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ON GENERALIZED SEMIDERIVATIONS OF Γ-RINGS

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Abstract. In this paper, we introduce the notion of a generalized semiderivation on Γ-ring, and we try to generalize some known results of derivations, semiderivations and generalized derivations to generalized semiderivations on a prime Γ-ring. We also prove that there exist no nontrivial generalized semiderivations which act as a homomorphism or as an antihomomorphism on a prime Γ-ring.

1. Introduction

J. C. Chang [6] studied on semiderivations of prime rings. He obtained some results of derivations of prime rings into semiderivations. H. E. Bell and W. S. Martindale III [1] investigated the commutativity property of a prime ring by means of semiderivations. C. L. Chuang [7] studied on the structure of semiderivations in prime rings. He obtained some remarkable results in connection with semiderivations. J. Bergen and P. Grzesczuk [3] obtained the commutatity properties of semiprime rings with the help of skew(semi)-derivations. A. Firat [8] generalized some results of prime rings with derivations to the prime rings with semiderivations. In this paper, we introduce the notion of a generalized semiderivation on Γ-ring, and we try to extend some known results of derivations, semiderivations and generalized derivations to generalized semiderivations on a prime Γ-ring. We also prove that there exist no nontrivial generalized semiderivations which act as a homomorphism or as an antihomomorphism on a prime Γ-ring.

2. Preliminaries

Let M and Γ be additive abelian groups. Then M is called a Γ-*ring* if the following conditions are satisfied:

(M1) $x\beta y \in M$,
(M2) $(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)y = x\alpha y + x\beta y, x\alpha(y+z) = x\alpha y + x\alpha z$,
(M3) $(x\alpha y)\beta z = x\alpha(y\beta z)$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

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Let M be a Γ-ring with center $Z(M)$. For any $x, y \in M$, the notation $[x, y]_\alpha$ denotes the commutator $x\alpha y - y\alpha x$ while the symbol (x, y) denotes $x + y - x - y$, respectively. We know that $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y + x[\beta, \alpha]_z y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z + y[\beta, \alpha]_x z$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. We take an assumption (*) $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Using the assumption (*), identities $[x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha\beta y$ and $[x, y\beta z]_\alpha = y\beta[x, z]_\alpha + [x, y]_\alpha\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$ are used extensively in our results.

Let M be a Γ-ring. A *right* (resp. *left*) *ideal* of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M \subset I$ (resp. $M\Gamma I \subset I$). If I is both a right and a left ideal, we say that I is an *ideal* of M.

Definition 2.1. Let M be a Γ-ring. Then

(1) M is said to be *prime* if $x\Gamma M\Gamma y = 0$ implies $x = 0$, or $y = 0$, for all $x, y \in M$.
(2) M is said to be *semiprime* if $x\Gamma M\Gamma x = 0$ implies $x = 0$, for all $x \in M$.
(3) M is said to be 2-*torsion free* if $2x = 0$ implies $x = 0$, for all $x \in M$.

Definition 2.2. Let M be a Γ-ring. An additive mapping $d : M \to M$ is called a *derivation* if

$$d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$$

for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.3. Let M be a Γ-ring. An additive mapping $d : M \to M$ is called a *semiderivation* associated with a map $g : M \to M$ if, for all $x, y \in M$ and $\alpha \in \Gamma$,

(1) $d(x\alpha y) = d(x)\alpha g(y) + x\alpha d(y) = d(x)\alpha y + g(x)\alpha d(y)$,
(2) $d(g(x)) = g(d(x))$.

Example 2.4. Let M_1 be a Γ_1-ring and M_2 be a Γ_2-ring. Consider $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$. Define addition and multiplication on M and Γ by

$$(m_1, m_2) + (m_3, m_4) = (m_1 + m_3, m_2 + m_4)$$

$$(\alpha_1, \alpha_2) + (\alpha_3, \alpha_4) = (\alpha_1 + \alpha_3, \alpha_2 + \alpha_4)$$

$$(m_1, m_2)(\alpha_1, \alpha_2)(m_3, m_4) = (m_1\alpha_1 m_3, m_2\alpha_2 m_4),$$

for every $(m_1, m_2), (m_3, m_4) \in M$ and $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4) \in \Gamma$. Under these addition and multiplication, M is a Γ-ring. Let $\delta : M_1 \to M_1$ be an additive map and $\tau : M_2 \to M_2$ be a left and right M_2^Γ-module which is not a derivation. Define $d : M \to M$ such that $d((m_1, m_2)) = (0, \tau(m_2))$ and $g : M \to M$ such that $g((m_1, m_2)) = (\delta(m_1), 0), m_1 \in M_1, m_2 \in M_2$. Then it is clear that d is a semiderivation of M associated with g, which is not a derivation on M. .

