there exist functions $a, b \in L^1([0,1])$ with $\|a\|_\infty = \sup\limits_{0 \le t \le 1} |a(t)| < \pi^2$ and there exists $R > 0$ big enough such that*

$$u[\, f(t,u) - a(t)u - b(t)] \le 0, \quad \text{for all } t \in [0,1] \text{ and all } u \in [0,R).$$

Then problem (3.1) has a solution $u \in C^2[0,1]$.

Proof. Integrating by parts, we obtain for all $u, v \in H_0^1(0,1)$

$$\begin{aligned} \varphi'(u)(v) &= \int_0^1 u'(t)v'(t)\,dt - \int_0^1 f(t,u(t))v(t)\,dt \\ &= \int_0^1 u'(t)v'(t)\,dt + \int_0^1 (Tu)''(t)v(t)\,dt \\ &= \int_0^1 \left(u'(t)v'(t)\,dt - (Tu)'(t)v'(t) \right)\,dt \\ &= (u,v) - (Tu,v) = (u - Tu, v) = ((I-T)u, v). \end{aligned}$$

Thus

$$\varphi' = I - T.$$

Let

$$Bu(t) = \int_0^1 G(t,s)a(s)u(s)\,ds.$$

We prove that the operator T verifies the hypotheses of Corollary 2.3.
Step 1: $(T(su),u) \le (B(su),u) + (v^*,u)$, $\forall s \in (0,1)$ for all $u \in \partial B(0,R)$ for some $R > \frac{2\|v^*\|}{1-\|B\|}$,
where

$$v^*(t) = \int_0^1 G(t,s)b(s)\,ds.$$

By Hypothesis (H), we have

$$\begin{aligned}\Big(Bv - Tv + v^*, u\Big) &= \int_0^1 (Bv - Tv + v^*)'(t)u'(t)\,dt\\ &= \int_0^1 \left(-(Bv)''(t) + (Tv)''(t) - (v^*)''(t)\right)u(t)\,dt\\ &= \int_0^1 \left(a(t)v(t) - f(t, v(t)) + b(t)\right)u(t)\,dt \geq 0.\end{aligned}$$

Then for $v = su$, we have $(T(su), u) \leq (B(su), u) + (v^*, u)$.
Also, we have

$$\begin{aligned}\|Bu\| &= \sup_{\|v\|\leq 1} |\langle Bu, v\rangle|\\ &= \sup_{\|v\|\leq 1} \left|(Bu, v)_{H_0^1}\right|\\ &= \sup_{\|v\|\leq 1} \left|\int_0^1 (Bu)'(t)v'(t)\,dt\right|\\ &= \sup_{\|v\|\leq 1} \left|\int_0^1 -(Bu)''v(t)\,dt\right|\\ &= \sup_{\|v\|\leq 1} \left|\int_0^1 a(t)u(t)v(t)\,dt\right|\\ &\leq \|a\|_\infty \sup_{\|v\|\leq 1} \int_0^1 |u(t)v(t)|\,dt\\ &\leq \|a\|_\infty \sup_{\|v\|\leq 1} \|u\|_{L^2}\|v\|_{L^2}\\ &\leq \|a\|_\infty \|u\|_{L^2} \sup_{\|v\|\leq 1} \|v\|_{L^2}\\ &\leq \|a\|_\infty \frac{1}{\sqrt{\lambda_1}}\|u\| \sup_{\|v\|\leq 1} \frac{1}{\sqrt{\lambda_1}}\|v\|\\ &\leq \frac{1}{\lambda_1}\|a\|_\infty\|u\|.\end{aligned}$$

Since B is a linear operator, i.e., $\|Bu\| \leq \|B\|\|u\|$, we get

$$\|B\| \leq \frac{\|a\|_\infty}{\lambda_1} = \frac{\|a\|_\infty}{\pi^2} < 1.$$

Here $\lambda_1 = \pi^2$ is the first eigenvalue of the problem

$$\begin{cases} -u''(t) = \lambda u(t), & t \in (0, 1),\\ u(0) = u(1) = 0.\end{cases}$$

Therefore, by Corollary 2.3, the operator T has a fixed point u, which is a weak solution of problem (3.1). Since $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, then $u \in C^2[0, 1]$. $\square$

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