ON THE M-POWER CLASS (N) OPERATORS

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ABSTRACT. A Hilbert space operator $T \in \mathcal{B}(H)$ is said to be M-Power class $N$ if there is a real number $M > 0$ such that $\| (T - \lambda)^n x \|^2 \leq M \| (T - \lambda)^{2n} x \| \| x \|$ for all $\lambda > 0$ and all $x \in H$. In this paper we prove the following assertions: (1) $T$ is M-Power class $N$ if and only if $M^2 (T - \lambda)^{2n} (T - \lambda)^n + r^2 I \geq 0$ for all $r > 0$ and all $\lambda \in \mathbb{C}$. (2) If $T$ is invertible M-Power class $(N)$, then $T^{-1}$ is also M-Power class $(N)$. (3) If $T$ is partial isometry M-Power class $(N)$ satisfies $\| T - \lambda \| \leq \frac{1}{M}$ then it is subnormal. (4) If $T$ is M-Power class $(N)$, then $T$ is an isoloid.

1. Introduction

Let $H$ be a complex separable Hilbert space and let $\mathcal{B}(H)$ denote the algebra of all bounded linear operators on $H$. If $T \in \mathcal{B}(H)$, we write $\ker(T)$, $\mathcal{R}(T)$, $\sigma(T)$, and $\sigma_a(T)$ for the null space, the range space, the spectrum, and the approximate point spectrum of $T$, respectively. An operator $T \in \mathcal{B}(H)$ is said hyponormal if $\| Tx \| \geq \| T^n x \|$ for all $x \in H$ [5]. $T$ is called $M$-hyponormal if there exists a positive real number $M$ such that $\| (T - z)^x \| \leq M \| (T - z) x \|$ for all $x \in H$ and all $z \in \mathbb{C}$. The following definition of $M$-Power class $(N)$ also appear in [3].

**Definition 1** An operator $T \in \mathcal{B}(H)$ is said to be of $M$-power class $(N)$ ($T \in \mathcal{MP}C(N)$ for short) if

$\| (T - \lambda)^n x \|^2 \leq M \| (T - \lambda)^{2n} x \| \| x \|$

for all $x \in H$, all $n \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$.

If $M = 1$ and $n = 1$, then the $M$-Power class $(N)$ becomes the class of totally paranormal operators as studied by [4, 7] and [6]. The purpose of the present paper is to study certain properties of $M$-Power class $(N)$ operators.

2. Main Results

In this section, we study some properties of $M$-Power class $(N)$ operators. We begin with the following lemma which is characterize the class of $M$-Power class...
(\mathcal{N}) operators.

**Lemma 1** Let $T \in (B)(\mathbb{H})$. Then $T$ is $M$-Power class $(\mathcal{N})$ if and only if

$$M^2(T - \lambda)^{2n}(T - \lambda)^{2n} - 2r(T - \lambda)^n(T - \lambda)^n + r^2I \geq 0$$

for all real number $r > 0$ and all $\lambda \in \mathbb{C}$.

**Proof.** In elementary algebra, we know that for positive real numbers $A$, $B$ and $C$, $A - 2Br + r^2C \geq 0$ for all $r > 0$ if and only if $B^2 \leq AC$. Therefore, $T$ is $M$-Power class $(\mathcal{N})$ operator if and only if

$$\langle (M^2(T - \lambda)^{2n}(T - \lambda)^{2n} - 2r(T - \lambda)^n(T - \lambda)^n + r^2I)x, x \rangle \geq 0$$

for all $x \in \mathbb{H}$ if and only if $M^2\|(T - \lambda)^{2n}x\|^2 - 2r\|(T - \lambda)^n x\|^2 + r^2\|x\|^2 \geq 0$ for all $x \in \mathbb{H}$ if and only if $\|(T - \lambda)^n x\|^2 \leq M\|(T - \lambda)^{2n}x\|\|x\|$ for all $x \in \mathbb{H}$.