Lemma 2.5. [9] *Let M be a prime Γ-ring and I be a nonzero ideal of M such that $I \subseteq Z(M)$. Then I is commutative.*

Let M be a Γ-ring and F be a semiderivation of M. If $F(x\alpha y) = F(x)\alpha F(y)$ and $F(x\alpha y) = F(y)\alpha F(x)$, for all $x, y \in M$ and $\alpha \in \Gamma$, then F is said to act as a homomorphism or antihomomorphism on M, respectively.

3. Generalized Semiderivations of Γ-rings

Definition 3.1. Let M be a Γ-ring. An additive mapping $F : M \to M$ is called a *generalized semiderivation* of M if there exists a semiderivation $d : M \to M$ associated with a map $g : M \to M$ if

1. $F(x\alpha y) = F(x)\alpha y + g(x)\alpha d(y) = d(x)\alpha g(y) + x\alpha F(y)$,
2. $F(g(x)) = g(F(x))$, for all $x, y \in M$ and $\alpha \in \Gamma$.

If $g = I$, i.e., an identity mapping of M, then all semiderivations associated with g are merely ordinary derivations. If g is any endomorphism, then semiderivations are of the form $f(x) = x - g(x)$.

Example 3.2. Let R be a commutative ring, $\Gamma = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} : b \in Z_2 \right\}$ and $M = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} : a, b, c \in R \right\}$. Then M is a Γ-ring. Define a map $F : M \to M$ by $F\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$ and $g : M \to M$ by $g\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right) = \begin{pmatrix} 0 & 0 \\ b & -c \end{pmatrix}$. Then F is a generalized semiderivation associated with nonzero semiderivation $d : M \to M$ defined by $d\left(\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \right) = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$, which is not a generalized derivation on M.

Lemma 3.3. *Let M be a 2-torsion free prime Γ-ring. If d is a non-zero semiderivation of M associated with an onto map g which is onto, then $d^2 \neq 0$.*

Proof. Suppose that $d^2(M) = 0$. Then, for all $x, y \in M$ and $\alpha \in \Gamma$, we have

$$\begin{aligned} 0 &= d^2(x\alpha y) \\ &= d(d(x\alpha y)) \\ &= d(d(x)\alpha y + g(x)\alpha d(y)) \\ &= d^2(x)\alpha y + d(x)\alpha d(y) + d(g(x))\alpha d(y) + g(x)\alpha d^2(y) \\ &= d(x)\alpha d(y) + d(g(x))\alpha d(y). \end{aligned}$$

Note that $g(d(x)) = d(g(x))$ and g is onto, we get

$$2d(x)\alpha d(y) = 0 \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma.$$

Since M is 2-torsion free, we get

$$d(x)\alpha d(y) = 0 \text{ for all } x, y \in M \text{ and } \alpha \in \Gamma.$$

Replacing y by $r\beta y$ in the above relation, we get

$$\begin{aligned} 0 &= d(x)\alpha d(r\beta y) \\ &= d(x)\alpha(d(r)\beta y + g(r)\beta d(y)) \\ &= d(x)\alpha d(r)\beta y + d(x)\alpha g(r)\beta d(y) \end{aligned}$$

for all $x, y, r \in S$ and $\alpha, \beta \in \Gamma$. This implies that

$$d(x)\alpha g(r)\beta d(y) = 0 \text{ for all } x, y, r \in M \text{ and } \alpha, \beta \in \Gamma.$$

Since g is onto, we have

$$d(x)\alpha r\beta d(y) = 0 \text{ for all } x, y, r \in M \text{ and } \alpha \in \Gamma.$$

Hence

$$d(M)\Gamma M\Gamma d(M) = \{0\}.$$

Thus we obtain $d = 0$, a contradiction. □

Lemma 3.4. *Let M be a prime Γ-ring and let d be a nonzero semiderivation associated with an onto map $g : M \to M$. If $d(M) \subseteq Z(M)$, then M is commutative.*

