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ON THE DYNAMICS OF A RECURSIVE SEQUENCE

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Abstract. In this paper we investigate the global asymptotic behavior of the solutions of the difference equation

$$x_{n+1}=\frac{\alpha x_{n-(2k-1)}}{\beta+\gamma x^{p}_{n-(2k+2)}x^{q}_{n-2k}x^{t}_{n-(2k-2)}},\ n=0,1,...,$$

where k is a positive integer number, the parameters α,β,γ and the initial conditions $x_{-2k-2},\ x_{-2k-1},\ ...,\ x_{-1},\ x_0$ are non-negative real numbers with $p,q,t\geqslant 1$ and $\beta+\gamma x^{p}_{n-(2k+2)}x^{q}_{n-2k}x^{t}_{n-(2k-2)}>0$.

1. Introduction

Recently there has been an increasing interest in the study of global behavior of rational difference equations. That is because difference equations appear naturally as discrete analogues and as numerical solutions of differential and delay differential equations having applications in biology, ecology, physics, etc. See [10]. Rational difference equations is an important class of difference equations where they have many applicatio ns in real life for example the difference equation $x_{n+1}=\frac{a+bx_n}{c+x_n}$ which is known by Riccati Difference Equation has an applications in Optics and Mathematical Biology (see [8]). For more results of the investigation of rational difference equation see ([1]-[10]) and the references therein

In this paper we investigate the global asymptotic behavior of the solutions of the difference equation

$$x_{n+1}=\frac{\alpha x_{n-(2k-1)}}{\beta+\gamma x^{p}_{n-(2k+2)}x^{q}_{n-2k}x^{t}_{n-(2k-2)}},\ n=0,1,... \tag{1}$$

where k is a positive integer number, the parameters α,β,γ and the initial conditions $x_{-2k-2},\ x_{-2k-1},...,\ x_{-1},\ x_0$ are non-negative real numbers with $p,q,t\geqslant 1$ and $\beta+\gamma x^{p}_{n-(2k+2)}x^{q}_{n-2k}x^{t}_{n-(2k-2)}>0$.

Remark
(i) If $\alpha=0$, then Eq.(1) is trivial.
(ii) If $\gamma=0$, then Eq.(1) is linear.

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(iii) If $\beta = 0$, then Eq.(1) can be also reduced to a linear difference equation by the change of variables $x_n = e^{y_n}$.

The following definitions and the well known theorem A will be useful in the investigation of Eq.(1).

Definition 1 Let I be an interval of real numbers and let $f : I^{k+1} \to I$ be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f\left(x_n, x_{n-1}, ..., x_{n-k}\right), \quad n = 0, 1, ..., \tag{2}$$

with $x_{-k}, ..., x_0 \in I$. Let $\overline{x}$ be the equilibrium point of Eq.(2).
The linearized equation of Eq.(2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1+} ... + c_{(k+1)} y_{n-k}, \quad n = 0, 1, ..., \tag{3}$$

where

$$c_1 = \frac{\partial f}{\partial x_n}(\overline{x}, ..., \overline{x}), c_2 = \frac{\partial f}{\partial x_{n-1}}(\overline{x}, ..., \overline{x}), ..., c_{(k+1)} = \frac{\partial f}{\partial x_{n-k}}(\overline{x}, ..., \overline{x})$$

The characteristic equation of Eq.(3) is

$$\lambda^{(k+1)} - c_1 \lambda^k - c_2 \lambda^{(k-1)} - ... - c_k \lambda - c_{(k+1)} = 0 \tag{4}$$

Definition 2 Let $\overline{x}$ be an equilibrium point of Eq.(2).

(a) The equilibrium $\overline{x}$ is called *locally stable* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_0, \ldots, x_{-k} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-k} - \overline{x}| < \delta$, then $|x_n - \overline{x}| < \varepsilon$, for all $n \geq -k$.

(b) The equilibrium $\overline{x}$ is called *locally asymptotically stable* if it is locally stable and if there exists $\gamma > 0$ such that if $x_0, \ldots, x_{-k} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-k} - \overline{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \overline{x}$.