**Proposition 1** Let $T \in \mathbb{B}((\mathbb{H})$ be $M$-Power class $(\mathcal{N})$ operator. Then $T - \alpha$ and $\alpha T$ are $M$-Power class $(\mathcal{N})$ operators for each $\alpha \in \mathbb{C}$.

**Proof.** Suppose that $T$ is $M$-Power class $(\mathcal{N})$ operator. Then for all $x \in \mathbb{H}$, we have

$$\|(T - \lambda)^n x\|^2 = \|(T - (\alpha + \lambda))^{\alpha}\|x\|^2 \leq M\|(T - (\alpha + \lambda))^{2n}\|\|x\| = \|(T - \lambda)^{2n}\|x\|.$$ 

Hence $T - \alpha$ is $M$-Power class $(\mathcal{N})$ operator. Now, To prove $\alpha T$ is $M$-Power class $(\mathcal{N})$ operator, we consider two cases:

Case I: If $\alpha = 0$, then $\alpha T = 0$ and so its $M$-Power class $(\mathcal{N})$ operator.

Case II: If $\alpha \neq 0$, then for all $x \in \mathbb{H}$

$$\|(\alpha T - \lambda)^n x\|^2 = |\alpha|^{2n}\|(T - \frac{\lambda}{\alpha})^n x\|^2 \leq |\alpha|^{2n}M_{\lambda/\alpha}\|(T - \frac{\lambda}{\alpha})^2n x\|\|x\| \leq M\|\|(\alpha T - \lambda)^{2n} x\|\|x\|.$$ 

Hence $\alpha T$ is $M$-Power class $(\mathcal{N})$ operator.

**Corollary 1** Let $T$ be a weighted shift with weights $\{\alpha_n\}$. Then $T$ satisfies the inequality $\|T^n x\| \leq M\|T^{2n}x\|$ if and only if

$$|\alpha_m \cdots \alpha_{m+n-1}| \leq M|\alpha_{m+n} \alpha_{m+n-1} \cdots \alpha_{m+2n-1}|$$

for all $m \in \mathbb{N}$.

**Proposition 2** Let $T \in (B)(\mathbb{H})$ be an $M$-Power class $(\mathcal{N})$ operator. If $\sigma(T) = \{\lambda\}$, then $T = \lambda$.

**Proof.** Suppose that $T$ is an $M$-Power class $(\mathcal{N})$ operator. Then $T$ has invariant translation property. But every quasinilpotent $M$-Power class $(\mathcal{N})$ operator is zero operator [1], hence $T - \lambda = 0$ and so $T = \lambda$. 
Proposition 3 If $T$ is invertible belongs to $\mathcal{MP}(N)$, then $T^{-1}$ is also belongs to $\mathcal{MP}(N)$.

Proof. We have
$$M \|(T - \lambda)^{2n}x\| \geq \|(T - \lambda)^n x\|^2$$
for each $x$ with $\|x\| = 1$. This can be replaced by
$$\frac{M \|x\|}{\|(T - \lambda)^n x\|} \geq \frac{\|(T - \lambda)^n x\|}{\|(T - \lambda)^{2n} x\|}$$
for each $x \in \mathbb{H}$ and all $\lambda \in \mathbb{C}$. Now replace $x$ by $(T - \lambda)^{-2n}x$, then
$$M \|x\| \|(T - \lambda)^{-2n}\| \geq \|(T - \lambda)^{-n} x\|^2$$
for each $x \in \mathbb{H}$ and all $\lambda \in \mathbb{C}$. This shows that $T^{-1}$ is M-Power class $(N)$.