Proof. By hypothesis, we have $d(x\alpha y) \in Z(M)$ for all $x, y \in M$. That is, $d(x)\alpha g(y) + x\alpha d(y) \in Z(M)$ for all $x, y \in M$ and $\alpha \in \Gamma$. Commuting this term with x, we obtain

$$\begin{aligned} 0 &= [d(x)\alpha g(y) + x\alpha d(y), x]_\gamma \\ &= [d(x)\alpha g(y), x]_\gamma + [x\alpha d(y), x]_\gamma \\ &= d(x)\alpha[g(y), x]_\gamma + [d(x), x]_\gamma \alpha g(y) + x\alpha[d(y), x]_\gamma + [x, x]_\gamma \alpha d(y) \\ &= d(x)\alpha[g(y), x]_\gamma. \end{aligned}$$

Since $d(x) \in Z(N)$, and g is surjective function of M, we have $d(x)\alpha s\beta[y, x]_\gamma = 0$ for all $x, y, s \in M$ and $\alpha, \beta \in \Gamma$. Since M is prime and $d(x) \neq 0$ for all $x \in M$, we have $[y, x]_\gamma = 0$ for all $x, y \in M$. That is, M is commutative. □

Lemma 3.5. *Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) and admitting a nonzero semiderivation d associated with an onto map $g : M \to M$. If $[d(x), d(y)]_\alpha = 0$ for all $x, y \in M$, then M is commutative.*

Proof. Suppose that

$$d(x)\alpha d(y) = d(y)\alpha d(x), \quad \forall\ x, y \in M, \alpha \in \Gamma. \tag{1}$$

Replacing y by $y\beta z$ in (1), we obtain

$$d(x)\alpha d(y\beta z) = d(y\beta z)\alpha d(x), \forall x, y, z \in M, \alpha, \beta \in \Gamma. \tag{2}$$

$$d(x)\alpha(d(y)\beta z + g(y)\beta d(z)) = (d(y)\beta z + g(y)\beta d(z))\alpha d(x), \forall\ x, y, z \in M, \alpha, \beta \in \Gamma. \tag{3}$$

$$\begin{aligned} d(x)\alpha d(y)\beta z + &d(x)\alpha g(y)\beta d(z) \\ &= d(y)\beta z\alpha d(x) + g(y)\beta d(z)\alpha d(x), \forall x, y, z \in M, \alpha, \beta \in \Gamma. \end{aligned} \tag{4}$$

Substituting $d(y)$ for y in (4) and using (1), we get

$$d(x)\alpha d^2(y)\beta z = d^2(y)\beta z\alpha d(x), \forall x, y, z \in M, \alpha, \beta \in \Gamma. \tag{5}$$

Taking $z\delta t$ instead of z in (5), we obtain

$$d(x)\alpha d^2(y)\beta z\delta t = d^2(y)\beta z\delta t\alpha d(x), \forall x, y, z \in M, \alpha, \beta, \delta \in \Gamma. \tag{6}$$

Using (5) in (6), we get

$$d^2(y)\beta z\alpha d(x)\delta t = d^2(y)\beta z\delta t\alpha d(x), \forall x, y, z \in M, \alpha, \beta, \delta \in \Gamma. \tag{7}$$

Hence we get

$$d^2(y)\beta z\alpha[d(x), t]_\delta = 0, \forall x, y, z \in M, \alpha, \beta, \delta \in \Gamma. \tag{8}$$

This implies

$$d^2(y)\Gamma M\Gamma[d(x),t]_\delta = 0, \forall x,y,z \in M, \alpha,\beta,\delta \in \Gamma. \tag{9}$$

Since M is prime, we have $d^2 = 0$ or $d(M) \subseteq Z(M)$. But $d^2 \neq 0$ by Lemma 3.3, and so $d(M)$ is contained in $Z(M)$, which implies M is commutative by Lemma 3.4. □

Theorem 3.6. *Let M be a 2-torsion free prime Γ-ring satisfying the condition (*) with a generalized semiderivation F associated with a nonzero semiderivation d and onto map g associated with d. If $F(M) \subseteq Z(M)$, then M is commutative.*

