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NONLINEAR INITIAL VALUE PROBLEMS WITH MEASURE SOLUTIONS

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Abstract. In this expository article, we discuss several initial value problems, involving extensions of Nemickiĭ operators to measures. For instance, we work with general nonnegative Carathéodory functions and with signed measures that are absolutely continuous with respect to a sigma finite measure. We work as well with Carathéodory functions that are piecewise linear, allowing us to extend the associated Nemickiĭ operator to general signed measures. We prove existence and uniqueness for the solution of the initial value problem using an extension of the Banach fixed point theorem. Our goal is to give a detailed exposition of the results.

1. Introduction

These notes comprise material taken from [1], plus several extensions and a few additions. Our aim is to give a detailed exposition, including abundant preliminary material, on the existence of signed measures solutions for non linear initial value problems.

The problems we consider have the form

$$\begin{cases} \frac{d\lambda}{dt} - \mathcal{A}(\lambda)(t) & = & 0 \text{ for } 0 < t < T \\ \lambda(0) & = & \lambda_0 \end{cases},$$

where $\mathcal{A} = \mathcal{A}(t)$ is associated with a particular Nemyckiĭ operator.

The notion of Nemyckiĭ operator was first introduced by Viktor V. Nemyckiĭ [10] and studied by A. Mark Krasnosel'skiĭ [9] and Mordukhaĭ M. Vaĭnberg [13], who was Nemyckiĭ's doctoral student at Moscow State University, among others. Since we are interested in extending this operator to signed measures, we begin by summarizing some of the definitions and results we need from measure theory. We present them in the context of the extended real number system $\mathbb{R}^*$, although for much of what we need it suffices to work in $\mathbb{R}$. Other results will be stated at the appropriate time. For more details, we refer to [11] and the references therein.

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2. Preliminary definitions and results

Let us recall that a family Σ of subsets of a non-empty set X is called a σ-algebra if it satisfies the following three properties:

(1) The empty set $\varnothing$ belongs to Σ.
(2) If $A \in \Sigma$, then the complement $X \backslash A$ also belongs to Σ.
(3) If $\{A_j\}_{j\geq 1} \subseteq \Sigma$, then $\bigcup_{j\geq 1} A_j \in \Sigma$.

Let $\mathbb{R}^*$ be the extended real line consisting of the real numbers and the symbols $-\infty$ and $+\infty$, with the usual operations. We adopt the convention $0.(\pm\infty) = (\pm\infty).0 = 0$. We leave $(+\infty) + (-\infty)$ and $(-\infty) + (+\infty)$ undefined. For more on the algebraic and topological structure of $\mathbb{R}^*$, see [14].

Given a σ-algebra Σ, we consider set functions $\lambda : \Sigma \to \mathbb{R}^*$ that take at most one of the two values $+\infty$ and $-\infty$.

Definition 1. *The set function $\lambda : \Sigma \to \mathbb{R}^*$ is called a signed measure if*

(1) $\lambda(\varnothing) = 0$
(2) $\lambda(\bigcup_{i\geq 1} A_i) = \sum_{i\geq 1} \lambda(A_i)$ *whenever $\{A_i\}_{i\geq 1} \subseteq \Sigma$ are pairwise disjoint.*

The signed measure λ is called finite if $\lambda : \Sigma \to \mathbb{R}$.

As a consequence of 2) in Definition 1, the series $\sum_{i\geq 1} \lambda(A_i)$ converges commutatively in $\mathbb{R}^*$ and, if $\lambda(\bigcup_{i\geq 1} A_i)$ is finite, it converges absolutely in $\mathbb{R}$. We also observe that if A and $B \in \Sigma$ with $B \subseteq A$ and $\mu(B)$ is finite,

$$\lambda(A - B) = \lambda(A) - \lambda(B).$$

Moreover, if $\lambda(A)$ is finite for some $A \in \Sigma$, then $\lambda(B)$ is finite for every $B \subseteq A$, $B \in \Sigma$. In fact, assume that this statement is not true for some set B. We can write

$$\lambda(A) = \lambda(B) + \lambda(A - B),$$

which contradicts the assumption that $\lambda(A)$ is finite.