Theorem 1 Let $\alpha$ be a non-zero eigenvalue of an M-Power class $(N)$ operator and $T = \begin{pmatrix} \alpha & A \\ 0 & B \end{pmatrix}$ on $\mathbb{H} = \ker(T - \alpha) \oplus \overline{\mathcal{R}(T - \alpha)}^* \ 2 \times 2$ expression.
Then $\|A(B - 1)^{-1}x\|^2 + \|(B - 1)^n x\|^2 \leq M \|(B - 1)^{2n}x\|$ for every unit vector $x \in \overline{\mathcal{R}(T - \alpha)}$. In particular $B$ belongs to $\mathcal{MP}(N)$.

Proof. Without loss of generality, we may assume $\alpha = 1$. By Lemma 2, $T$ satisfies
$$M^2(T - 1)^*2n(T - 1)^{2n} - 2r(T - 1)^*n(T - 1)^n + r^2 I \geq 0$$
for all $r > 0$. Set $S := T - 1$. Then
$$0 \leq M^2 S^{2n} S^{2n} - 2r S^{n} S^n + r^2$$
$$= \begin{pmatrix} M^2 r^2 \\ 0 & M^2 B_1^{(2n-1)*} A^* A B_1^{2n-1} - 2r B_1^{(n-1)*} A^* AB_1^{(n-1)} + M^2 B_1^{2n*} B_1^{2n} - 2r B_1^{n*} B_1^n + r^2 \end{pmatrix},$$
where $B_1 = B - 1$. Recall the above characterization of positive $2 \times 2$ matrix with operator entries. For each $r \neq 1$ there exists a contraction $D(r)$ such that
$$AB_1^{2n-1} = D(r)(L(r))^\frac{1}{2},$$
where $L(r) = M^2 B_1^{(2n-1)*} A^* AB_1^{2n-1} - 2r B_1^{(n-1)*} (A^* A + B^* B) B_1^{(n-1)} + M^2 B_1^{2n*} B_1^n + r^2$. Since $(L(r))^\frac{1}{2} D(r)^* D(r)(L(r))^\frac{1}{2} \leq L(r)$, we have
$$M^2 B_1^{2n*} B_1^{2n} - 2r B_1^{(n-1)*} (A^* A + B^* B) B_1^{n-1} + r^2 \geq 0$$
for every $r \neq 1$. Since the left of the above inequality is norm continuous as a function of $r$, that inequality holds for every $r > 0$. For every unit vector $x \in \overline{\mathcal{R}(T - 1)}^*$,
$$0 \leq r^2 - 2r(\|AB^{n-1}x\|^2 + \|B_1^n x\|^2) + M^2 \|B_1^{2n} x\|^2$$
for all $r > 0$. This is equivalent to
$$\|A(B - 1)^{n-1}x\|^2 + \|(B - 1)^n x\|^2 \leq M \|(B - 1)^{2n}x\|.$$
This completes the proof.

The sum of two $M$-Power class $(N)$ even commuting or double commuting $(A$ and $B$ are said to be double commuting if $A$ commutes with $B$ and $B^*)$ operators may not be $M$-Power class $(N)$ as can seen by the following example:
Example 1 Let
\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}
\]
be operators on 2-dimensional space. Then \(T\) and \(S\) are both \(\sqrt{2}\)-Power class \((N)\) while \(T + S\) is not so.

Theorem 2 Let \(T \in \mathcal{B}(\mathbb{H})\). Suppose that \(T\) belongs to \(\mathcal{MP}(N)\) and \(S = (T - \lambda)^n\) is partial isometry satisfies \(\|S\| \leq \frac{1}{M}\). Then \(T\) is subnormal.