Proof. Assume that

$$F(x) \in Z(M), \quad \forall x \in M. \tag{10}$$

For all $x,n \in M$, $z \in Z(M)$ and $\alpha,\beta \in \Gamma$, we have

$$F(x\alpha z)\beta n = F(x)\alpha z\beta n + g(x)\alpha d(z)\beta n,$$

$$\begin{aligned} n\beta F(x\alpha z) &= n\beta F(x)\alpha z + n\beta g(x)\alpha d(z) \\ &= F(x)\alpha z\beta n + n\beta g(x)\alpha d(z). \end{aligned} \tag{11}$$

Since $F(x\alpha z)\beta n = n\beta F(x\alpha z)$, we have $g(x)\alpha d(z)\beta n = n\beta g(x)\alpha d(z)$ for all $x,n \in M, z \in Z(M)\backslash\{0\}$. Thus $g(x)\alpha d(z) \in Z(M)$ for all $x \in M$ and $\alpha,\beta \in \Gamma$. Let $d(Z) \neq \{0\}$. Choosing z such that $d(z) \neq 0$ and noting that $d(z) \in Z(M)$, we have $g(x) \in Z(M)$. Since g is onto, we have $M \in Z(M)$. Hence M is commutative by Lemma 2.5. On the other hand, if $d(z) = 0$, then for all $x,y \in M$,

$$\begin{aligned} 0 &= d(F(x\alpha y)) \\ &= d(F(x)\alpha y + g(x)\alpha d(y)) \\ &= F(x)\alpha d(y) + g(x)\alpha d^2(y) + d(g(x))g(d(y)), \end{aligned}$$

for all $x,y \in M$ and $\alpha \in \Gamma$. Hence $F(x\alpha d(y)) = -d(g(x))\alpha g(d(y)) \in Z(M)$ for all $x,y \in M$ and $\alpha \in \Gamma$. Since g is onto, we have we have $d(x)\alpha d(y) \in Z(M)$ for all $x,y \in M$ and $\alpha \in \Gamma$. This implies that

$$d(x)\alpha(d(x)\beta d(y) - d(y)\beta d(x)) = 0$$

for all $x,y \in M$ and $\alpha,\beta \in \Gamma$. Multiplying by $d(y)$ on left side of the above relation, we obtain

$$d(y)\gamma d(x)\alpha M\delta((d(x)\beta d(y) - d(y)\beta d(x)) = \{0\},$$

for all $x,y \in M$ and $\alpha,\beta,\delta\gamma \in \Gamma$. Since M is prime, we have

$$[d(x),d(y)]_\beta = 0,$$

for all $x,y \in M$. We conclude that M is commutative by Lemma 3.5. □

Theorem 3.7. *Let M be a prime Γ-ring satisfying the condition (*) and admitting a generalized semiderivation F associated with a nonzero semiderivation d and an onto map g associated with d such that $g(x\alpha y) = g(x)\alpha g(y)$ for all $x,y \in M$ and $\alpha \in \Gamma$. If $[F(M),F(M)]_\alpha = 0$, then M is commutative.*

Proof. By the hypothesis, we have

$$F(x)\alpha F(y) = F(y)\alpha F(x) \tag{12}$$

for all $x,y \in M$ and $\alpha \in \Gamma$. Replacing y by $F(z)\beta y$ in the above relation, we get

$$F(x)\alpha F(F(z)\beta y) = F(F(z)\beta y)\alpha F(x) \tag{13}$$

for all $x, y, z \in N$ and $\alpha, \beta \in \Gamma$. This implies that

$$F(x)\alpha(d(F(z))\beta g(y) + F(z)F(y)) \\ = (d(F(z))\beta g(y) + F(z)\beta F(y)) = \alpha F(x) \tag{14}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Hence we get

$$F(x)\alpha d(F(z))\beta g(y) = d(F(z))\beta g(y)\alpha F(x), \tag{15}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Taking $y\gamma w$ instead of y in (14) and using (14), we obtain

$$d(F(z))\beta g(y)\alpha F(x)\gamma g(w) = d(F(z))\beta g(y)\gamma g(w)\alpha F(x),$$

for all $x, y, z \in M$ and $\alpha, \beta, \gamma \in \Gamma$. Since g is onto, we get