Definition 2. *A measure μ is a signed measure that only takes nonnegative values in $\mathbb{R}^*$. It is a finite measure if $\mu : \Sigma \to [0, +\infty)$.*

Definition 3. *An ordered triple (X, Σ, μ) consisting of a non-empty set X, a σ-algebra Σ of subsets of X and a measure $\mu : \Sigma \to \mathbb{R}^*$, is called a measure space.*

Definition 4. *A measure space (X, Σ, μ) is complete if $A \in \Sigma$ and $\mu(A) = 0$ imply that $B \in \Sigma$ for every $B \subseteq A$. As a consequence, $\mu(B) = 0$.*

Measurable sets of measure zero are called null sets, o μ-null sets, if it is necessary to identify the measure.

Definition 5. *A measure space (X, Σ, μ) is σ-finite if we can write*

$$X = \bigcup_{i\geq 1} A_i,$$

where $\{A_i\}_{i\geq 1} \subseteq \Sigma$ are pairwise disjoint and $\mu(A_i)$ is finite for all $i \geq 1$.

Definition 6. *Given a measure space (X, Σ, μ) and a Σ-measurable function $f : X \to \mathbb{R}^*$, we say that f has a μ-integral if at least one of $\int_X f^+ d\mu$ and $\int_X f^- d\mu$ is finite. If this is the case, we write*

$$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.$$

If a function f has a μ-integral, then the integral $\int_E f d\mu$ exists in $\mathbb{R}^*$ for every $E \in \Sigma$.

Definition 7. *Given a signed measure $\lambda : \Sigma \to \mathbb{R}^*$, the set functions $\lambda^+,\lambda^- : \Sigma \to \mathbb{R}^*$ are defined as*

$$\lambda^+(E) = \sup\{\lambda(A) : A \subseteq E,\ A \in \Sigma\}$$
$$\lambda^-(E) = -\inf\{\lambda(A) : A \subseteq E, A \in \Sigma\}.$$

Since $\lambda(\emptyset) = 0$, this implies that λ^+ and λ^- are nonnegative. They are, in fact, measures and they are called, respectively, the positive part or upper variation of λ and the negative part or lower variation of λ. We recall the following properties of the measures λ^+ and λ^-:

(1) $\lambda^+ \geq \lambda$ and $\lambda^- \geq -\lambda$
(2) λ^+ and λ^- are increasing, and $\lambda^- = (-\lambda)^+$
(3) Given $E \in \Sigma$, if one of the numbers $\lambda^+(E)$ and $\lambda^-(E)$ is finite, then $\lambda(E) = \lambda^+(E) - \lambda^-(E)$.
As a consequence, if $\lambda : \Sigma \to \mathbb{R}$,
$$\lambda(E) = \lambda^+(E) - \lambda^-(E), \tag{1}$$
for all $E \in \Sigma$.
(4) Let $\lambda : \Sigma \to \mathbb{R}^*$ be a signed measure and let $E \in \Sigma$. Then,
$$\lambda^+(E) = +\infty \text{ implies } \lambda(E) = +\infty \tag{2}$$
$$\lambda^-(E) = +\infty \text{ implies } \lambda(E) = -\infty. \tag{3}$$
Therefore, if $\lambda(E)$ is finite, both $\lambda^+(E)$ and $\lambda^-(E)$ are finite. We can conclude that every signed measure $\lambda : \Sigma \to \mathbb{R}$ is bounded, in the sense that
$$\sup_{E \in \Sigma} |\lambda(E)| < +\infty.$$

As a consequence of 3) and 4) we have

Theorem 1. *(Jordan decomposition) Given a signed measure $\lambda : \Sigma \to \mathbb{R}^*$,*
$$\lambda = \lambda^+ - \lambda^-.$$

Definition 8. *If $\lambda : \Sigma \to \mathbb{R}^*$ is a signed measure, the total variation or variation of λ is defined as*
$$|\lambda| = \lambda^+ + \lambda^-.$$

Remark 1. *The properties of λ^+ and λ^- imply that $|\lambda|$ is a measure satisfying the following properties:*

(1) $|\lambda(A)| \leq |\lambda|(A)$, *for any* $A \in \Sigma$
(2) $|\lambda(A)| = \sup\{|\lambda(B)| + |\lambda(A \setminus B)| : B \subseteq A, B \in \Sigma\}$
(3) $|\lambda|(A) = \sup\left\{\sum_i |\lambda(A_i)|\right\}$, *where* $\{A_i\}_i$ *is any finite partition of* A *with* $A_i \in \Sigma$.

Property 3 can be adopted as the definition of $|\lambda|$, avoiding any reference to the Jordan Decomposition.

Thus, a signed measure $\lambda : \Sigma \to \mathbb{R}^*$ is bounded if and only if it has finite total variation.