(c) The equilibrium $\overline{x}$ is called *global attractor* if for every $x_0, \ldots, x_{-k} \in I$ we have $\lim_{n\to\infty} x_n = \overline{x}$.

(d) The equilibrium $\overline{x}$ is called *globally asymptotically stable* if it is locally stable and is a global attractor.

Definition 3 A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(2) is called nonoscillatory if there exists $N \geq -k$ such that either

$$x_n > \overline{x} \quad for \quad \forall n \geq N \quad or \quad x_n < \overline{x} \quad for \quad \forall n \geq N,$$

and it is called oscillatory if it is not nonoscillatory.

Theorem A [6]**:** (i) If all roots of Eq.(4) have absolute values less than one, then the equilibrium point $\overline{x}$ of Eq.(2) is locally asymptotically stable.
(ii) If at least one of the roots of Eq.(4) has absolute value greater than one, then the equilibrium point $\overline{x}$ of Eq.(2) is unstable.

2. Main Results

In this section, we investigate the dynamics of Eq.(1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = (\frac{\beta}{\gamma})^{\frac{1}{p+q+t}} y_n$ reduces Eq.(1) to the difference equation

$$y_{n+1} = \frac{r y_{n-(2k-1)}}{1 + y^p_{n-(2k+2)} y^q_{n-2k} y^t_{n-(2k-2)}} \quad \text{for } n = 0, 1, ..., \tag{5}$$

where $r = \frac{\alpha}{\beta} > 0$. Note that $\overline{y}_1 = 0$ is always an equilibrium point of Eq.(5). When $r > 1$, Eq.(5) also possesses the unique positive equilibrium $\overline{y}_2 = (r-1)^{\frac{1}{p+q+t}}$.

Lemma The following statements are true:

(i) If $r < 1$, then the equilibrium point $\overline{y}_1 = 0$ of Eq.(5) is locally asymptotically stable,

(ii) If $r > 1$, then the equilibrium point $\overline{y}_1 = 0$ of Eq.(5) is unstable,

(iii) If $r > 1$, then the equilibrium point $\overline{y}_2 = (r-1)^{\frac{1}{p+q+t}}$ of Eq.(5) is unstable.

Proof. The linearized equation of Eq.(5) about the equilibrium point $\overline{y}_1 = 0$ is

$$z_{n+1} = r z_{n-(2k-1)} \quad \text{for } n = 0, 1, ...,$$

so the associated characteristic equation about $\overline{y}_1$ is

$$\lambda^3(\lambda^{2k} - r) = 0.$$

Then the proof of (i) and (ii) follows by Theorem A. Now the linearized equation of Eq.(5) about the equilibrium point $\overline{y}_2 = (r-1)^{\frac{1}{p+q+t}}$ is

$$z_{n+1} = z_{n-(2k-1)} - \frac{p(r-1)}{r} z_{n-(2k+2)} - \frac{q(r-1)}{r} z_{n-2k} - \frac{t(r-1)}{r} z_{n-(2k-2)}$$

for $n = 0, 1, ...$ and the associated characteristic equation about $\overline{y}_2$ is

$$F(\lambda) = \lambda^{2k+3} - \lambda^3 + \frac{p(r-1)}{r} + \frac{q(r-1)}{r}\lambda^2 + \frac{t(r-1)}{r}\lambda^4 = 0.$$

Since $F(-1) > 0$ and $\lim_{\lambda \to -\infty} F(\lambda) = -\infty$, then there is $\lambda < -1$ such that $F(\lambda) = 0$. Consequently, $\overline{y}_2$ is unstable. This completes the proof.

Theorem 1 Assume that $r > 1$ and let $\{y_n\}_{n=-(2k+2)}^{\infty}$ be a solution of Eq.(5) such that

$$y_{-(2k+2)}, y_{-2k} ..., ..., y_0 \geq \overline{y}_2 \quad \text{and } y_{-(2k+1)}, y_{-(2k-1)}, ..., y_{-1} < \overline{y}_2, \tag{6}$$

or

$$y_{-(2k+2)}, y_{-2k} ..., y_{-2m}, ..., y_0 < \overline{y}_2 \quad \text{and} \quad y_{-(2k+1)}, y_{-(2k-1)}, ..., y_{-1} \geq \overline{y}_2. \tag{7}$$

Then, $\{y_n\}_{n=-(2k+2)}^{\infty}$ oscillates about $\overline{y}_2 = (r-1)^{\frac{1}{p+q+t}}$ with semicycles of length one.

