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PERIODIC CHARACTER AND BOUNDEDNESS NATURE OF POSITIVE SOLUTIONS OF A MAX-TYPE SYSTEM OF DIFFERENCE EQUATIONS

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ABSTRACT. In this paper, we investigate the periodic character and the boundedness nature of the positive solutions of the system of max-type difference equations

$$x_{n+1} = \frac{\max\{A, y_n\}}{x_{n-1}}, \quad y_{n+1} = \frac{\max\{A, x_n\}}{y_{n-1}}, \quad n = 0, 1, ...,$$

where the parameter A and the initial conditions x_{-1}, y_{-1}, x_0, and y_0 are positive real numbers.

1. Introduction

max-type difference equations appeared for the first time in control theory. Recently, there is an increasing interest in the study of systems of max-type difference equations see, for example, [2], [5], [8], [13], [14].
In [3] the authors investigated global behavior of solutions of the max-type equation

$$x_{n+1} = \frac{\max\{A, x_n\}}{x_{n-1}}, \quad n = 0, 1, ..., \tag{1}$$

where the parameter A is a positive real number and the initial conditions x_{-1}, and x_0 are arbitrary positive constants. Inspired and motivated by above mentioned paper, our aim in this paper is to investigate the periodic character and boundedness nature of positive solutions of the system

$$x_{n+1} = \frac{\max\{A, y_n\}}{x_{n-1}}, \quad y_{n+1} = \frac{\max\{A, x_n\}}{y_{n-1}}, \quad n = 0, 1, ..., \tag{2}$$

where the parameter A and the initial conditions x_{-1}, y_{-1}, x_0, and y_0 are positive real numbers.
We note that if $x_{-1} = y_{-1}$ and $x_0 = y_0$ then system (2) reduces to the Eq.(1).

Let I and J be two intervals of real numbers and

$$f : I^2 \times J^2 \to I, \quad g : I^2 \times J^2 \to J,$$

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be two continuously differentiable functions. Then for every set of initial conditions $(x_i, y_i) \in I \times J,\ \ i = -1, 0$, the system of difference equations

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \\ y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}), \end{cases} \qquad n = 0, 1, ..., \tag{3}$$

has a unique solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$.
we need the following definitions.
Definition 1 A point $(\overline{x}, \overline{y}) \in I \times J$ is called an equilibrium point of system (3) if

$$\overline{x} = f(\overline{x}, \overline{x}, \overline{y}, \overline{y}) \text{ and } \overline{y} = g(\overline{x}, \overline{x}, \overline{y}, \overline{y}).$$

Definition 2 A solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (3) is called positive if $x_n > 0$ and $y_n > 0$ for all $n \geq -1$.
Definition 3 A solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (3) is said to be periodic with period p if there is an integer $p \geq 1$ such that $x_{n+p} = x_n$ and $y_{n+p} = y_n$ for all $n \geq -1$.
Definition 4 A solution $\{(x_n, y_n)\}_{n=-1}^{\infty}$ of system (3) is said that is bounded if there exists a positive constant M such that $\| (x_n, y_n) \|_{IR^2} \leq M$.

2. Boundedness nature of solutions

The following lemma will be used in the proof of Theorem 1.
Lemma 1 Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (2), and for $n \geq 0$, set $I_n = \max\{1, \frac{1}{x_{n-1}}\} \times \max\{1, \frac{1}{y_{n-1}}\} \times \max\{1, \frac{1}{x_n}\} \times \max\{1, \frac{1}{y_n}\} \times \max\{A, x_n, y_{n-1}\}$ $\times \max\{A, y_n, x_{n-1}\}$. Then $I_n = I_0$ for all $n \geq 0$.
Proof. Observe that for $n \geq 0$,

