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ON THE M-POWER CLASS (N) OPERATORS

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Abstract. A Hilbert space operator $T \in \mathbb{B}(\mathbb{H})$ is said to be M-Power class N if there is a real number $M > 0$ such that $\|(T-\lambda)^n x\|^2 \leq M \left\|(T-\lambda)^{2n} x\right\| \|x\|$ for all $\lambda > 0$ and all $x \in \mathbb{H}$. In this paper we prove the following assertions:(1) T is M-Power class N if and only if $M^2(T-\lambda)^{*2n}(T-\lambda)^{2n} - 2r(T-\lambda)^{*n}(T-\lambda)^n + r^2 I \geq 0$ for all $r > 0$ and all $\lambda \in \mathbb{C}$. (2) If T is invertible M-Power class (N), then T^{-1} is also M-Power class (N). (3) If T is partial isometry M-Power class (N) satisfies $\|T-\lambda\| \leq \dfrac{1}{M}$, then it is subnormal (4)If T is M-Power class (N), then T is an isoloid.

1. Introduction

Let $\mathbb{H}$ be a complex separable Hilbert space and let $\mathbb{B}(\mathbb{H})$ denote the algebra of all bounded linear operators on $\mathbb{H}$. If $T \in \mathbb{B}(\mathbb{H})$, we write $\ker(T)$, $\mathcal{R}(T)$, $\sigma(T)$, and $\sigma_a(T)$ for the null space, the range space, the spectrum, and the approximate point spectrum of T, respectively. An operator $T \in \mathbb{B}(\mathbb{H})$ is said hyponormal if $\|Tx\| \geq \|T^* x\|$ for all $x \in \mathbb{H}$ [5]. T is called M-hyponormal if there exists a positive real number M such that $\|(T-z)^* x\| \leq M \|(T-z)x\|$ for all $x \in \mathbb{H}$ and all $z \in \mathbb{C}$. The following definition of M-Power class (N) also appear in [3].

Definition 1 An operator $T \in \mathbb{B}(\mathbb{H})$ is said to be of M-power class (N) ($T \in \mathcal{M}\mathbf{P}\mathfrak{C}(N)$ for short) if

$$\|(T-\lambda)^n x\|^2 \leq M \left\|(T-\lambda)^{2n} x\right\| \|x\|$$

for all $x \in \mathbb{H}$, all $n \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$.

If $M = 1$ and $n = 1$, then the M-Power class (N) becomes the class of totally paranormal operators as studied by [4, 7] and [6]. The purpose of the present paper is to study certain properties of M-Power class (N) operators.

2. **Main Results**

In this section, we study some properties of M-Power class (N) operators. We begin with the following lemma which is characterize the class of M-Power class

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(N) operators.

Lemma 1 Let $T \in (B)(\mathbb{H})$. Then T is M-Power class (N) if and only if

$$M^2(T-\lambda)^{*2n}(T-\lambda)^{2n} - 2r(T-\lambda)^{*n}(T-\lambda)^n + r^2 I \geq 0$$

for all real number $r > 0$ and all $\lambda \in \mathbb{C}$.

Proof. In elementary algebra, we know that for positive real numbers A, B and C, $A - 2Br + r^2C \geq 0$ for all $r > 0$ if and only if $B^2 \leq AC$. Therefore, T is M-Power class (N) operator if and only if

$$\left\langle (M^2(T-\lambda)^{*2n}(T-\lambda)^{2n} - 2r(T-\lambda)^{*n}(T-\lambda)^n + r^2 I)x, x \right\rangle \geq 0$$

for all $x \in \mathbb{H}$ if and only if $M^2 \left\|(T-\lambda)^{2n}x\right\|^2 - 2r\left\|(T-\lambda)^n x\right\|^2 + r^2\left\|x\right\|^2 \geq 0$ for all $x \in \mathbb{H}$ if and only if $\left\|(T-\lambda)^n x\right\|^2 \leq M\left\|(T-\lambda)^{2n}x\right\|\left\|x\right\|$ for all $x \in \mathbb{H}$.

Proposition 1 Let $T \in \mathbb{B}(\mathbb{H})$ be M-Power class (N) operator. Then $T - \alpha$ and αT are M-Power class (N) operators for each $\alpha \in \mathbb{C}$.