Definition 9. *Two measures* $\lambda, \nu : \Sigma \to [0, \infty]$ *are mutually singular, denoted* $\lambda \perp \nu$*, if there is a partition* $X = A \cup B$*, with* $A, B \in \Sigma$*, such that* $\lambda(A) = \nu(B) = 0$*.*

If $\lambda : \Sigma \to \mathbb{R}^*$ is a signed measure, then $\lambda^+ \perp \lambda^-$. Moreover, if there are measures λ_1 and λ_2 such that $\lambda = \lambda_1 - \lambda_2$ and $\lambda_1 \perp \lambda_2$, then $\lambda_1 = \lambda^+$ and $\lambda_2 = \lambda^-$. In particular, the Jordan decomposition of a signed measure is unique.

Remark 2. *We extend Definition 9 to signed measures in the following way: Two signed measures* $\lambda, \nu : \Sigma \to \mathbb{R}^*$ *are mutually singular, denoted* $\lambda \perp \nu$*, if there is a partition* $X = A \cup B$*, with* $A, B \in \Sigma$*, such that* $\lambda(A_1) = 0$ *for every* Σ*-measurable subset* A_1 *of* A *and* $\nu(B_1) = 0$ *for every* Σ*-measurable subset* B_1 *of* B*. Equivalently, two signed measures* $\lambda, \nu : \Sigma \to \mathbb{R}^*$ *are mutually singular, if* $|\lambda| \perp |\nu|$*.*

Remark 3. *If* $\lambda_1, \lambda_2 : \Sigma \to \mathbb{R}^*$ *are mutually singular signed measures and we assume that* $\lambda_1(E) + \lambda_2(E)$ *is well defined for all* $E \in \Sigma$*, then*

$$|\lambda_1 + \lambda_2| = |\lambda_1| + |\lambda_2| .$$

The inequality $|\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2|$ *follows from 3) in Remark 1, while the proof of the reverse inequality,* $|\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2|$*, uses Remark 2, and the definition of supremum adapted to the various cases,* $|\lambda_1|(E), |\lambda_2|(E) < +\infty$*,* $|\lambda_1|(E) = +\infty, |\lambda_2|(E) < +\infty$*, etc.*

Definition 10. *If* λ *and* ν *are signed measures, we say that* ν *is absolutely continuous with respect to* λ*, denoted* $\nu \ll \lambda$*, if* $E \in \Sigma$ *and* $|\lambda|(E) = 0$ *implies that* $|\nu|(E) = 0$*.*

If λ is a measure, we have that $\nu \ll \lambda$ if and only if $\nu^+ \ll \lambda$ and $\nu^- \ll \lambda$. Moreover, we can say that $\nu \ll \lambda$ if and only if $E \in \Sigma$ and $\lambda(E) = 0$ implies that $\nu(E_1) = 0$ for every Σ-measurable subset E_1 of E.

Given a measure space (X, Σ, μ) and a function $f : X \to \mathbb{R}^*$ that has a μ-integral, the set function λ, defined for $E \in \Sigma$ as $\lambda(E) = \int_E f d\mu$, is a signed measure and $\lambda \ll \mu$. We will denote this signed measure λ as $f d\mu$.

Conversely, we have the following result:

Theorem 2. *(Radon-Nikodym theorem) Let* (X, Σ, μ) *be a* σ*-finite measure space and let* $\lambda : \Sigma \to \mathbb{R}^*$ *be a signed measure such that* $\lambda << \mu$*. Then, there exists a* Σ*-measurable function* $f : X \to \mathbb{R}$ *such that* $\lambda = f d\mu$*. The function* f *is unique up to* μ*-a.e. That is to say, if* $f d\mu = g d\mu$ *then* f *and* g *are equal, except on a* μ*-null set.*

We remark that this result does not require the signed measure λ to be σ-finite. Moreover, there are known conditions characterizing those measures μ for which every signed measure λ can be represented as $f d\mu$.

Observe that given $E \in \Sigma$,

$$(d\mu)(E) = \int_E d\mu = \mu(E) . \tag{4}$$

As for the total variation of $f d\mu$,

$$|f d\mu| = |f| \, d\mu, \tag{5}$$

where $|f|$ denotes de absolute value of the function f. The context should make clear the distinction between the absolute value of a function and the total variation of a signed measure. If the signed measure λ is finite, then $f \in L^1(\mu)$.