Proof. Assume that (6) holds. Then,

$$y_1 = \frac{r y_{-(2k-1)}}{1 + y^p_{-(2k+2)} y^q_{-2k} y^t_{-(2k-2)}} < \overline{y}_2,$$

$$y_2 = \frac{r y_{1-(2k-1)}}{1 + y^p_{1-(2k+2)} y^q_{1-2k} y^t_{1-(2k-2)}} > \overline{y}_2$$

and

$$y_3 = \frac{r y_{2-(2k-1)}}{1 + y^p_{2-(2k+2)} y^q_{2-2k} y^t_{2-(2k-2)}} < \overline{y}_2.$$

The proof follows by induction. The case where (7) holds is similar and will be omitted. This completes the proof.

Theorem 2 Assume that $r < 1$, then the equilibrium point $\overline{y}_1 = 0$ of Eq.(5) is globally asymptotically stable.

Proof. By Lemma 1, the equilibrium point $\overline{y_1} = 0$ of Eq.(5) is locally asymptotically stable, when $r < 1$. Let $\{y_n\}_{n=-(2k+2)}^{\infty}$ be a solution of Eq.(5). We have $y_{n+1} \leq r y_{n-(2k-1)}$. This implies that $y_{2nk+m} \leq r^{n+1} y_{(m-1)-(2k-1)}$, $n = 1, 2, ...,$ $m = 1, 2, ..., 2k$. Hence $\lim_{n\to\infty} y_n = 0$. This completes the proof.

As a confirmation of the result obtained in Theorem 2 above, we present the following figures:

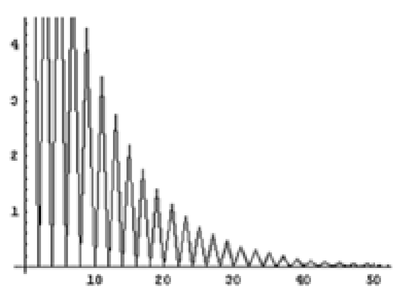

Fig 1: k =1 and r = 0.8

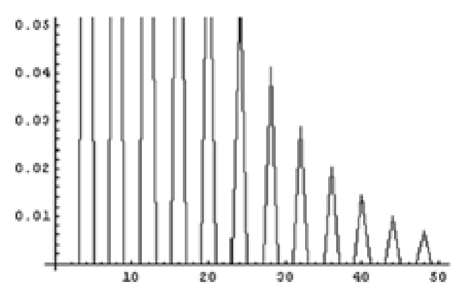

Fig 2: k = 2 and r = 0.7

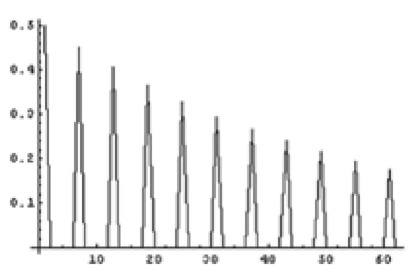

Fig.3: k =3 and r = 0.9

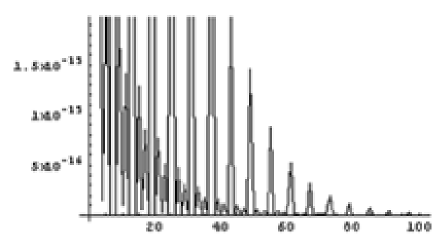

Fig.4: k =4 and r = 0.6

Theorem 3 Assume that $r > 1$, then Eq.(5) possesses an unbounded solution $\{y_n\}_{n=-(2k+2)}^{\infty}$ such that $\{y_{2nm+i}\}_n$ increases to ∞ whenever i is odd and decreases to 0 whenever i is even, where $i = 0, 1, ..., 2m - 1$.