Proof. Suppose that T is M-Power class (N) operator. Then for all $x \in \mathbb{H}$, we have

$$\begin{aligned}\left\|[(T-\alpha)-\lambda]^n x\right\|^2 &= \left\|(T-(\alpha+\lambda))^n x\right\|^2 \\ &\leq M\left\|(T-(\alpha+\lambda))^{2n}x\right\|\left\|x\right\| = \left\|[(T-\alpha)-\lambda]^{2n}x\right\|\left\|x\right\|.\end{aligned}$$

Hence $T-\alpha$ is M-Power class (N) operator. Now, To prove αT is M-Power class (N) operator, we consider two cases:
Case I: If $\alpha = 0$, then $\alpha T = 0$ and so its M-Power class (N) operator.
Case II: If $\alpha \neq 0$, then for all $x \in \mathbb{H}$

$$\begin{aligned}\left\|(\alpha T-\lambda)^n x\right\|^2 &= |\alpha|^{2n}\left\|(T-\frac{\lambda}{\alpha})^n x\right\|^2 \\ &\leq |\alpha|^{2n} M_{\lambda/\alpha}\left\|(T-\frac{\lambda}{\alpha})^{2n}x\right\|\left\|x\right\| \\ &\leq M\left\|(\alpha T-\lambda)^{2n}x\right\|\left\|x\right\|.\end{aligned}$$

Hence αT is M-Power class (N) operator.

Corollary 1 Let T be a weighted shift with weights $\{\alpha_n\}$. Then T satisfies the inequality $\left\|T^n x\right\| \leq M\left\|T^{2n}x\right\|$ if and only if

$$|\alpha_m \cdots \alpha_{m+n-1}| \leq M|\alpha_{m+n}\alpha_{m+n-1}\cdots\alpha_{m+2n-1}|$$

for all $m \in \mathbb{N}$.

Proposition 2 Let $T \in (B)(\mathbb{H})$ be an M-Power class (N) operator. If $\sigma(T) = \{\lambda\}$, then $T = \lambda$.

Proof. Suppose that T is an M-Power class (N) operator. Then T has invariant translation property. But every quasinilpotent M-Power class (N) operator is zero operator [1], hence $T - \lambda = 0$ and so $T = \lambda$.

Proposition 3 If T is invertible belongs to $\mathcal{M}\mathbf{P}\mathfrak{C}(N)$, then T^{-1} is also belongs to $\mathcal{M}\mathbf{P}\mathfrak{C}(N)$.

Proof. We have

$$M\left\|(T-\lambda)^{2n}x\right\| \geq \|(T-\lambda)^n x\|^2$$

for each x with $\|x\| = 1$. This can be replaced by

$$\frac{M\|x\|}{\|(T-\lambda)^n x\|} \geq \frac{\|(T-\lambda)^n x\|}{\|(T-\lambda)^{2n} x\|}$$

for each $x \in \mathbb{H}$ and all $\lambda \in \mathbb{C}$. Now replace x by $(T-\lambda)^{-2n}x$, then

$$M\|x\|\left\|(T-\lambda)^{-2n}\right\| \geq \left\|(T-\lambda)^{-n}x\right\|^2$$

for each $x \in \mathbb{H}$ and all $\lambda \in \mathbb{C}$. This shows that T^{-1} is M-Power class (N).

Theorem 1 Let α be a non-zero eigenvalue of an M-Power class (N) operator and $T = \begin{pmatrix} \alpha & A \\ 0 & B \end{pmatrix}$ on $\mathbb{H} = \ker(T-\alpha) \oplus \overline{\mathcal{R}(T-\alpha)^*}$ be 2×2 expression. Then $\left\|A(B-1)^{n-1}x\right\|^2 + \|(B-1)^n x\|^2 \leq M\left\|(B-1)^{2n}x\right\|$ for every unit vector $x \in \overline{\mathcal{R}(T-\alpha)}$. In particular B belongs to $\mathcal{M}\mathbf{P}\mathfrak{C}(N)$.