Theorem 3. *(Lebesgue decomposition) Let* (X, Σ, μ) *be a measure space and let* $\lambda : \Sigma \to \mathbb{R}^*$ *be a* σ*-finite signed measure. Then, there exist unique signed measures* λ_a *and* λ_s *defined on* Σ *such that* $\lambda = \lambda_a + \lambda_s$*,* $\lambda_a \ll \mu$ *and* $\lambda_s \perp \mu$*.*

The Jordan and Lebesgue decompositions are related in the following way: Given the Jordan decomposition $\lambda = \lambda^+ - \lambda^-$ of a signed measure $\lambda : \Sigma \to \mathbb{R}^*$, if λ is σ-finite, then

$$\lambda_a = \left(\lambda^+\right)_a - \left(\lambda^-\right)_a, \tag{6}$$

$$\lambda_s = \left(\lambda^+\right)_s - \left(\lambda^-\right)_s. \tag{7}$$

This concludes our review of signed measures. Additional results will be presented at the appropriate time.

We are now ready to introduce the notion of Nemyckiĭ operator. We follow, for the most part, the presentation in ([13], Chapter VI), which we extend from the Lebesgue measure space on $\mathbb{R}^n$, to the setting of an abstract measure space.

3. The Nemyckiĭ operator

To begin, we fix a complete measure space (X, Σ, μ).

Definition 11. *(Carathéodory function) A function* $g : X \times \mathbb{R} \to \mathbb{R}$ *is called a Carathéodory function, or N-function, if it satisfies the following two properties, called Carathéodory conditions:*

(1) *The function* $u \to g(x, u)$ *is continuous for* μ*-a.e.* $x \in X$*.*
(2) *The function* $x \to g(x, u)$ *is* Σ*-measurable for each* $u \in \mathbb{R}$*.*

We denote

$$L^0(\Sigma) = \{f : X \to \mathbb{R};\ f \text{ is } \Sigma\text{-measurable}\}.$$

Lemma 4. *Given an N-function g and given* $f \in L^0(\Sigma)$*, the composite function* $g(x, f(x))$ *belongs to* $L^0(\Sigma)$*.*

Proof. We first assume that f is a simple function,

$$f = \sum_{i=1}^{k} c_i \chi_{E_i},$$

with $c_i \in \mathbb{R}$ and $\{E_i\} \subseteq \Sigma$, pairwise disjoint. Given $a \in \mathbb{R}$,

$$\{x \in X : g(x, f(x)) < a\} = \left\{x \in X \setminus \bigcup_{i=1}^{k} E_i : g(x, 0) < a\right\} \cup \bigcup_{i=1}^{k} \{x \in E_i : g(x, c_i) < a\},$$

which is Σ-measurable because of Definition 11. If $f \in L^0(\Sigma)$, there exists a sequence $\{\varphi_n\}$ of simple functions, $\varphi_n : X \to \mathbb{R}$, such that $\varphi_n(x) \to f(x)$ in $\mathbb{R}$ for each $x \in X$ ([11], p. 78). Then, Definition 11 implies that $g(x, \varphi_n(x)) \to g(x, f(x))$ for μ-a.e. $x \in X$. Thus, the function $g(x, f(x))$ is Σ-measurable and the lemma is proved. □

Remark 4. *When the function g only depends on u, the operator* N_g *is called autonomous. In this case,* N_g *reduces to a simple composition of two functions,* $g \circ f$*.*

It was Constantin Carathéodory [4] *who studied the measurability of this composition. He observed that the composition of two measurable functions might no be measurable and proved that $g \circ f$ is measurable if f is measurable and g is continuous, thus suggesting the correct assumptions on a general function g as stated in Definition 11. This is the reason for N-functions to be called Carathéodory functions and for the two conditions formulated in Definition 11 to be referred to as Carathéodory conditions.*

Definition 12. *(Nemyckiĭ operator) Given an N-function g, the Nemyckiĭ operator N_g is defined for $f \in L^0(\Sigma)$ as*

$$N_g(f)(x) = g(x, f(x)). \tag{8}$$

Lemma 4 implies that the Nemyckiĭ operator maps $L^0(\Sigma)$ into itself. When μ is a finite measure, $L^0(\Sigma)$ becomes a complete semi-metric space defining

$$d(f,g) = \int_X \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} d\mu(x),$$

for which convergence of a sequence is the convergence in μ-measure ([11], p. 110). When X is a bounded subset of $\mathbb{R}^k$ for some $k \geq 1$ and μ is the Lebesgue measure defined on the Lebesgue σ-algebra of X, Vainberg proves that N_g is continuous from $\left(L^0(\Sigma), d\right)$ to itself ([13], p. 153). In the abstract case, the continuity is proved in ([7], p. 343, Proposition 7.18).