Proof. From Theorem 2, we may choose solution $\{y_n\}_{n=-(2k+2)}^{\infty}$ of Eq.(5) such that $y_{2n-1} < \overline{y}_2$ and $y_{2n} > \overline{y}_2$ for each n. We can see that the sequence $\{y_{2nk}\}_n$

increases to $l_0 > \overline{y}_2$, where we allow for l_0 to be infinity. Indeed, we have

$$y_{2(n+1)k} = \frac{r y_{2nk}}{1 + y_{2nk-3}^{p} y_{2nk-1}^{q} y_{2nk+1}^{t}} > y_{2nk} > \overline{y}_2. \tag{8}$$

So it is easy to see that the sequences

$$\left\{y_{2nk-3}\right\}_n, \ \left\{y_{2nk-1}\right\}_n \ \text{ and } \ \left\{y_{2nk+1}\right\}_n$$

decrease to three non-negative numbers, say a_1, b_1 and c_1 respectively which are less than $\overline{y}_2$. Assume as the sake of contradiction that $l_0 < \infty$. Then taking the limit as $n \to \infty$ in Eq.(8), to obtain $1 = r \ / \ (1 + a_1^p b_1^q c_1^t)$.Hence $r - 1 = a_1^p b_1^q c_1^t < \overline{y}_2^{p+q+t}$ which is a contradiction. We conclude that $l_0 = \infty$. Now the sequence $\left\{y_{2nk+1}\right\}_n$ is decreasing to a non-negative number l_1. Indeed, we have

$$y_{2(n+1)k+1} = \frac{r y_{2nk+1}}{1 + y_{2nk-2}^{p} y_{2nk}^{q} y_{2nk+2}^{t}} < y_{2nk+1} < \overline{y}_2. \tag{9}$$

Similarly, $\left\{y_{2nk+i}\right\}_n$ is increasing (decreasing) to $l_i < \overline{y_2}$ ($l_i > \overline{y_2}$), i is even (i is odd). As before we can see that the sequences

$$\left\{y_{2nk-2}\right\}_n, \left\{y_{2nk}\right\}_n \ \text{ and } \ \left\{y_{2nk+2)}\right\}_n$$

increase to three elements a_2, b_2 and c_2 respectively which are greater than $\overline{y}_2$. Assume that $l_1 \neq 0$. Taking the limit as $n \to \infty$ in Eq.(9), we get $r - 1 = a_2^p b_2^q c_2^t > \overline{y}_2^{p+q+t}$ which is a contradiction. This implies that $l_1 = 0$. Similarly, we can show that $l_i = 0, i = 3, 5, ..., 2m-1$ and $l_i = \infty, i = 2, 4, ..., 2m-2$.This completes the proof.

Again as a confirmation of the result obtained in Theorem 3 above, we present the following figures:

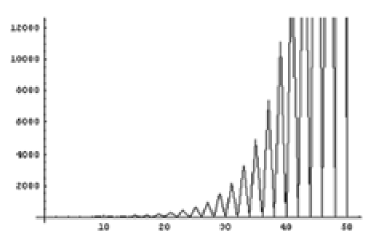

Fig.5: k =1 and r = 1.5

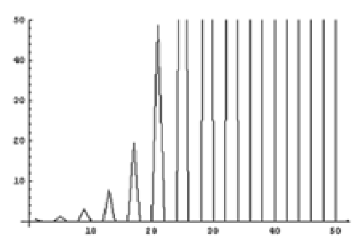

Fig.6: k =2 and r = 2.5

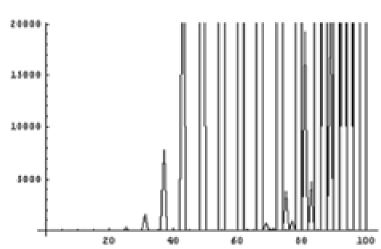

Fig.7: k =3 and r = 5

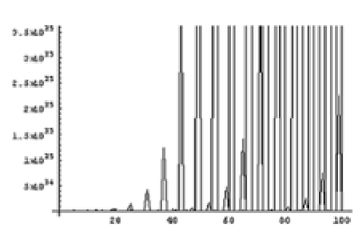

Fig.8: k =4 and r = 3

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