Proof. Since \(S\) is partial isometry, \(SS^*S = S\) [2, Corollary 3, Problem 98], also \(T \in \mathcal{MP}(N)\), therefore by Lemma 2
\[
M^2S^*S^2 - 2rS^*S + r^2 \geq 0
\]
for each \(r > 0\). Using \(SS^*S = S\) we obtain
\[
M^2S^*S^2 - 2rS^*S + r^2 = S^*S(M^2S^*S^2 - 2rS^*S + r^2)S \geq 0.
\]
This is true for each \(r > 0\) and hence for \(r = 1\),
\[
M^2S^*S^2 - S^*S \geq 0.
\]
This means
\[
\|S\| = M^2 \|S^2\| \leq M^2 \|S\|^2 \|S\| \leq \|S\|^2
\]
because \(\|S\| \leq \frac{1}{M}\). This shows
\[
S^*S = M^2S^*S^2
\]
which on repeated us yields \(S^*S = M^2(m-1)S^mS^m\) for each \(m \geq 1\). Now let \(x_0, x_1, \ldots, x_m\) be a finite collection of vectors in \(\mathbb{H}\)
\[
M^4m \sum_{i,j=0}^m \langle S^{i+j}x_i, S^{i+j}x_j \rangle = \sum_{i,j=0}^m M^{4m-2(i+j-1)} \langle M^{2(i+j-1)}S^*(i+j)S^{i+j}x_i, x_j \rangle
\]
\[
= \sum_{i,j=0}^m M^{2n+1-i-j} \langle S^*Sx_i, x_j \rangle
\]
Since \(S^*S\) is a projection [2, Problem 98], we obtain
\[
M^4m \sum_{i,j=0}^m \langle S^{i+j}x_i, S^{i+j}x_j \rangle = \sum_{i,j=0}^m M^{2(2m+1-i-j)} \langle (S^*S)^{i+j}x_i, (S^*S)^{i+j}x_j \rangle
\]
\[
= M^{2(2m+1)} \langle x_0, x_0 \rangle + M^{4m} \sum_{i,j=1}^1 \langle (S^*S)x_i, (S^*S)x_j \rangle
\]
\[
+ M^{2(2m-1)} \sum_{i,j=2}^2 \langle (S^*S)^2x_i, (S^*S)^2x_j \rangle + \cdots +
\]
\[
+ M^{2} \sum_{i,j=2m}^{2m} \langle (S^*S)^{2m}x_i, (S^*S)^{2m}x_j \rangle.
\]
As \(M \geq 1\), we get that
\[
M^{2(2m+1)} \langle x_0, x_0 \rangle \geq M^{4m} \langle x_0, x_0 \rangle.
\]
Thus
\[ M^{2(2m+1)} \langle x_0, x_0 \rangle + M^{4m} \sum_{i,j=1}^1 \langle (S^* S)x_i, (S^* S)x_j \rangle \geq M^{4m} \langle x_0, x_0 \rangle + M^{4m} \sum_{i,j=1}^1 \langle (S^* S)x_i, (S^* S)x_j \rangle \]
\[ = M^{4m} \sum_{i,j=0}^1 \langle (S^* S)^{i+j}x_i, (S^* S)^{i+j}x_j \rangle \geq 0, \]
since \( S^* S \) being self-adjoint is subnormal. Again
\[ M^{4m} \sum_{i,j=0}^1 \langle (S^* S)^{i+j}x_i, (S^* S)^{i+j}x_j \rangle \geq M^{2(2m-1)} \sum_{i,j=0}^1 \langle (S^* S)^{i+j}x_i, (S^* S)^{i+j}x_j \rangle. \]
Hence
\[ M^{2(2m+1)} \langle x_0, x_0 \rangle + M^{4m} \sum_{i+j=1}^{i+j=1} \langle (S^* S)x_i, (S^* S)x_j \rangle + M^{2(2m-1)} \sum_{i+j=2}^{i+j=2} \langle (S^* S)^2x_i, (S^* S)^2x_j \rangle \]
\[ \geq M^{2(2m-1)} \sum_{i,j=0}^1 \langle (S^* S)^{i+j}x_i, (S^* S)^{i+j}x_j \rangle \]
\[ + M^{2(2m-1)} \sum_{i+j=2}^2 \langle (S^* S)^2x_i, (S^* S)^2x_j \rangle \]
\[ = M^{2(2m-1)} \sum_{i,j=0}^2 \langle (S^* S)^{i+j}x_i, (S^* S)^{i+j}x_j \rangle \geq 0. \]
Continuing in this way, we would have
\[ M^{4m} \sum_{i,j=0}^m \langle S^{i+j}x_i, S^{i+j}x_j \rangle \geq M^2 \sum_{i,j=0}^m \langle (S^* S)^{i+j}x_i, (S^* S)^{i+j}x_j \rangle. \]
This gives
\[ \sum_{i,j=0}^m \langle S^{i+j}x_i, S^{i+j}x_j \rangle \geq 0. \]
Hence \( S \) is subnormal and consequently \( T \) is subnormal.