Remark 5. *Definition 11 can be extended to a function $g : X \times \mathbb{R}^n \to \mathbb{R}^m$ in the natural way, with the associated Nemyckiĭ operator defined as*

$$\begin{aligned} N_{\mathbf{g}}(\mathbf{f})(x) &= \mathbf{g}(x, \mathbf{f}(x)) \\ &= (g_1(x, f_1(x), ..., f_n(x)), ..., g_m(x, f_1(x), ..., f_n(x))). \end{aligned}$$

Then, if $f_1, ..., f_n \in L^0(\Sigma)$, each of the functions $g_j(x, f_1(x), ..., f_n(x))$ belongs to $L^0(\Sigma)$, for $1 \leq j \leq m$, extending Lemma 4. To prove this assertion, we observe that if we fix $(u_1, ..., u_{n-1}) \in \mathbb{R}^{n-1}$, the function $(x, u_n) \to g_j(x, u_1, ..., u_{n-1}, u_n)$ is an N-function from $X \times \mathbb{R}$ to $\mathbb{R}$. Thus, using Lemma 4, the function $x \to g_j(x, u_1, ..., u_{n-1}, f_n(x))$ is measurable for each $(u_1, ..., u_{n-1}) \in \mathbb{R}^{n-1}$. Then, the function $(x, u_1, ..., u_{n-1}) \to g_j(x, u_1, ..., u_{n-1}, f_n(x))$ from $X \times \mathbb{R}^{n-1}$ to $\mathbb{R}$ is an N-function as well, and so on.

Under appropriate hypotheses, the Nemyckiĭ operator has interesting continuity and boundedness properties in L^p spaces and in Sobolev spaces, among others. As an example, we state the following result: Consider the Lebesgue measure space on $\mathbb{R}^k$. Given $1 \leq p, q < \infty$, (8) defines a continuous and bounded operator from L^p into L^q if and only if ([13], p. 155) there exist a function $a \in L^q$ and a constant $b \geq 0$ so that

$$|g(x, u)| \leq a(x) + b|u|^{p/q}.$$

This result applies as well to $0 < p, q < 1$, if L^p and L^q are endowed with their standard structure of metric spaces ([13], p. 155).

For much more on L^p-continuity, see ([7], Chapter 7).

There is an extensive literature in which various forms of the Nemyckiĭ operator are studied in connection with nonlinear problems involving integral operators as well as partial differential equations and ordinary differential equations, (for instance, see the references mentioned in [1]). In some applications, the need arises to study the Fréchet and Gâteau differentiability of particular Nemyckiĭ operators (see [5], p. 267; [6], pp. 318, 342; [8], pp. 96-97; [12]). Other results on differentiability can be formulated using Sobolev spaces ([6], p. 342). For a treatment of the differentiability of a general Nemyckiĭ operator, see ([2], Chapter 1; [7], Chapter 6).

In the following sections we will describe extensions of particular Nemyckiĭ operators to various spaces of signed measures, under appropriate conditions on the function g.

From now on we will assume that (X, Σ, μ) is a complete σ-finite measure space.

4. Extension of the Nemyckiĭ operator for non-negative N-functions

If we denote

$$L(\mu) = \left\{ f \in L^0(\Sigma) ; \, f \text{ has a } \mu\text{-integral} \right\}$$

(see Definition 6) and

$$\mathcal{M}_a = \{ \lambda : \Sigma \to \mathbb{R}^* \text{ signed measure; } \lambda << \mu \},$$

there is a map $\Lambda : L(\mu) \to \mathcal{M}_a$, defined as $\Lambda(f) = f d\mu$. Moreover, if g is a non-negative N-function, N_g maps $L(\mu)$ into itself. In fact, N_g maps $L^0(\Sigma)$ into $L^0(\Sigma)$ according to Lemma 4, and then the non-negative measurable function $g(x, f(x))$ has a μ-integral, for every $f \in L(\mu)$.