$$\begin{aligned} I_{n+1} &= \max\{1, \frac{1}{x_n}\} \times \max\{1, \frac{1}{y_n}\} \times \max\{1, \frac{1}{x_{n+1}}\} \times \max\{1, \frac{1}{y_{n+1}}\} \\ &\quad \times \max\{A, x_{n+1}, y_n\} \times \max\{A, y_{n+1}, x_n\} \\ &= \max\{1, \frac{1}{x_n}\} \times \max\{1, \frac{1}{y_n}\} \times \max\{1, \frac{x_{n-1}}{\max\{A, y_n\}}\} \times \max\{1, \frac{y_{n-1}}{\max\{A, x_n\}}\} \\ &\quad \times \max\{A, y_n, \frac{\max\{A, y_n\}}{x_{n-1}}\} \times \max\{A, x_n, \frac{\max\{A, x_n\}}{y_{n-1}}\} \\ &= \max\{1, \frac{1}{x_n}\} \times \max\{1, \frac{1}{y_n}\} \frac{\max\{\max\{A, y_n\}, x_{n-1}\}}{\max\{A, y_n\}} \times \frac{\max\{\max\{A, x_n\}, y_{n-1}\}}{\max\{A, x_n\}} \\ &\quad \times \max\{\max\{A, y_n\}, \frac{\max\{A, y_n\}}{x_{n-1}}\} \times \max\{\max\{A, x_n\}, \frac{\max\{A, x_n\}}{y_{n-1}}\} \\ &= \max\{1, \frac{1}{x_n}\} \times \max\{1, \frac{1}{y_n}\} \frac{1}{\max\{A, y_n\}} \times \max\{A, y_n, x_{n-1}\} \\ &\quad \times \frac{1}{\max\{A, x_n\}} \times \max\{A, x_n, y_{n-1}\} \times \max\{A, y_n\} \times \max\{1, \frac{1}{x_{n-1}}\} \\ &\quad \times \max\{A, x_n\} \times \max\{1, \frac{1}{y_{n-1}}\}. \end{aligned}$$

Hence

$$I_{n+1} = I_n \text{ for all } n \geq 0.$$

Then

$$I_n = I_0 \text{ for all } n \geq 0.$$

Theorem 1 Every solution of system (2) is bounded. Furthermore,

$$(x_n, y_n) \in [\frac{1}{I_0}, \frac{1}{A}I_0] \times [\frac{1}{I_0}, \frac{1}{A}I_0], \text{ for all } n \geq 1,$$

where

$$I_0 = \max\{1, \frac{1}{x_{-1}}\} \times \max\{1, \frac{1}{y_{-1}}\} \times \max\{1, \frac{1}{x_0}\} \times \max\{1, \frac{1}{y_0}\}$$
$$\times \max\{A, x_0, y_{-1}\} \times \max\{A, y_0, x_{-1}\}.$$

Proof. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (2). We have

$$\max\{A, x_n\} \times \max\{1, \frac{1}{x_{n-1}}\} \times \max\{1, \frac{1}{y_{n-1}}\} \times \max\{1, \frac{1}{x_n}\}$$
$$\times \max\{1, \frac{1}{y_n}\} \times \max\{A, x_n, y_{n-1}\} \times \max\{A, y_n, x_{n-1}\} \geq y_{n-1}.$$

Then

$$\frac{\max\{A, x_n\}}{y_{n-1}} \geq \frac{1}{I_n} \text{ for all } n \geq 0,$$

so,

$$y_{n+1} \geq \frac{1}{I_0} \text{ for all } n \geq 0.$$

Similarly, we obtain

$$x_{n+1} \geq \frac{1}{I_0} \text{ for all } n \geq 0.$$

We have also

$$A \times \max\{A, x_n\} \leq \times \max\{A, y_n, x_{n-1}\} \max\{A, x_n, y_{n-1}\} \times \max\{1, \frac{1}{x_n}\}$$
$$\times \max\{1, \frac{1}{y_n}\} \times \max\{1, \frac{1}{x_{n-1}}\},$$

and

$$\frac{1}{y_{n-1}} \leq \max\{1, \frac{1}{y_{n-1}}\},$$

then

$$y_{n+1} \leq \frac{1}{A}I_n \text{ for all } n \geq 0.$$

Hence

$$y_{n+1} \leq \frac{1}{A}I_0 \text{ for all } n \geq 0.$$

Similarly, we get

$$x_{n+1} \leq \frac{1}{A}I_0 \text{ for all } n \geq 0.$$

Hence

$$(x_n, y_n) \in [\frac{1}{I_0}, \frac{1}{A}I_0] \text{ for all } n \geq 1.$$

Lemma 2 System (2) has a unique positive equilibrium point $(\overline{x}, \overline{y})$. Furthermore,

$$(\overline{x}, \overline{y}) = \begin{cases} (1, 1) & \text{if } \quad A \leq 1 \\ (\sqrt{A}, \sqrt{A}) & \text{if } \quad A > 1. \end{cases}$$

Lemma 3 Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (2), and suppose that there exists $N \geq -1$ such that $(x_n, y_n) = (\overline{x}, \overline{y})$ for all $n \geq N$. Then $(x_n, y_n) = (\overline{x}, \overline{y})$ for all $n \geq -1$.

