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EXISTENCE OF SOLUTIONS TO NONLINEAR SINGULAR DIFFERENTIAL EQUATIONS ARISING IN THE THEORY OF POWER LAW FLUIDS

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Abstract. We consider the nonlinear singular boundary layer problem

$$u''(t) = -\frac{t}{u^p(t)}$$

subject to the conditions $u'(h) = b, u(1) = 0$, which describes the steady laminar boundary layer flow of power law fluids on a moving flat plate. This paper establishes the new existence results for the boundary value problem for the case: $p > 0$, $0 \leq h < 1$, $b \in R$.

1. Introduction

The laminar boundary layer flow has been extensively investigated since the early 20th century. In 1908, Blasius used similarity transformations to transform the steady boundary layer flow of Newtonian fluids past a flat plate into the famous Blasius equation. Applying the Crocco variable transformation to the classical Blasius equation, Crocco obtained nonlinear singular second order differential equation. Since the non-Newtonian fluid has a wide range of application in industry, it has been considered by many researchers. With the help of the techniques developed by Blasius and Crocco, the steady boundary layer flows for power law fluids past a moving porous plate with suction or injection are transformed into nonlinear singular ordinary differential equations of the form

$$u''(t) = -\frac{t}{u^p(t)} \tag{1}$$

with the conditions

$$u'(h) = b, u(1) = 0, \tag{2}$$

where h means the ratio of the velocity of the flat plate to that of the uniform flow and b is the suction or injection parameter(see [2]). If $h > 0$, then the uniform flow and the plate move in the same direction. $h < 0$ implies that their moving directions are opposite. The case $b > 0$ means that there is injection of the fluid into

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the boundary layer and $b < 0$ implies the suction of the fluid from the boundary layer. If $p = 1$, then the equation (1)-(2) corresponds to a Newtonian fluid. If $p > 1$, then it describes the motion of a pseudoplastic fluid. Dilatant fluids have $0 < p < 1$.

The equation (1)-(2) has been investigated by many mathematicians. Callegari et al. [1] used the standardization technique to prove the existence of uniqueness solution of (1)-(2) for the case $p = 1, h = 0,\ b = 0$ and obtained an analytical solution. Callegari et al. [2] established uniqueness and analyticity of solutions in the case $p = 1, h \geq 0,\ b = 0$. Vajravelu et al. [11] obtained the existence, uniqueness and analyticity results for the case $p = 1,\ 0 \leq h < 1,\ b \in R$. Nachman et al. [7, 8] studied the case $p > 1,\ h = 0,\ b = 0$ and the case $p > 0,\ h = 0,$ $b < 0$. Zheng et al. [14] proved existence and uniqueness of solutions for the case $0 < p < 1, h = 0,\ b = 0$ and obtained an estimate for the skin friction coefficient. In [3, 4, 5, 10, 9, 12], the authors established sufficient conditions for existence and nonuniqueness of solutions for the case $p = 1,\ h < 0,\ b \in R$. Zheng et al. [13] used the shooting technique to obtain sufficient conditions for the existence of bifurcation solutions for the case $p \geq 1,\ h < 0,\ b \in R$. Lu [6] proved new results for the existence of solutions for the case $p \geq 1,\ h < 0,\ b \in R$.

In spite of efforts of many mathematicians, this existence problem has not yet been completely solved. In the present paper, we consider the case $p > 0, 0 \leq h < 1$, $b \in R$ which is more general than the previous works [1, 7, 8, 11, 14].

2. Existence and uniqueness of solutions of boundary layer problems for power law fluids

In this section we discuss the existence and uniqueness of solutions of the boundary value problem (1)-(2). In the paper the equation (1)-(2) is investigated by considering the singular ordinary differential equation (1) subject to initial conditions

$$u(h) = a, u'(h) = b \tag{3}$$

where $0 \leq h < 1,\ a > 0$ and $b \in R$. By Peano Theorem, the initial value problem (1)-(3) has at least one local solution $u_{h,a,b,p}(t)$. The solution $u_{h,a,b,p}(t)$ can be continuously extended to a maximal interval of existence $[h, T_{h,a,b,p})$ where $T_{h,a,b,p} \in R \cup \{\infty\}$.

Lemma 1 Let $0 \leq h < 1$,$a > 0$,$b \in R$ and $p > 0$. Then $u'_{h,a,b,p}(t)$ is monotone decreasing in $[h, T_{h,a,b,p})$. If $T_{h,a,b,p} = \infty$, then $u'_{h,a,b,p}(t) > 0$ for $t > h$.