$$d(F(z))\beta y\alpha F(x)\gamma w = d(F(z)\beta y\gamma w\alpha F(x),$$

for all $x, y, z \in M$ and $\alpha, \beta, \gamma \in \Gamma$. This implies that

$$d(F(z))\Gamma M\Gamma(F(x)\gamma w - w\gamma F(x)) = \{0\},$$

for all $x, z, w \in M$ and $\gamma \in \Gamma$. Since M is prime, we have $d(F(M)) = \{0\}$ or $F(M) \subseteq Z(M)$. If $F(M)$ is contained in $Z(M)$, then M is commutative by Theorem 3.6. On the other hand, if $d(F(M)) = 0$, then

$$d(F(x\alpha y)) = d((d(x)\alpha g(y) + x\alpha F(y)) = 0$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Thus

$$d^2(x)\alpha g(y) + d(x)\alpha d(g(y)) + d(x)\alpha F(y) + x\alpha d(F(y)) = 0,$$

for all $x, y \in M$ and $\alpha \in \Gamma$. This implies that

$$d^2(x)\alpha g(y) + d(x)\alpha d(g(y)) + d(x)\alpha F(y) = 0 \tag{16}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replacing y by $y\beta z$ and g is onto, we have

$$d^2(x)\alpha g(y\beta z) + d(x)\alpha d(g(y\beta z)) + d(x)\alpha F(y\beta z) = 0,$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, which implies that

$$d^2(x)\alpha y\beta z + d(x)\alpha d(y\beta z) + d(x)\alpha F(y\beta z) = 0,$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Hence

$$d^2(x)\alpha y\beta z + d(x)\alpha(y\beta d(z) + d(y)\beta g(z)) + d(x)\alpha(F(y)\beta z + g(y)\beta d(z)) = 0,$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

$$d^2(x)\alpha y\beta z + d(x)\alpha y\beta d(z) + d(x)\alpha d(y)\beta g(z) + d(x)\alpha F(y)\beta z + d(x)\alpha g(y)\beta d(z) = 0,$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Since g is onto, we have

$$d^2(x)\alpha y\beta z + d(x)\alpha y\beta d(z) + d(x)\alpha d(y)\beta z + d(x)\alpha F(y)\beta z + d(x)\alpha y\beta d(z) = 0,$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Thus

$$\{d^2(x)\alpha y + d(x)\alpha d(y) + d(x)\alpha F(y)\}\beta z + 2d(x)\alpha y\beta d(z) = 0,$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Since M is a 2-torsion free and using (16), we get $d(x)\alpha y\beta d(z) = 0$ for all $y, z \in M$ and $\alpha, \beta \in \Gamma$. That is, $d(M)\Gamma M\Gamma d(M) = \{0\}$. Thus we get $d = 0$, which is a contradiction. This completes the proof. □

Corollary 3.8. *Let M be a 2-torsion free prime Γ-ring. If M admits a generalized derivation F associated with a nonzero derivation d such that $[F(x), F(y)]_\alpha = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$, then M is commutative.*

Theorem 3.9. *Let M be a 2-torsion free prime Γ-ring satisfying the condition (*). If F is a generalized semiderivation of M associated with a nonzero semiderivation d and an automorphism g associated with d, then the following conditions are equivalent:*

1. $F([x,y]_\alpha) = [F(x),y]_\alpha$, *for all* $x,y \in M$ *and* $\alpha \in \Gamma$,
2. $F([x,y]_\alpha) = -[F(x),y]_\alpha$, *for all* $x,y \in M$ *and* $\alpha \in \Gamma$,
3. M *is commutative.*

Proof. It is obvious that (3) implies both (1) and (2).