Proof. Without loss of generality, we may assume $\alpha = 1$. By Lemma 2, T satisfies

$$M^2(T-1)^{*2n}(T-1)^{2n} - 2r(T-1)^{*n}(T-1)^n + r^2 I \geq 0$$

for all $r > 0$. Set $S := T - 1$. Then

$$\begin{aligned} 0 \leq & M^2 S^{2n*} S^{2n} - 2r S^{n*} S^n + r^2 \\ = & \begin{pmatrix} M^2 r^2 & 0 \\ 0 & M^2 B_1^{(2n-1)*} A^* A B_1^{2n-1} - 2r B_1^{(n-1)*} A^* A B_1^{(n-1)} + M^2 B_1^{2n*} B_1^{2n} - 2r B_1^{n*} B_1^n + r^2 \end{pmatrix}, \end{aligned}$$

where $B_1 = B - 1$. Recall the above characterization of positive 2×2 matrix with operator entries. For each $r \neq 1$ there exists a contraction $D(r)$ such that $AB_1^{2n-1} = D(r)(L(r))^{\frac{1}{2}}$, where $L(r) = M^2 B_1^{(2n-1)*} A^* A B_1^{2n-1} - 2r B_1^{(n-1)*}(A^*A + B^*B)B_1^{(n-1)} + M^2 B_1^{2n*} B_1^{2n} + r^2$. Since $(L(r))^{\frac{1}{2}} D(r)^* D(r) (L(r))^{\frac{1}{2}} \leq L(r)$, we have

$$M^2 B_1^{2n*} B_1^{2n} - 2r B_1^{(n-1)*}(A^*A + B^*B)B^{n-1} + r^2 \geq 0$$

for every $r \neq 1$. Since the left of the above inequality is norm continuous as a function of r, that inequality holds for every $r > 0$. For every unit vector $x \in \overline{\mathcal{R}(T-1)^*}$,

$$0 \leq r^2 - 2r(\left\|AB^{n-1}x\right\|^2 + \|B_1^n x\|^2) + M^2\left\|B_1^{2n}x\right\|^2$$

for all $r > 0$. This is equivalent to

$$\left\|A(B-1)^{n-1}x\right\|^2 + \|(B-1)^n x\|^2 \leq M\left\|(B-1)^{2n}x\right\|.$$

This completes the proof.

The sum of two M-Power class (N) even commuting or double commuting (A and B are said to be double commuting if A commutes with B and B^*) operators may not be M-Power class (N) as can seen by the following example:

Example 1 Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } S = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

be operators on 2-dimensional space. Then T and S are both $\sqrt{2}$-Power class (N) while $T + S$ is not so.

Theorem 2 Let $T \in \mathbb{B}(\mathbb{H})$. Suppose that T belongs to $\mathcal{M}\mathbf{P}\mathfrak{C}(N)$ and $S = (T - \lambda)^n$ is partial isometry satisfies $\|S\| \leq \dfrac{1}{M}$. Then T is subnormal.

Proof. Since S is partial isometry, $SS^*S = S$ [2, Corollary 3, Problem 98], also $T \in \mathcal{M}\mathbf{P}\mathfrak{C}(N)$, therefore by Lemma 2

$$M^2 S^{*2} S^2 - 2rS^*S + r^2 \geq 0$$

for each $r > 0$. Using $SS^*S = S$ we obtain

$$M^2 S^{*2} S^2 - 2rS^*S + r^2 = S^*S[M^2 S^{*2} S^2 - 2rS^*S + r^2]S^*S \geq 0.$$

This is true for each $r > 0$ and hence for $r = 1$,

$$M^2 S^{*2} S^2 - S^*S \geq 0.$$

This means

$$\|Sx\|^2 \leq M^2 \left\|S^2 x\right\|^2 \leq M^2 \|S\|^2 \|Sx\|^2 \leq \|Sx\|^2$$

because $\|S\| \leq \dfrac{1}{M}$. This shows

$$S^*S = M^2 S^{*2} S^2$$

which on repeated us yields $S^*S = M^{2(m-1)} S^{*m} S^m$ for each $m \geq 1$. Now let $x_0, x_1, \cdots, x_m$ be a finite collection of vectors in $\mathbb{H}$

$$\begin{aligned} M^{4m} \sum_{i,j=0}^{m} \left\langle S^{i+j} x_i, S^{i+j} x_j \right\rangle &= \sum_{i,j=0}^{m} M^{4m-2(i+j-1)} \left\langle M^{2(i+j-1)} S^{*(i+j) S^{i+j} x_i, x_j} \right\rangle \\ &= \sum_{i,j=0}^{m} M^{[2n+1-i-j]} \left\langle S^*S x_i, x_j \right\rangle \end{aligned}$$