Proof.

- Suppose $A > 1$. Then
$$x_{N+1} = \frac{\max\{A, y_N\}}{x_{N-1}} = \overline{x} \text{ and } y_{N+1} = \frac{\max\{A, x_N\}}{y_{N-1}} = \overline{y},$$
hence
$$x_{N-1} = \sqrt{A} = \overline{x} \text{ and } y_{N-1} = \sqrt{A} = \overline{y}.$$
By induction we obtain $(x_n, y_n) = (\overline{x}, \overline{y})$ for all $n < N$.
- If $A \leq 1$, the proof is similar and will be omitted.

3. Periodicity of solutions of system (2)

Theorem 2 Suppose $A = 1$. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (2). Then $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is periodic with period 10 and it is given as follows.

- Case 1 $\beta \leq 1,\ \delta \leq 1,\ \gamma \geq 1,$ and $\alpha \geq 1$.
$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$
- Case 2 $\beta \leq 1,\ \delta \leq 1,\ \gamma \geq 1,$ and $\alpha \leq 1$.
$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{1}{\alpha\delta}),(\frac{1}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\alpha\delta},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$
- Case 3 $\beta \leq 1,\ \delta \leq 1,\ \gamma \leq 1,$ and $\alpha \geq 1$.
$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{1}{\gamma}),(\frac{1}{\gamma\beta},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\gamma\beta}),(\frac{1}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$
- Case 4 $\beta \leq 1,\ \delta \leq 1,\ \gamma \leq 1,$ and $\alpha \leq 1$.
$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{1}{\gamma}),(\frac{1}{\gamma\beta},\frac{1}{\alpha\delta}),(\frac{1}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\alpha\delta},\frac{1}{\gamma\beta}),(\frac{1}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$
- Case 5 $\beta \leq 1,\ 1 \leq \delta \leq \alpha,$ and $\gamma \geq 1$.
$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$
- Case 6 $\beta \leq 1,\ 1 \leq \alpha \leq \delta,$ and $\gamma \geq 1$.
$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{1}{\alpha}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 7 $\beta \leq 1,\ \alpha \leq 1 \leq \delta,\ \text{and}\ \gamma \geq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{1}{\alpha}),(\frac{1}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 8 $\beta \leq 1,\ 1 \leq \delta \leq \alpha,\ \text{and}\ \gamma \leq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{1}{\gamma}),(\frac{1}{\gamma\beta},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\gamma\beta}),(\frac{1}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 9 $\beta \leq 1,\ \delta \geq 1,\ \gamma \leq 1,\ \text{and}\ \alpha \leq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{1}{\gamma}),(\frac{1}{\gamma\beta},\frac{1}{\alpha}),(\frac{1}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\gamma\beta}),(\frac{1}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 10 $\beta \leq 1,\ \gamma \leq 1,\ \text{and}\ 1 \leq \alpha \leq \delta.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{1}{\gamma}),(\frac{1}{\gamma\beta},\frac{1}{\alpha}),(\frac{\alpha}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{1}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\gamma\beta}),(\frac{1}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 11 $1 \leq \beta \leq \gamma,\ \delta \leq 1,\ \text{and}\ \alpha \geq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\beta},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 12 $1 \leq \gamma \leq \beta,\ \delta \leq 1,\ \text{and}\ \alpha \geq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\gamma}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 13 $\beta \geq 1,\ \delta \leq 1,\ \gamma \leq 1,\ \text{and}\ \alpha \geq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 14 $1 \leq \beta \leq \gamma,\ \delta \leq 1,\ \text{and}\ \alpha \leq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\beta},\frac{1}{\alpha\delta}),(\frac{1}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\alpha\delta},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 15 $\delta \leq 1 \leq \beta,\ \delta \leq 1,\ \gamma \leq 1,$ and $\alpha \leq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\alpha\delta}),(\frac{1}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\alpha\delta},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 16 $1 \leq \gamma \leq \beta,\ \delta \leq 1,$ and $\alpha \leq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{1}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\alpha\delta}),(\frac{1}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{1}{\alpha}),$$
$$(\frac{1}{\alpha\delta},\frac{1}{\gamma}),(\frac{\gamma}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 17 $1 \leq \beta \leq \gamma$ and $1 \leq \delta \leq \alpha.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\beta},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 18 $1 \leq \beta \leq \gamma$ and $1 \leq \alpha \leq \delta.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\beta},\frac{1}{\alpha}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 19 $1 \leq \beta \leq \gamma,\ \delta \geq 1,$ and $\alpha \leq 1.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\beta},\frac{1}{\alpha}),(\frac{1}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\beta}),(\frac{\gamma}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 20 $1 \leq \gamma \leq \beta$ and $1 \leq \delta \leq \alpha.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\gamma}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 21 $\beta \geq 1,\ \gamma \leq 1,$ and $1 \leq \delta \leq \alpha.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\delta}),(\frac{\alpha}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\delta},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 22 $1 \leq \gamma \leq \beta$ and $1 \leq \alpha \leq \delta.$