Proof. Since $u_{h,a,b,p}(t)$ is initially positive, it is clear that $u_{h,a,b,p}(t) > 0$ for any $t \in [h, T_{h,a,b,p})$. Integrating both sides of (1) over $[h, t]$, we obtain

$$u'_{h,a,b,p}(t) = b - \int_h^t \frac{s}{u^p_{h,a,b,p}(s)} ds. \tag{4}$$

By (4), $u'_{h,a,b,p}(t)$ is monotone decreasing in $[h, T_{h,a,b,p})$. By integrating both sides of (4) over $[h, t]$, we have

$$u_{h,a,b,p}(t) = a + b(t - h) - \int_h^t \frac{(t - s)s}{u^p_{h,a,b,p}(s)} ds. \tag{5}$$

Assume that $T_{h,a,b,p} = \infty$ and there exists $t_0 > 0$ such that $u'_{h,a,b,p}(t_0) < 0$. Then $u'_{h,a,b,p}(t) < u'_{h,a,b,p}(t_0) < 0$ for $t > t_0$ and, by (4), we have that for $t > t_0$,

$$b - u'_{h,a,b,p}(t_0) < \int_h^t \frac{s}{u^p_{h,a,b,p}(s)} ds. \tag{6}$$

Integrating both sides of (6) over $[h, t]$, we have that for $t > t_0$,

$$(b - u'_{h,a,b,p}(t_0))(t - h) < \int_h^t \frac{(t - s)s}{u^p_{h,a,b,p}(s)} ds. \tag{7}$$

By (5) and (7), we have that for $t > t_0$,

$$u_{h,a,b,p}(t) < a + b(t - h) - (b - u'_{h,a,b,p}(t_0))(t - h) = a + u'_{h,a,b,p}(t_0)(t - h).$$

Thus $\lim_{t\to\infty} u_{h,a,b,p}(t) = -\infty$. This contradicts $u_{h,a,b,p}(t) > 0$ for any $t \geq h$.

Lemma 2 Let $0 \leq h < 1$, $a > 0$, $b \in R$ and $p > 0$. If $T_{h,a,b,p} < \infty$, then $u_{h,a,b,p}(T_{h,a,b,p}) = 0$.

Proof. From the fact that $u'_{h,a,b,p}(t)$ is monotone decreasing in $[h, T_{h,a,b,p})$, the sign of $u'_{h,a,b,p}(t)$ is not changed in $[T_{h,a,b,p} - \epsilon, T_{h,a,b,p})$ for a sufficiently small positive number ϵ. Therefore $\lim_{t\to T_{h,a,b,p}} u_{h,a,b,p}(t)$ exists. Let $\lim_{t\to T_{h,a,b,p}} u_{h,a,b,p}(t) = u_{h,a,b,p}(T_{h,a,b,p}) = d$. If $d > 0$, then, by Peano Theorem, there exists a continuous solution in some neighbourhood of $T_{h,a,b,p}$. Thus $u_{h,a,b,p}(T_{h,a,b,p}) = 0$.

Lemma 3 If $b < 0$, then $T_{h,a,b,p} < \infty$ for $0 \leq h < 1$, $a > 0$, $p > 0$.
If $0 < p \leq 2$, then $T_{h,a,b,p} < \infty$ for $0 \leq h < 1$, $a > 0$, $b \in R$. If $p > 2$, then for any $0 \leq h < 1$ and $b \in R$, there exists $a > 0$ such that $T_{h,a,b,p} < \infty$.

Proof. If $b \leq 0$, then, by (4), $u'_{h,a,b,p}(t) < 0$ for any $t > h$ and by Lemma 1, $T_{h,a,b,p} < \infty$. Now we consider the case $b > 0$. In order to use the method of proof by contradiction to prove the second assertion of the lemma, we assume that there exists $0 < p \leq 2$ such that $T_{h,a,b,p} = \infty$. Then, by Lemma 1, $u'_{h,a,b,p}(t) > 0$ for any $t > h$. By (5), $u_{h,a,b,p}(t) < b(t - h) + a$.
If $p = 1$, then we have

$$\lim_{t\to\infty} u'_{h,a,b,1}(t) < b - \lim_{t\to\infty} \int_h^t \frac{s}{b(s - h) + a} ds = b - \lim_{t\to\infty} \Bigg[\frac{t - h}{b} + \frac{1}{b}\Big(h - \frac{a}{b}\Big) ln\Big(t - h + \frac{a}{b}\Big) - \frac{1}{b}\Big(h - \frac{a}{b}\Big) ln\frac{a}{b}\Bigg] = -\infty.$$

If $p \in (0, 1) \cup (1, 2)$, then we have

$$\lim_{t\to\infty} u'_{h,a,b,p}(t) < b - \lim_{t\to\infty} \int_h^t \frac{s}{(b(s - h) + a)^p} ds = b - \lim_{t\to\infty} \Bigg[\frac{1}{b^p(2 - p)}\Bigg(\Big(t - h + \frac{a}{b}\Big)^{2-p} - \Big(\frac{a}{b}\Big)^{2-p}\Bigg) + \frac{1}{b^p(1 - p)}\Big(h - \frac{a}{b}\Big)\Bigg(\Big(t - h + \frac{a}{b}\Big)^{1-p} - \Big(\frac{a}{b}\Big)^{1-p}\Bigg)\Bigg] = -\infty.$$