Now we prove that (1) implies (3). By hypothesis,

$$F([x,y]_\alpha) = [F(x),y]_\alpha \tag{17}$$

for all $x,y \in M$ and $\alpha \in \Gamma$. Taking $x\beta y$ instead of y in (17) and noting that $[x,x\beta y]_\alpha = x\beta[x,y]_\alpha$, we get

$$F([x,x\beta y]_\alpha) = [F(x),x\beta y]_\alpha,$$

for all $x,y \in M$ and $\alpha \in \Gamma$. Hence we get

$$F(x\beta[x,y]_\alpha) = [F(x),x\beta y]_\alpha,$$

for all $x,y \in M$ and $\alpha \in \Gamma$. This implies that

$$x\beta F([x,y]_\alpha) + d(x)\beta g([x,y]_\alpha) = F(x)\alpha x\beta y - x\beta y\alpha F(x),$$

for all $x,y \in M$ and $\alpha \in \Gamma$. Using (17) and noting that $F(x)\alpha x = x\alpha F(x)$ by (17), the last equation yields

$$d(x)\alpha g([x,y]_\alpha) = 0$$

for all $x,y \in M$ and $\alpha \in \Gamma$. Since g is automorphism, we get

$$d(x)\alpha g(x)\alpha g(y) = d(x)\alpha g(y)\alpha g(x)$$

for all $x,y \in M$ and $\alpha \in \Gamma$. Replacing y by $y\beta t$, in the last equation, we get

$$d(x)\alpha y\alpha[x,t]_\beta = 0,$$

for all $x,y,t \in M$ and $\alpha,\beta \in \Gamma$. This implies that

$$d(x)\Gamma M\Gamma[x,t]_\beta = 0$$

for all $x,t \in M$ and $\beta \in \Gamma$. Since M is prime, we have $M \subseteq Z(M)$ or $d(M) = \{0\}$. In both cases, M is commutative Lemma 2.5 and Lemma 3.4. By the similar fashion, we can show that (2) implies (3). □

Theorem 3.10. *Let M be a 2-torsion free prime Γ-ring satisfying the condition (*). If F is a generalized semiderivation of M associated with a nonzero semiderivation d and an automorphism g associated with d, then the following conditions are equivalent:*

1. $F([x,y]_\alpha) = [x,F(y)]_\alpha$, *for all* $x,y \in M$ *and* $\alpha \in \Gamma$,
2. $F([x,y]_\alpha) = -[x,F(y)]_\alpha$, *for all* $x,y \in M$ *and* $\alpha \in \Gamma$,
3. M *is commutative.*

Proof. It is obvious that (3) implies both (1) and (2). Now we prove that (1) implies (3). By hypothesis,

$$F([x,y]_\alpha) = [x,F(y)]_\alpha \tag{18}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Taking $y\beta x$ instead of x in (18) and noting that $[x, x\beta y]_\alpha = x\beta[x, y]_\alpha$, we get

$$y\beta F([x, y]_\alpha) + d(y)\beta g([x, y]_\alpha) = y\beta x\alpha F(y) - F(y)\alpha x\beta y,$$

for all x, y, and $\alpha, \beta \in \Gamma$. Using (18) and noting that $y\beta F(y) = F(y)\beta y$ by (1), we have

$$d(y)\alpha g([x, y]) = 0$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Arguing in the similar manner as in the Theorem 3.9, we get the results. Similarly, we can prove that (2) implies (3). □

Theorem 3.11. *Let M be a prime Γ-ring satisfying the condition (*). Suppose that F is a generalized semiderivation of M associated with a nonzero semiderivation d and an onto map g associated with d such that $g(x\alpha y) = g(x)\alpha g(y)$. If F acts as a homomorphism on M, then either F is an identity map or $F = 0$.*

Proof. By the hypothesis, we have

$$F(x\alpha y) = d(x)\alpha g(y) + x\alpha F(y) = F(x)\alpha F(y), \tag{19}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replacing y by $y\beta z$ in the above relation, we get

$$F(x\alpha y\beta z) = d(x)\alpha g(y\beta z) + x\alpha F(y\beta z)$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Thus

$$F(x\alpha y)\beta F(z) = d(x)\alpha g(y\beta z) + x\alpha F(y\beta z)$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. By (1), we get

$$(d(x)\alpha g(y) + x\alpha F(y))\beta F(z) = d(x)\alpha g(y\beta z) + x\alpha(d(y)\beta g(z) + y\beta F(z)),$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Hence we have

$$d(x)\alpha g(y)\beta F(z) + x\alpha F(y)\beta F(z) = d(x)\alpha g(y\beta z) + x\alpha d(y)\beta g(z) + x\alpha y\beta F(z),$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, and

$$d(x)\alpha g(y)\beta F(z) + x\alpha F(y\beta z) = d(x)\alpha g(y\beta z) + x\alpha d(y)\beta g(z) + x\alpha y\beta F(z),$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, and