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\alpha}),(\frac{\alpha}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\gamma}),(\frac{\gamma}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 23 $\beta \geq 1,\ \gamma \leq 1,$ and $1 \leq \alpha \leq \delta$.

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\alpha}),(\frac{\alpha}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{\alpha}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 24 $1 \leq \gamma \leq \beta,\ \alpha \leq 1,$ and $\delta \geq 1$.

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\alpha}),(\frac{1}{\delta},\frac{\gamma}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\gamma}),(\frac{\gamma}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

- Case 25 $\beta \geq 1,\ \delta \geq 1,\ \alpha \leq 1,$ and $\gamma \leq 1$.

$$\{(x_n, y_n)\}_{n=-1}^{\infty} = \{(\alpha,\gamma),(\beta,\delta),(\frac{\delta}{\alpha},\frac{\beta}{\gamma}),(\frac{1}{\gamma},\frac{1}{\alpha}),(\frac{1}{\delta},\frac{1}{\beta}),(\gamma,\alpha),(\delta,\beta),(\frac{\beta}{\gamma},\frac{\delta}{\alpha}),$$
$$(\frac{1}{\alpha},\frac{1}{\gamma}),(\frac{1}{\beta},\frac{1}{\delta}),(\alpha,\gamma),(\beta,\delta),...\}.$$

Where $x_{-1} = \alpha,\ x_0 = \beta,\ y_{-1} = \gamma,$ and $y_0 = \delta$.

Proof. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (2), and let $x_{-1} = \alpha$, $x_0 = \beta$, $y_{-1} = \gamma$, and $y_0 = \delta$.

- Case 1 Assume that $\beta \leq 1,\ \delta \leq 1,\ \gamma \geq 1,$ and $\alpha \geq 1$. Then

$$x_1 = \frac{max\{1,\delta\}}{\alpha} = \frac{1}{\alpha} \text{ and } y_1 = \frac{max\{1,\beta\}}{\gamma} = \frac{1}{\gamma},$$
$$x_2 = \frac{max\{1,\frac{1}{\gamma}\}}{\beta} = \frac{1}{\beta} \text{ and } y_2 = \frac{max\{1,\frac{1}{\alpha}\}}{\delta} = \frac{1}{\delta},$$
$$x_3 = \frac{max\{1,\frac{1}{\delta}\}}{\frac{1}{\alpha}} = \frac{\alpha}{\delta} \text{ and } y_3 = \frac{max\{1,\frac{1}{\beta}\}}{\frac{1}{\gamma}} = \frac{\gamma}{\beta},$$
$$x_4 = \frac{max\{1,\frac{\gamma}{\beta}\}}{\frac{1}{\beta}} = \gamma \text{ and } y_4 = \frac{max\{1,\frac{\alpha}{\delta}\}}{\frac{1}{\delta}} = \alpha,$$
$$x_5 = \frac{max\{1,\alpha\}}{\frac{\alpha}{\delta}} = \delta \text{ and } y_5 = \frac{max\{1,\gamma\}}{\frac{\gamma}{\beta}} = \beta,$$
$$x_6 = \frac{max\{1,\beta\}}{\gamma} = \frac{1}{\gamma} \text{ and } y_6 = \frac{max\{1,\delta\}}{\alpha} = \frac{1}{\alpha},$$
$$x_7 = \frac{max\{1,\frac{1}{\alpha}\}}{\delta} = \frac{1}{\delta} \text{ and } y_7 = \frac{max\{1,\frac{1}{\gamma}\}}{\beta} = \frac{1}{\beta},$$
$$x_8 = \frac{max\{1,\frac{1}{\beta}\}}{\frac{1}{\gamma}} = \frac{\gamma}{\beta} \text{ and } y_8 = \frac{max\{1,\frac{1}{\delta}\}}{\frac{1}{\alpha}} = \frac{\alpha}{\delta},$$
$$x_9 = \frac{max\{1,\frac{\alpha}{\delta}\}}{\frac{1}{\delta}} = \alpha \text{ and } y_9 = \frac{max\{1,\frac{\gamma}{\beta}\}}{\frac{1}{\beta}} = \gamma,$$
$$x_{10} = \frac{max\{1,\gamma\}}{\frac{\gamma}{\beta}} = \beta \text{ and } y_{10} = \frac{max\{1,\alpha\}}{\frac{\alpha}{\delta}} = \delta.$$