If $p = 2$, then we have

$$\lim_{t\to\infty} u'_{h,a,b,2}(t) < b - \lim_{t\to\infty} \int_h^t \frac{s}{(b(s - h) + a)^2} ds = b - \frac{1}{b^2} \lim_{t\to\infty} \Bigg[ln\Big(t - h + \frac{a}{b}\Big) - ln\frac{a}{b} + \Big(\frac{a}{b} - h\Big)\Bigg(\Big(t - h + \frac{a}{b}\Big)^{-1} - \frac{b}{a}\Bigg)\Bigg] = -\infty.$$

The above inequalities contradict $\lim_{t\to\infty} u'_{h,a,b,p}(t) > 0$ for $t > h$. Thus $T_{h,a,b,p} < \infty$ for any $0 < p \le 2$. Now we will prove the last assertion of the lemma. Assume that there exists $p \in (2,\infty)$ such that $T_{h,a,b,p} = \infty$ for any $a > 0$. Then, by Lemma 1, $u'_{h,a,b,p}(t) > 0$ for any $a > 0$ and $t > h$. We have

$$\lim_{a\to 0} u'_{h,a,b,p}(t) < b - \lim_{a\to 0}\int_h^t \frac{s}{(b(s-h)+a)^p}ds = b - \frac{1}{b^p(2-p)}\lim_{a\to 0}\left[\left(t-h+\frac{a}{b}\right)^{2-p} - \left(\frac{a}{b}\right)^{2-p}\right] + \lim_{a\to 0}\left[\frac{1}{b^p(p-1)}\left(h-\frac{a}{b}\right)\left(\left(t-h+\frac{a}{b}\right)^{1-p} - \left(\frac{a}{b}\right)^{1-p}\right)\right] = b - \lim_{a\to 0}\left(\frac{a}{b}\right)^{2-p}\left[\frac{1}{b^p(p-2)} - \frac{1}{b^p(p-1)}\right] - \lim_{a\to 0}\frac{ha^{1-p}}{b(p-1)} + \frac{h(t-h)^{1-p}}{b^p(p-1)} - \frac{(t-h)^{2-p}}{b^p(2-p)} = -\infty.$$

This contradiction implies that for any $p > 2$, there exists $a > 0$ such that $T_{h,a,b,p} < \infty$.

Remark 1 In the case $p > 2$, it is difficult to prove whether $T_{h,a,b,p} < \infty$ for any $a > 0$. Thus we need to improve the previous techniques.

For $0 \le h < 1$, $a > 0$, $b > 0$ and $p > 0$, if $T_{h,a,b,p} < \infty$, then there exists $E_{h,a,b,p} > h$ such that $u_{h,a,b,p}(E_{h,a,b,p}) = a$. If $T_{h,a,b,p} = \infty$, then we set $E_{h,a,b,p} = \infty$.

Lemma 4 Let $0 \le h_0 < 1$, $a > 0$, $b > 0$, $p > 0$ and $T_{h_0,a,b,p} < \infty$. Then $E_{h,a,b,p}$ is continuous at h_0 with respect to h.

Proof. We have that for any $t \in [max\{h, h_0\}, min\{E_{h,a,b,p}, E_{h_0,a,b,p}\}]$,