Since S^*S is a projection [2, Problem 98], we obtain

$$\begin{aligned} M^{4m} \sum_{i,j=0}^{m} \left\langle S^{i+j} x_i, S^{i+j} x_j \right\rangle &= \sum_{i,j=0}^{m} M^{2[2m+1-i-j]} \left\langle (S^*S)^{i+j} x_i, (S^*S)^{i+j} x_j \right\rangle \\ &= M^{2(2m+1)} \langle x_0, x_0 \rangle + M^{4m} \sum_{i,j=1}^{1} \left\langle (S^*S) x_i, (S^S) x_j \right\rangle \\ &+ M^{2(2m-1)} \sum_{i,j=2}^{2} \left\langle (S^*S)^2 x_i, (S^S)^2 x_j \right\rangle + \cdots + \\ &+ M^2 \sum_{i,j=2m}^{2m} \left\langle (S^*S)^{2m} x_i, (S^S)^{2m} x_j \right\rangle. \end{aligned}$$

As $M \geq 1$, we get that

$$M^{2(2m+1)} \langle x_0, x_0 \rangle \geq M^{4m} \langle x_0, x_0 \rangle .$$

Thus

$$M^{2(2m+1)}\langle x_0,x_0\rangle+M^{4m}\sum_{i,j=1}^{1}\langle (S^*S)x_i,(S^*S)x_j\rangle\geq M^{4m}\langle x_0,x_0\rangle+M^{4m}\sum_{i,j=1}^{1}\left\langle (S^*S)x_i,(S^S)x_j\right\rangle$$

$$=M^{4m}\sum_{i,j=0}^{1}\left\langle (S^*S)^{i+j}x_i,(S^*S)^{i+j}x_j\right\rangle\geq 0,$$

since S^*S being self-adjoint is subnormal. Again

$$M^{4m}\sum_{i,j=0}^{1}\left\langle (S^*S)^{i+j}x_i,(S^*S)^{i+j}x_j\right\rangle\geq M^{2(2m-1)}\sum_{i,j=0}^{1}\left\langle (S^*S)^{i+j}x_i,(S^*S)^{i+j}x_j\right\rangle.$$

Hence

$$M^{2(2m+1)}\langle x_0,x_0\rangle+M^{4m}\sum_{i+j=1}^{1}\langle (S^*S)x_i,(S^*S)x_j\rangle+M^{2(2m-1)}\sum_{i+j=2}^{2}\left\langle (S^*S)^2x_i,(S^S)^2x_j\right\rangle$$

$$\geq M^{2(2m-1)}\sum_{i,j=0}^{1}\left\langle (S^*S)^{i+j}x_i,(S^S)^{i+j}x_j\right\rangle$$

$$+M^{2(2m-1)}\sum_{i+j=2}^{2}\left\langle (S^*S)^2x_i,(S^S)^2x_j\right\rangle$$

$$=M^{2(2m-1)}\sum_{i,j=0}^{2}\left\langle (S^*S)^{i+j}x_i,(S^S)^{i+j}x_j\right\rangle\geq 0.$$

Continuing in this way, we would have

$$M^{4m}\sum_{i,j=0}^{m}\left\langle S^{i+j}x_i,S^{i+j}x_j\right\rangle\geq M^{2}\sum_{i,j=0}^{m}\left\langle (S^*S)^{i+j}x_i,(S^S)^{i+j}x_j\right\rangle.$$

This gives

$$\sum_{i,j=0}^{m}\left\langle S^{i+j}x_i,S^{i+j}x_j\right\rangle\geq 0.$$

Hence S is subnormal and consequently T is subnormal.