$$d(x)\alpha g(y)\beta F(z)+x\alpha d(y)\beta g(z)+x\alpha y\beta F(z) = d(x)\alpha g(y\beta z)+x\alpha d(y)\beta g(z)+x\alpha y\beta F(z),$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. This implies that

$$d(x)\alpha g(y)\beta F(z) = d(x)\alpha g(y)\beta g(z),$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Since g is onto, we obtain

$$d(x)\alpha y\beta(F(z) - z) = 0,$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Thus,

$$d(x)\Gamma M\Gamma(F(z) - z) = \{0\},$$

for all $x, z \in M$. Therefore, $d(M) = 0$ or $F(z) = z$ for all $z \in M$. In the later case, F is an identity map. On the other hand, assume that $d(M) = 0$. Then $F(x\alpha y) = F(x)\alpha y = F(x)\alpha F(y)$, that is, $F(x)\alpha(y - F(y)) = 0$ for all $x, y \in M$ and $\alpha \in \Gamma$. Replacing y by $z\beta y, z \in M$, and noting that $F(z\beta y) = z\beta F(y)$, we have $F(x)\alpha z\beta(y - F(y)) = 0$, for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. That is, $F(x)\Gamma M\Gamma(y - F(y)) = \{0\}$, for all $x, y \in M$. Therefore $F(M) = \{0\}$ or F is an identity map. □

Theorem 3.12. *Let M be a 2-torsion free prime Γ-ring satisfying the condition (*). Suppose that F is a generalized semiderivation of M associated with a nonzero semiderivation d and an onto map g associated with d such that $g(x\alpha y) = g(x)\alpha g(y)$. If F acts as an antihomomorphism on M, then either F is an identity map or $F = 0$ and M is commutative.*

Proof. By the hypothesis, we have

$$F(x\alpha y) = d(x)\alpha g(y) + x\alpha F(y) = F(y)\alpha F(x), \tag{20}$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Thus

$$F(y)\alpha F(x) = d(x)\alpha g(y) + x\alpha F(y) = F(y)\alpha F(x),$$

for all $x, y \in M$ and $\alpha \in \Gamma$. Replacing y by $x\beta y$ in the above relation, we get

$$F(x\beta y)\alpha F(x) = d(x)\alpha g(x\beta y) + x\alpha F(x\beta y)$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. That is,

$$(d(x)\beta g(y) + x\beta F(y))\alpha F(x) = d(x)\alpha g(x\beta y) + x\alpha F(x\beta y)$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. This implies that $d(x)\beta g(y)\alpha F(x) + x\beta F(y)\alpha F(x) = d(x)\alpha g(x\beta y) + x\alpha F(x\beta y)$ for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Hence we get

$$d(x)\beta g(y)\alpha F(x) + x\beta F(y)\alpha F(x) = d(x)\alpha g(x\beta y) + x\alpha F(y)\beta F(x),$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, which implies that

$$d(x)\beta g(y)\alpha F(x) = d(x)\alpha g(x)\beta g(y) \tag{21}$$

for all $x, y \in M$ and $\alpha, \beta \in \Gamma$. Replacing y by $y\gamma t$ in the above relation, we get

$$d(x)\beta g(y\gamma t)\alpha F(x) = d(x)\alpha g(x)\beta g(y\gamma t)$$

for all $x, y, t \in M$ and $\alpha, \beta \in \Gamma$, and so

$$d(x)\beta g(y)\gamma g(t)\alpha F(x) = d(x)\alpha g(x)\beta g(y)\gamma g(t),$$

for all $x, y, t \in M$ and $\alpha, \beta \in \Gamma$. Using (21) in the above relation, we get

$$d(x)\beta g(y)\gamma g(t)\alpha F(x) = d(x)\beta g(y)\alpha F(x)\gamma g(t),$$

for all $x, y, t \in M$ and $\alpha, \beta \in \Gamma$. Since g is onto, we have

$$d(x)\Gamma M\Gamma[F(x), t]_\alpha = \{0\}$$

for all $x, t \in M$. Therefore either $d(M) = \{0\}$ or $F(M) \subseteq Z(M)$. Hence in either case, F acts as a homomorphism on M. Thus this completes the proof by Theorem 3.11 and Lemma 3.4. □

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