By induction we obtain

$$x_{n+10} = x_n \text{ and } y_{n+10} = y_n \text{ for all } n \geq 0.$$

- The proof of the other cases is similar and will be omitted.

If $A > 1$, set

$$D_4 = \{(u, v, w, t) \in (0, \infty)^4, \quad 1 \le u, v, w, t \le A\},$$
$$CD_4 = (0, \infty)^4 - D_4,$$

and if $0 < A < 1$, set

$$D_6 = \{(u, v, w, t) \in [A, \frac{1}{A}]^4, \quad Av \le w \le \frac{1}{A}v, \quad Au \le t \le \frac{1}{A}u\},$$
$$CD_6 = (0, \infty)^4 - D_6.$$

Theorem 3 Suppose $A > 1$. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (2). Then the following statements are true

(1) If $(x_{-1}, x_0, y_{-1}, y_0) \in D_4$. Then $(x_n, x_{n+1}, y_n, y_{n+1}) \in D_4$ for all $n \ge 0$. Moreover $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is periodic with period 4 and is given by

$$(x_{-1}, y_{-1}), (x_0, y_0), (\frac{A}{x_{-1}}, \frac{A}{y_{-1}}), (\frac{A}{x_0}, \frac{A}{y_0}), (x_{-1}, y_{-1}), ...$$

(2) If $(x_{-1}, x_0, y_{-1}, y_0) \in CD_4$. Then $(x_n, x_{n+1}, y_n, y_{n+1}) \in CD_4$ for all $n \ge 0$.

Proof. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (2).

(1) Suppose $(x_{-1}, x_0, y_{-1}, y_0) \in D_4$. Then

$$x_1 = \frac{A}{x_{-1}} \in [1, A] \text{ and } y_1 = \frac{A}{y_{-1}} \in [1, A],$$
$$x_2 = \frac{A}{x_0} \in [1, A] \text{ and } y_2 = \frac{A}{y_0} \in [1, A],$$
$$x_3 = x_{-1} \in [1, A] \text{ and } y_3 = y_{-1} \in [1, A],$$
$$x_4 = x_0 \in [1, A] \text{ and } y_4 = y_0 \in [1, A].$$

By induction we get $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is periodic with period 4 and $(x_n, x_{n+1}, y_n, y_{n+1}) \in D_4$ for all $n \ge 0$.

(2) Suppose that $(x_{-1}, x_0, y_{-1}, y_0) \in CD_4$ and we shall show that $(x_n, x_{n+1}, y_n, y_{n+1}) \in CD_4$ for all $n \ge 0$. Suppose that there exists an integer $N \ge 0$ such that $(x_N, x_{N+1}, y_N, y_{N+1}) \in D_4$. Then

$$x_{N+1} = \frac{\max\{A, y_N\}}{x_{N-1}} = \frac{A}{x_{N-1}} \text{ and } y_{N+1} = \frac{\max\{A, x_N\}}{y_{N-1}} = \frac{A}{y_{N-1}}.$$

Hence

$$x_{N-1} = \frac{A}{x_{N+1}} \in [1, A] \text{ and } y_{N-1} = \frac{A}{y_{N+1}} \in [1, A].$$

Then $(x_{N-1}, x_N, y_{N-1}, y_N) \in D_4$. By induction it follows that $(x_{-1}, x_0, y_{-1}, y_0) \in D_4$ which is a contradiction, So $(x_n, x_{n+1}, y_n, y_{n+1}) \in CD_4$ for all $n \ge 0$.