$$|u_{h_0,a,b,p}(t) - u_{h,a,b,p}(t)| \le |b(h-h_0)| + \int_h^t \frac{|u^p_{h_0,a,b,p}(s) - u^p_{h,a,b,p}(s)|}{u^p_{h,a,b,p}(s)u^p_{h_0,a,b,p}(s)}(t-s)sds + \left|\int_{h_0}^h \frac{(t-s)s}{u^p_{h_0,a,b,p}(s)}ds\right| = |b(h-h_0)| + \int_h^t \frac{1}{u^p_{h,a,b,p}(s)}\left|1 - \frac{u^p_{h,a,b,p}(s)}{u^p_{h_0,a,b,p}(s)}\right|(t-s)sds + \left|\int_{h_0}^h \frac{(t-s)s}{u^p_{h_0,a,b,p}(s)}ds\right| \le |b(h-h_0)| + \int_h^t \frac{1}{u^p_{h,a,b,p}(s)}\left|1 - \frac{u^{\lceil p\rceil}_{h,a,b,p}(s)}{u^{\lceil p\rceil}_{h_0,a,b,p}(s)}\right|(t-s)sds + \left|\int_{h_0}^h \frac{(t-s)s}{u^p_{h_0,a,b,p}(s)}ds\right| = |b(h-h_0)| + \int_h^t \frac{|u^{\lceil p\rceil-1}_{h_0,a,b,p}(s) + \cdot + u^{\lceil p\rceil-1}_{h,a,b,p}(s)|}{u^p_{h,a,b,p}(s)u^{\lceil p\rceil}_{h_0,a,b,p}(s)} |u_{h_0,a,b,p}(s) - u_{h,a,b,p}(s)|(t-s)sds + \left|\int_{h_0}^h \frac{(t-s)s}{u^p_{h_0,a,b,p}(s)}ds\right| \le |b(h-h_0)| + \frac{\lceil p\rceil}{a^{p+1}}\int_h^t |u_{h_0,a,b,p}(s) - u_{h,a,b,p}(s)|(t-s)sds + \left|\int_{h_0}^h \frac{(t-s)s}{u^p_{h_0,a,b,p}(s)}ds\right| \le |b(h-h_0)| + \frac{\lceil p\rceil E^2_{h_0,a,b,p}}{a^{p+1}}\int_h^t |u_{h_0,a,b,p}(s) - u_{h,a,b,p}(s)|ds + \frac{E^2_{h_0,a,b,p}}{a^p}|h-h_0| \le |h-h_0|\left(b + \frac{E^2_{h_0,a,b,p}}{a^p}\right) + \frac{\lceil p\rceil E^2_{h_0,a,b,p}}{a^{p+1}}\int_h^t |u_{h_0,a,b,p}(s) - u_{h,a,b,p}(s)|ds.$$

By Gronwall's inequality, for any $t \in [max\{h, h_0\}, min\{E_{h,a,b,p}, E_{h_0,a,b,p}\}]$,

$$|u_{h_0,a,b,p}(t) - u_{h,a,b,p}(t)| \leq |h_0 - h|\bigg(b + \frac{E^2_{h_0,a,b,p}}{a^p}\bigg)exp\bigg(\frac{\lceil p\rceil E^3_{h_0,a,b,p}}{a^{p+1}}\bigg). \quad (8)$$

Now we will use the method of proof by contradiction to prove the assertion of the lemma. If there exists a number sequence $\{h_n\}$ such that $\lim_{n\to\infty} h_n = h_0$ and $E_{h_0,a,b,p} - E_{h_n,a,b,p} > \delta$ for any $n \in N$, then

$$|u_{h_0,a,b,p}(E_{h_n,a,b,p}) - a| \leq |h_0 - h_n|\bigg(b + \frac{E^2_{h_0,a,b,p}}{a^p}\bigg)exp\bigg(\frac{\lceil p\rceil E^3_{h_0,a,b,p}}{a^{p+1}}\bigg).$$

Since $u'_{h_0,a,b,p}(E_{h_0,a,b,p}) \neq 0$, by the implicit function theorem, $u^{-1}_{h_0,a,b,p}(t)$ is continuous at a with respect to t and thus $\lim_{n\to\infty} E_{h_n,a,b,p} = E_{h_0,a,b,p}$ which contradicts $E_{h_0,a,b,p} - E_{h_n,a,b,p} > \delta$. If there exists a number sequence $\{h_n\}$ such that $\lim_{n\to\infty} h_n = h_0$ and $E_{h_n,a,b,p} - E_{h_0,a,b,p} > \delta$ for any $n \in N$, then

$$|a - u_{h_n,a,b,p}(E_{h_0,a,b,p})| \leq |h_0 - h_n|\bigg(b + \frac{E^2_{h_0,a,b,p}}{a^p}\bigg)exp\bigg(\frac{\lceil p\rceil E^3_{h_0,a,b,p}}{a^{p+1}}\bigg).$$

Using the same discussion as the above one, we can prove that for any $t \in [max\{h, h_0\}, min\{E_{h,a,b,p}, E_{h_0,a,b,p}\}]$,

$$|u'_{h_0,a,b,p}(t) - u'_{h,a,b,p}(t)| \leq |h_0 - h|\frac{E_{h_0,a,b,p}}{a^p}exp\bigg(\frac{\lceil p\rceil E^2_{h_0,a,b,p}}{a^{p+1}}\bigg).$$

Thus $\lim_{n\to\infty} u'_{h_n,a,b,p}(E_{h_0,a,b,p}) = u'_{h_0,a,b,p}(E_{h_0,a,b,p}) < 0$. But since $E_{h_n,a,b,p} - E_{h_0,a,b,p} > \delta$ and $u'_{h_n,a,b,p}(t)$ is monotone decreasing,

$$\lim_{n\to\infty} u'_{h_n,a,b,p}(E_{h_0,a,b,p}) \geq \lim_{n\to\infty} \frac{a - u_{h_n,a,b,p}(E_{h_0,a,b,p})}{E_{h_n,a,b,p} - E_{h_0,a,b,p}} = 0.$$

This contradiction implies that $\lim_{h\to h_0} E_{h,a,b,p} = E_{h_0,a,b,p}$.