**Proposition 3** Let \( T \in B(\mathbb{H}) \). If \( T \) belongs to \( \mathcal{MP}(N) \) and \( \mathfrak{M} \) is an invariant subspace for \( T \), then \( T|_{\mathfrak{M}} \) belongs to \( \mathcal{MP}(N) \).

**Proof.** Since \( T \) has the invariant translation property, we may assume \( \lambda = 0 \). Let \( P \) be the orthogonal projection onto \( \mathfrak{M} \). Then \( TP = PTP \), so that \( T|_{\mathfrak{M}} = PTP \). Hence, for \( x \in \mathfrak{M} \) we have
\[ \| (T|_{\mathfrak{M}})^n \|^2 = \| P T^n x \|^2 \leq \| T^n x \|^2 \leq \| T^{2n} x \| \| x \| = \| (T|_{\mathfrak{M}})^{2n} x \| \| x \|. \]
Thus \( T|_{\mathfrak{M}} \in \mathcal{MP}(N) \).

**Theorem 3** Any isolated point in the spectrum of an an \( M \)-Power class \((N)\) operator is its eigenvalue.
Proof. Since $T - z$ is an $M$-Power class $(N)$ for each complex number $z$, therefore we can assume the isolated point in the spectrum $\sigma(T)$ to be zero. Choose $R > 0$ such that the only point of $\sigma(T)$ strictly within $\{z : |z| = R\}$ is zero and $\{z : |z| = R\} \cap \sigma(T) = \emptyset$. Set

$$E = \int_{|z|=R} \frac{1}{T - z} \, dz.$$ 

Then $E$ is a non-zero projection commuting with $T$ and hence its range span $N$ is invariant under $T$. This implies that $T|_N$ is an $M$-Power class $(N)$ by Proposition 2. Also then

$$\sigma(T|_N) = \sigma(T) \cap \{z : |z| < R\} = \{0\}.$$ 

Thus $T|_N$ is an $M$-Power class $(N)$ quasinilpotent operator by Proposition 2 is zero. Let $0 \neq x \in N$. Then $Tx = 0$. This proves the theorem.

The following example shows the $m$-Power class $N$ contains the class of hyponormal properly.

**Example 2** Let $\{e_i\}$ be an orthonormal basis for $\mathbb{H}$, and define

$$Te_i = \begin{cases} e_2, & \text{if } i = 1; \\ 2e_3, & \text{if } i = 2; \\ e_{i+1}, & \text{if } i \geq 3. \end{cases}$$

That is, $T$ is a weight shift. From the definition of $T$ we see that $T$ is similar to the unilateral shift $U$ ([2, Problem 90]). Thus there exists an operator $S$ such that $T = SUS^{-1}$. In our case $\|S\| = 2, \|S^{-1}\| = 1$. Since $U$ is the unilateral shift, $U$ is hyponormal operator, and thus for any $n$ and $\lambda \in \mathbb{C}$ the operator $(U - \lambda)^n$ is a paranormal operator. It follows that

$$\|(U - \lambda)^n x\|^2 \leq \|(U - \lambda)^{2n} x\|^2$$

for all $x \in \mathbb{H}$ with $\|x\| = 1$, and hence $T$ belongs to $\mathcal{MP}(N)$ with $M = 4$.

**References**


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