Theorem 4 Suppose $0 < A < 1$. Let $\{(x_n, y_n)\}_{n=-1}^{\infty}$ be a solution of system (2). Then the following statements are true

(1) If $(x_{-1}, x_0, y_{-1}, y_0) \in D_6$. Then $(x_n, x_{n+1}, y_n, y_{n+1}) \in D_6$ for all $n \ge 0$. Moreover $\{(x_n, y_n)\}_{n=-1}^{\infty}$ is periodic with period 6 and is given by

$$(x_{-1}, y_{-1}), (x_0, y_0), (\frac{y_0}{x_{-1}}, \frac{x_0}{y_{-1}}), (\frac{1}{y_{-1}}, \frac{1}{x_{-1}}), (\frac{1}{y_0}, \frac{1}{x_0}), (\frac{y_{-1}}{x_0}, \frac{x_{-1}}{y_0}), (x_{-1}, y_{-1}), ...$$

(2) If $(x_{-1}, x_0, y_{-1}, y_0) \in CD_6$. Then $(x_n, x_{n+1}, y_n, y_{n+1}) \in CD_6$ for all $n \ge 0$.

Proof. The proof is similar to that of Theorem 3 and will be omitted.
Remark 1 Observe that if $A > 1$, then $(\overline{x}, \overline{x}, \overline{y}, \overline{y}) = (\sqrt{A}, \sqrt{A}, \sqrt{A}, \sqrt{A})$ is an interior point of D_4 and if $0 < A < 1$, then $(\overline{x}, \overline{x}, \overline{y}, \overline{y}) = (1, 1, 1, 1)$ is an interior point of D_6.
Corollary 1 The equilibrium point $(\overline{x}, \overline{y})$ of system (2) is stable but is not locally asymptotically stable.

References

[1] E.A. Grove and G. Ladas, Periodicities in nonlinear difference equations, Advances in Discrete Mathematics and Applications 4, Chapman and Hall, CRS Press, 2005.
[2] B.D. Iričănin and N. Touafek, On a second order max-type system of difference equations, Indian Journal of Mathematics 54, 119-142, 2012.
[3] E.J. Janowski and V.L. Kocic and G. Ladas and S.W. Schultz, Global behavior of solutions of $x_{n+1} = \frac{\max\{x_n, A\}}{x_{n-1}}$, Proceeding of the First International Conference on Difference Equations, May 25-28, 1994, San Antonio, Texas, USA, Gordon and Breach Science Publishers, Basel, 1995.
[4] V.L. Kocic and G. Ladas, Global behavior of nonlinear difference equations of higher order with applications, Kluwer Academic Publishers, Dordrecht, 1993.
[5] G. apaschinopoulos and J. Schinas and V. Hatzifilippidis, Global behavior of the solutions of a max-equation and a system of two max-equations, J. Comput. Anal. Appl,5, 2, 237-254, 2003.
[6] W.T. Patula and H.D. Voulov, On a max type recurrence relation with periodic coefficients, J. Differ. Equ. Appl, 10, 3, 329-338, 2004.
[7] G. Stefanidou and G. Papaschinopoulos, Behavior of the positive solutions of a fuzzy max-difference equation, Advance in Difference Equations, 2, 153-172, 2005.
[8] G. Stefanidou and G. Papaschinopoulos and C.J. Schinas, On a system of max-difference equations, Commun. Appl. Nonlinear Anal, 17, 2, 1-13, 2010.
[9] S. Stevic, Boundedness character of a class of difference equations, Nonlinear Anal. TMA, 70, 839-848, 2009.
[10] S. Stevic, On a generalized max-type difference equation from automatic control theory, Nonlinear Anal. TMA, 72, 1841-1849, 2010.
[11] S. Stevic, Periodicity of max difference equations, Util. Math, 83, 69-71, 2010.
[12] S. Stevic, Periodicity of a class of nonautonomous max-type difference equations, Appl. Math. Comput, 217, 9562-9566, 2011.
[13] S. Stevic, Solution of a max-type system of difference equations, Appl. Math. Comput, 218, 9825-9830, 2012.
[14] S. Stevic, On some periodic systems of max-type difference equations, Appl. Math. Comput, 218, 11483-11487, 2012.
[15] H.D. Voulov, On the periodic character of some difference equations, J. Differ. Equ. Appl, 8, 9, 799-810, 2002.
[16] H.D. Voulov, Periodic solutions to a difference equation with maximum, Proc. Am. Math. Soc, 131, 7, 2155-2160, 2003.

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