Lemma 5 Let $0 \leq h < 1$, $a_1 > a_2 > 0$, $b \in R$ and $p > 0$. Then $T_{h,a_1,b,p} \geq T_{h,a_2,b,p}$ and $u_{h,a_1,b,p}(t) > u_{h,a_2,b,p}(t)$ for any $t \in [h, T_{h,a_2,b,p})$.

Proof. In order to use the method of proof by contradiction to prove the lemma, we assume there exists $t_0 > h$ such that $u_{h,a_1,b,p}(t_0) = u_{h,a_2,b,p}(t_0)$ and $u_{h,a_1,b,p}(t) > u_{h,a_2,b,p}(t)$ for any $t < t_0$. By (5), we have that for $t \in [h, t_0]$,

$$u_{h,a_1,b,p}(t) - u_{h,a_2,b,p}(t) = a_1 - a_2 + \int_h^t (t-s)s\bigg[\frac{1}{u^p_{h,a_2,b,p}(s)} - \frac{1}{u^p_{h,a_1,b,p}(s)}\bigg]ds$$

$> a_1 - a_2$, which contradicts $u_{h,a_1,b,p}(t_0) = u_{h,a_2,b,p}(t_0)$.

Lemma 6 Let $0 \leq h < 1$, $a_0 > 0$, $b > 0$, $p > 0$ and $T_{h,a_0,b,p} < \infty$. Then $E_{h,a,b,p}$ is continuous at a_0 with respect to the initial value a. Moreover there exist $a_1, h_1 > 0$ such that $\{E_{h,a,b,p}|a \in (0, a_1]\}$ converges uniformly to $E_{h,0,b,p}$ on $[0, h_1]$ as $a \to 0$ where $E_{h,a,b,p}$ is considered as a function of h and $E_{h,0,b,p} = \lim_{a\to 0} E_{h,a,b,p}$.

Proof. Let $|a - a_0| < \epsilon$ where ϵ is a positive number which is less than a_0. Similar to Lemma 4, we have that for $t \in [h, min\{E_{h,a_0,b,p}, E_{h,a,b,p}\}]$,

$$|u_{h,a_0,b,p}(t) - u_{h,a,b,p}(t)| \leq |a - a_0| + \int_h^t \frac{|u^p_{h,a_0,b,p}(s) - u^p_{h,a,b,p}(s)|}{u^p_{h,a,b,p}(s)u^p_{h,a_0,b,p}(s)}(t-s)sds$$
$$\leq |a - a_0| + \frac{\lceil p\rceil E^2_{h,a_0,b,p}}{(a_0-\epsilon)^{p+1}} \int_h^t |u_{h_0,a,b,p}(s) - u_{h,a,b,p}(s)|ds.$$

By Gronwall's inequality, we have that for $t \in [h, min\{E_{h,a_0,b,p}, E_{h,a,b,p}\}]$,

$$|u_{h,a_0,b,p}(t) - u_{h,a,b,p}(t)| \leq |a - a_0| exp\bigg(\frac{\lceil p\rceil E^3_{h,a_0,b,p}}{(a_0-\epsilon)^{p+1}}\bigg). \quad (9)$$

Similar to Lemma 4, by using the proof by contradiction, we can prove that $\lim_{a\to a_0} E_{h,a,b,p} = E_{h,a_0,b,p}$. Now we will prove the second assertion of the lemma. By Lemma 3, there exists $a_1 > 0$ such that $E_{0,a_1,b,p} < \infty$. Since $E_{h,a,b,p}$ is continuous with respect to h, there exists $h_1 > 0$ such that $E_{h,a_1,b,p} < \infty$ for $h \in [0, h_1]$. Then we have that for $a \leq a_1$ and $h \leq h_1$,

$$|E_{h,a,b,p} - E_{h,0,b,p}| \leq max\{|E_{h,a,b,p} - E_{h,0,b,p}| \big| h \in [0, h_1]\}.$$

Since $E_{h,a,b,p}$ is monotone decreasing as $a \to 0$, $\lim_{a\to 0} max\{|E_{h,a,b,p} - E_{h,0,b,p}| \big| h \in [0, h_1]\}$=0 which implies that $\{E_{h,a,b,p} | a \in (0, a_1]\}$ converges uniformly to $E_{h,0,b,p}$ on $[0, h_1]$ as $a \to 0$. The proof is completed.