Proposition 3 Let $T\in\mathbb{B}(\mathbb{H})$. If T belongs to $\mathcal{M}\mathbf{P}\mathfrak{C}(N)$ and $\mathfrak{M}$ is an invariant subspace for T, then $T|_{\mathfrak{M}}$ belongs to $\mathcal{M}\mathbf{P}\mathfrak{C}(N)$.

Proof. Since T has the invariant translation property, we may assume $\lambda=0$. Let P be the orthogonal projection onto $\mathfrak{M}$. Then $TP=PTP$, so that $T|_{\mathfrak{M}}=PTP$. Hence, for $x\in\mathfrak{M}$ we have

$$\|(T|_{\mathfrak{M}})^n\|^2=\|PT^nx\|^2\leq\|T^nx\|^2\leq\left\|T^{2n}x\right\|\|x\|=\left\|(T|_{\mathfrak{M}})^{2n}x\right\|\|x\|.$$

Thus $T|_{\mathfrak{M}}\in\mathcal{M}\mathbf{P}\mathfrak{C}(N)$.

Theorem 3 Any isolated point in the spectrum of an an M-Power class (N) operator is its eigenvalue.

Proof. Since $T - z$ is an M-Power class (N) for each complex number z, therefore we can assume the isolated point in the spectrum $\sigma(T)$ to be zero. Choose $R > 0$ such that the only point of $\sigma(T)$ strictly within $\{z : |z| = R\}$ is zero and $\{z : |z| = R\} \cap \sigma(T) = \emptyset$. Set

$$E = \int_{|z|=R} \frac{1}{T - z}\, dz.$$

Then E is a non-zero projection commuting with T and hence its range span N is invariant under T. This implies that $T|_N$ is an M-Power class (N) by Proposition 2. Also then

$$\sigma(T|_N) = \sigma(T) \cap \{z : |z| < R\} = \{0\}.$$

Thus $T|_N$ is an M-Power class (N) quasinilpotent operator by Proposition 2 is zero. Let $0 \neq x \in N$. Then $Tx = 0$. This proves the theorem.

The following example shows the m-Power class N contains the class of hyponormal properly.

Example 2 Let $\{e_i\}$ be an orthonormal basis for $\mathbb{H}$, and define

$$Te_i = \begin{cases} e_2, & \text{if } i = 1; \\ 2e_3, & \text{if } i = 2; \\ e_{i+1}, & \text{if } i \geq 3. \end{cases}$$

That is, T is a weight shift. From the definition of T we see that T is similar to the unilateral shift U ([2, Problem 90]). Thus there exists an operator S such that $T = SUS^{-1}$. In our case $\|S\| = 2, \left\|S^{-1}\right\| = 1$. Since U is the unilateral shift, U is hyponormal operator, and thus for any n and $\lambda \in \mathbb{C}$ the operator $U - \lambda)^n$ is a paranormal operator. It follows that

$$\|(U - \lambda)^n x\|^2 \leq \left\|(U - \lambda)^{2n} x\right\|$$

for all $x \in \mathbb{H}$ with $\|x\| = 1$, and hence T belongs to $\mathcal{M}\mathbf{P}\mathfrak{C}(N)$ with $M = 4$.

References

[1] S.C. Arora, Ramesh Kumar, M-Paranormal operators, Publ. Inst. Math., Nouvelle serie 29, 43 (1981), 5–13.

[2] P.R. Halmos, A Hilbert Space Problem Book, Van Nostrand, Princeton, 1967.

[3] V. Istratescu, Some result on M-hyponormal operators, Math. Seminar Notes, 6 (1978), 77–86.

[4] K. B. Laursen, Operators with finite ascent, Pacific J. Math., 152 (1992), 323–336.

[5] J. G. Stampfli, Hyponormal operators and spectrum density, Trans. Amer. Math. Soc., 117 (1965), 469476.

[6] C.Schmoeger, On totally paranormal operators, Bull. Austral. Math. Soc., 66 (2002), 425–441.

[7] K. Tanahashi and A. Uchiyama, A note on $*$-paranormal operators and related classes of operators, Bull. Korean Math. Soc., 51 (2014), no.2, 357–371.

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