For $0 \leq h < 1$, $a > c > 0$, $b \in R$ and $p > 0$, if $T_{h,a,b,p} < \infty$, then there exists $E_{h,a,b,p,c}$ such that $u_{h,a,b,p}(E_{h,a,b,p,c}) = c$. If $T_{h,a,b,p} = \infty$, then we set $E_{h,a,b,p,c} = \infty$.

Remark 2 Similar to Lemma 4 and Lemma 6, we can prove that $E_{h,a,b,p,c}$ is continuous with respect to initial value a and h. And we can show that $u_{h,a,b,p}(t)$, $u'_{h,a,b,p}(t)$ are continuous with respect to the initial value a and h for $t \in [h, E_{h,a,b,p,c}]$.

Lemma 7 Let $h \geq 0$, $b \in R$ and $p > 0$. Then $T_{h,a,b,p}$ is continuous with respect to the initial value a in $G_{h,b,p}$ where $G_{h,b,p} = (0, sup\{a | T_{h,a,b,p} < \infty\})$.

Proof. If $a \in G_{h,b,p}$, by Lemma 2 and Lemma 5, $u_{h,a,b,p}(T_{h,a,b,p}) = 0$. Since $u'_{h,a,b,p}(t)$ is continuous with respect to t and $u'_{h,a,b,p}(T_{h,a,b,p}) = -\infty$ for $a \in G_{h,b,p}$, there exists $\epsilon_a \in (0, a)$ such that $|T_{h,a,b,p} - E_{h,a,b,,p,\epsilon}| < \epsilon/4$ and $u'_{h,a,b,p}(E_{h,a,b,p,\epsilon}) < -1$ for any $0 < \epsilon < \epsilon_a$. Now we will prove that for any $a_0 \in G_{h,b,p}$ and $0 < \epsilon_0 < \epsilon_{a_0}$, there exists $\delta > 0$ such that $|T_{h,a_0,b,p} - T_{h,a,b,p}| < \epsilon_0$ for any $a \in (a_0, a_0 + \delta)$. Since $u'_{h,a,b,p}(t)$ is continuous with respect to the initial value a, there exists $\delta_1 > 0$ such that $|T_{h,a,b,p} - E_{h,a,b,p,\epsilon_0}| < \epsilon_0/4$ for any $a \in (a_0, a_0 + \delta_1)$. Meanwhile since $u'_{h,a_0,b,p}(E_{h,a_0,b,p,\epsilon_0}) < -1$ and $u_{h,a,b,p}(t)$ is continuous with respect to initial value a, there exists $\delta_2 > 0$ such that $|E_{h,a,b,p,\epsilon_0} - E_{h,a_0,b,p,\epsilon_0}| < |u_{h,a,b,p}(E_{h,a,b,p,\epsilon_0}) - u_{h,a,b,p}(E_{h,a_0,b,p,\epsilon_0})| = |u_{h,a_0,b,p}(E_{h,a_0,b,p,\epsilon_0}) - u_{h,a,b,p}(E_{h,a_0,b,p,\epsilon_0})| < \epsilon_0/4$ for any $a \in (a_0, a_0 + \delta_2)$. Set $\delta = min\{\delta_1, \delta_2\}$. Then, by the triangular inequality, we have that $|T_{h,a_0,b,p} - T_{h,a,b,p}| < |T_{h,a_0,b,p} - E_{h,a_0,b,p,\epsilon_0}| + |E_{h,a_0,b,p,\epsilon_0} - E_{h,a,b,p,\epsilon_0}| + |E_{h,a,b,p,\epsilon_0} - T_{h,a,b,p}| < \epsilon_0$ for any $a \in (a_0, a_0 + \delta)$. The proof is completed.

Lemma 8 Let $0 \leq h < 1$. If $b \leq 0$, then $\lim_{a\to 0} T_{h,a,b,p} = h$ for any $p > 0$. If $b > 0$, then $\lim_{a\to 0} T_{h,a,b,p} = h$ for any $p \geq 1$.

Proof. If $b < 0$, then, since $u'_{h,a,b,p}(t) < 0$ for $t > h$, by Lemma 1 and Lemma 2, $T_{h,a,b,p} < \infty$ and $u_{h,a,b,p}(T_{h,a,b,p}) = 0$. Since $u_{h,a,b,p}(t) \leq a + b(t-h)$ for $t \geq h$, $h \leq T_{h,a,b,p} \leq h - a/b$. Thus $T_{h,a,b,p} \to h$ as $a \to 0$. If $b = 0$ and $h > 0$, then, by

$u'_{h,a,0,p}(t) < 0$ for $t > h$, $T_{h,a,0,p} < \infty$ and $u_{h,a,0,p}(T_{h,a,0,p}) = 0$. We obtain

$$u_{h,a,b,p}(t) \le a - \frac{h}{a^p}\int_h^t (t-s)ds = a - \frac{h}{2a^p}(t-h)^2.$$

Then $h \le T_{h,a,0,p} \le h + 2^{1/2}a^{(1+p)/2}/h^{1/2}$ and thus $T_{h,a,0,p} \to h$ as $a \to 0$.
If $b = 0$ and $h = 0$, then we have

$$u_{0,a,0,p}(t) \le a - \frac{1}{a^p}\int_0^t (t-s)sds = a - \frac{1}{6a^p}t^3.$$

Then $0 \le T_{0,a,0,p} \le 6^{1/3}a^{(1+p)/3}$ and thus $T_{0,a,0,p} \to 0$ as $a \to 0$.
Now we consider the case $b > 0$. By Lemma 3, there exists $a_1 > 0$ such that $T_{h,a,b,p} < \infty$ for any $a \in (0, a_1)$. Then for any $a \in (0, a_1)$, there exists $S_{h,a,b,p} > h$ such that $u'_{h,a,b,p}(S_{h,a,b,p}) = 0$. We have

$$b = \int_h^{S_{h,a,b,p}} \frac{s}{u^p_{h,a,b,p}(s)}ds. \tag{10}$$

Let $S_p = \lim_{a\to 0} S_{h,a,b,p}$. By (10) and Lemma 5, $S_p < S_{h,a,b,p}$. In order to use the method of proof by contradiction to prove that $S_p = h$, we assume that $S_p > h$. If $p = 1$ and $0 < h < 1$, then we have

$$\begin{aligned} b &\ge \lim_{a\to 0}\int_h^{S_{h,a,b,1}} \frac{s}{a+b(s-h)}ds > \lim_{a\to 0}\int_h^{S_1} \frac{s}{a+b(s-h)}ds \\ &= \frac{S_1-h}{b} + \lim_{a\to 0}\left[\Big(h-\frac{a}{b}\Big)ln\Big(\frac{b(S_1-h)+a}{a}\Big)\right] = \infty. \end{aligned}$$

If $p \in (1, 2)$ and $0 < h < 1$, then we obtain

$$\begin{aligned} b &\ge \lim_{a\to 0}\int_h^{S_{h,a,b,p}} \frac{s}{(a+b(s-h))^p}ds > \lim_{a\to 0}\int_h^{S_p} \frac{s}{(a+b(s-h))^p}ds \\ &= \lim_{a\to 0}\left[\frac{1}{b^p(p-2)}\left(\Big(\frac{a}{b}\Big)^{2-p} - \Big(S_p - h + \frac{a}{b}\Big)^{2-p}\right) + \frac{1}{b^p(p-1)}\Big(h-\frac{a}{b}\Big)\right. \\ &\left.\left(\Big(\frac{a}{b}\Big)^{1-p} - \Big(S_p - h + \frac{a}{b}\Big)^{1-p}\right)\right] = \frac{1}{b^p(2-p)}\Big(S_p - h\Big)^{2-p} + \frac{h}{b^p(p-1)} \\ &\lim_{a\to 0}\Big(\frac{a}{b}\Big)^{1-p} - \frac{h}{b^p(p-1)}\Big(S_p - h\Big)^{1-p} = \infty. \end{aligned}$$

If $p \in (2, \infty)$, then we have

$$\begin{aligned} b &\ge \lim_{a\to 0}\int_h^{S_{h,a,b,p}} \frac{s}{(a+b(s-h))^p}ds > \lim_{a\to 0}\int_h^{S_p} \frac{s}{(a+b(s-h))^p}ds \\ &= \lim_{a\to 0}\Big(\frac{a}{b}\Big)^{2-p}\left(\frac{1}{b^p(p-2)} - \frac{1}{b^p(p-1)}\right) + \frac{h}{b^p(p-1)}\lim_{a\to 0}\Big(\frac{a}{b}\Big)^{1-p} - \frac{1}{b^p(p-2)} \\ &\Big(S_p - h\Big)^{2-p} - \frac{1}{b^p(p-1)}\Big(S_p - h\Big)^{1-p} = \infty. \end{aligned}$$

If $p = 2$, then we obtain

$$b \ge \lim_{a\to 0}\int_h^{S_{h,a,b,p}} \frac{s}{(a+b(s-h))^2}ds > \lim_{a\to 0}\int_h^{S_2} \frac{s}{(a+b(s-h))^2}ds$$

$$= \frac{1}{b^2}\lim_{a\to 0}\left[ln\Big(S_2 - h + \frac{a}{b}\Big) - ln\frac{a}{b} + \Big(\frac{a}{b} - h\Big)\left(\Big(S_2 - h + \frac{a}{b}\Big)^{-1} - \frac{b}{a}\right)\right]$$
$$= \frac{1}{b^2}\lim_{a\to 0} ln\frac{b}{a} + \frac{1}{b^2}ln(S_2 - h) + \lim_{a\to 0}\frac{h}{ab} - \frac{h}{b^2(S_2 - h)} - \frac{1}{b^2} = \infty.$$

The above contradictions imply that $\lim_{a\to 0} S_{h,a,b,p} = 0$ and $\lim_{a\to 0} T_{h,a,b,p} = 0$. If $p \in [1,2)$ and $h = 0$, then, by Lemma 4 and Lemma 6, $\lim_{a\to 0} E_{0,a,b,p} = \lim_{h\to 0}\lim_{a\to 0} E_{h,a,b,p} = 0$ and thus $\lim_{a\to 0} T_{0,a,b,p} = 0$. The proof is completed.

Remark 3 The work of [11] is not sufficient for proving the existence of solutions of the boundary layer problem for the case $h = 0$. In fact, that paper established the existence for the case $0 < h < 1$. In the present paper we use properties of $E_{h,a,b,p}$ to complete the proof.

Lemma 9 Let $0 \le h < 1$, $b \in R$ and $p > 0$. Then $sup\{T_{h,a,b,p}|T_{h,a,b,p} < \infty, a > 0\} = \infty$.

Proof. First we will prove that $\lim_{a\to\infty} T_{h,a,b,p} = \infty$. If $a > 1$ and $T_{h,a,b,p} < \infty$, then $E_{h,a,b,p,1} < \infty$ and $u_{h,a,b,p}(t) \ge 1$ for $t \in [h, E_{h,a,b,p,1}]$. By (5), we obtain

$$1 \le a + b(E_{h,a,b,p,1} - h) - \int_h^{E_{h,a,b,p,1}} (E_{h,a,b,p,1} - s)sds$$

which yields that $\lim_{a\to\infty} E_{h,a,b,p,1} = \infty$ and thus $\lim_{a\to\infty} T_{h,a,b,p} = \infty$.

Assume that $N = sup\{T_{h,a,b,p}|T_{h,a,b,p} < \infty, a > 0\} < \infty$.

If $T_{h,a,b,p} < \infty$ for any $a > 0$, then, since $\lim_{a\to\infty} T_{h,a,b,p} = \infty$, $N = \infty$ which contradicts $N < \infty$. If there exists $a_1 > 0$ such that $T_{h,a_1,b,p} = N$, then by Lemma 2, $u_{h,a_1,b,p}(N) = 0$ and $T_{h,a,b,p} = \infty$ for any $a > a_1$. Then, by Lemma 1, $u'_{h,a,b,p}(t) > 0$ for $a > a_1$ and $t > h$. Thus $u_{h,a,b,p}(t) \ge a > a_1$ for $a > a_1$ and $t > h$. For any $a > a_1$, there exists $\alpha_a > 0$ such that $u_{h,a_1,b,p}(t) < a - a_1$ for $t \in [N - \alpha_a, N]$. Thus for $a > a_1$ and $t \in [N - \alpha_a, N]$, $u_{h,a,b,p}(t) - u_{h,a_1,b,p}(t) > a_1$ which contradicts the continuous dependence of $u_{h,a,b,p}(t)$ on a.

If there exists $a_2 > 0$ such that $T_{h,a_2,b,p} = \infty$ and $T_{h,a,b,p} < N$ for any $a < a_2$, then $u_{h,a_2,b,p}(t) > a_2$ for $t > h$. For any $a < a_2$, since $u_{h,a,b,p}(T_{h,a,b,p}) = 0$, there exists $\beta_a > 0$ such that for $t \in [T_{h,a,b,p} - \beta_a, T_{h,a,b,p}]$, $u_{h,a_2,b,p}(t) - u_{h,a,b,p}(t) \ge a_2$ which contradicts the continuous dependence of $u_{h,a,b,p}(t)$ on a. The proof is completed.

Theorem 1 The boundary layer problem for power law flows (1)-(2) has at most one solution in $C[h,1]$.

Proof. By Lemma 6, this result is obvious.

Theorem 2 The boundary layer problem for pseudoplastic flows and Newtonian flows (1)-(2) has a unique solution in $C[h,1]$.

Proof. By Lemma 7, Lemma 8, Lemma 9 and intermediate value theorem, for $0 \le h < 1$, $b \in R$ and $p \ge 1$, there exists $a > 0$ such that $T_{h,a,b,p} = 1$.

Theorem 3 If $b \le 0$, then the boundary layer problem for dilatant flows (1)-(2) has a unique solution in $C[h,1]$. If $b > 0$ and there exists a real number $a > 0$ such that $T_{h,a,b,p} \le 1$, then the boundary layer problem for dilatant flows (1)-(2) has a unique solution in $C[h,1]$.

Proof. Similar to Theorem 2, this result can be proved.

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