Proposition 5. *Let g be a non-negative N-function. Then, there exists a unique operator $\overline{N}_g : \mathcal{M}_a \to \mathcal{M}_a$ such that*

$$\Lambda \circ N_g(f) = \overline{N}_g \circ \Lambda(f) \tag{9}$$

for all $f \in L^0(\Sigma)$. That is to say, there is a unique operator $\overline{N}_g : \mathcal{M}_a \to \mathcal{M}_a$ that makes the following diagram commutative:

$$\begin{array}{ccc} L(\mu) & \overset{N_g}{\to} & L(\mu) \\ \Lambda \downarrow & & \downarrow \Lambda \\ \mathcal{M}_a & \overset{\overline{N}_g}{\to} & \mathcal{M}_a \end{array} \tag{10}$$

Proof. According to Theorem 2, given $\lambda \in \mathcal{M}_a$, $\lambda = f d\mu$ for $f \in L(\mu)$. Thus, we propose the definition

$$\overline{N}_g(\lambda) = g(x, f(x)) \, d\mu. \tag{11}$$

Since Theorem 2 assures the uniqueness of f up to μ-a.e., we need to show that $\overline{N}_g$ is well defined. Indeed, if $f = h$ outside of a null set $O \in \Sigma$, and $E \in \Sigma$,

$$\int_E g(x, h(x)) \, d\mu = \int_{E \cap (X \setminus O)} g(x, f(x)) \, d\mu = \int_E g(x, f(x)) \, d\mu,$$

where we have used the completeness of (X, Σ, μ). So, $\overline{N}_g(\lambda) = \overline{N}_g(h d\mu)$. The definition given by (11) implies that (9) holds. Concerning the uniqueness of $\overline{N}_g$, suppose that $T : \mathcal{M}_a \to \mathcal{M}_a$ is another operator satisfying

$$\Lambda \circ N_g(f) = T \circ \Lambda(f).$$

Then,

$$T(f d\mu) = T \circ \Lambda(f) = \Lambda \circ N_g(f) = \overline{N}_g \circ \Lambda(f) = \overline{N}_g(f d\mu).$$

This completes the proof of the proposition. □

5. Properties of the map $(g, \lambda) \to \overline{N}_g(\lambda)$

If the operator $\overline{N}_g$ is given by (11), the map $(g, \lambda) \to \overline{N}_g(\lambda)$ has several properties resembling those expected from a functional calculus. We gather them in the following proposition:

Proposition 6. (1) *When λ is the measure μ,*

$$\overline{N}_g(\mu) = g(x, 1)\, d\mu,$$

for every non-negative N-function g.

(2) *If g_1 and g_2 are two N-functions, the sum $g_1 + g_2$ is an N-function. Moreover, when they are non-negative, the operator $\overline{N}_g$ is additive in g. That is,*

$$\overline{N}_{g_1+g_2} = \overline{N}_{g_1} + \overline{N}_{g_2}.$$

(3) *The multiplicative product $g_1 g_2$ of two N-functions g_1 and g_2, is an N-function. Given two non-negative N-functions g_1 and g_2,*

$$\overline{N}_{g_1 g_2}(f d\mu) = g_1(x, f)\, g_2(x, f)\, d\mu.$$

As a particular case, given a non-negative Σ-measurable function $\alpha : X \to [0, +\infty)$ and a non-negative N-function g, the multiplicative product αg is also a non-negative N-function and

$$\overline{N}_{\alpha g} = \alpha \overline{N}_g,$$

where $\left(\alpha \overline{N}_g\right)(f d\mu)$ is the measure defined on $E \in \Sigma$ as $\int_E \alpha(x)\, g(x, f(x))\, d\mu$.

(4) *Given two N-functions $g_1, g_2 : X \times \mathbb{R} \to \mathbb{R}$, the function $(g_2 \circ g_1)(x, u) = g_2(x, g_1(x, u))$ is also an N-function. Furthermore, if g_1 and g_2 are non-negative,*

$$\overline{N}_{g_2} \circ \overline{N}_{g_1} = \overline{N}_{g_2 \circ g_1}.$$

(5) *If $g(x, u)$ does not depend on u, then*

$$\overline{N}_g(\lambda) = g(x)\, d\mu,$$

for all $\lambda = f d\mu \in \mathcal{M}_a$. In particular,

$$\overline{N}_g(\mu) = g(x)\, d\mu. \tag{12}$$

(6) *If $g(x, u) = |u|$, then g is a non-negative N-function and*

$$\overline{N}_g(\lambda) = |\lambda|,$$

the total variation of λ.

Proof. The assertion in 1) follows from (4) and the definition of $\overline{N}_g$. As a consequence, the operator $\overline{N}_g$ can be defined on measures μ, finite or not, for a fairly general class of N-functions.

The main statements in 2) and 3) are direct applications of Definition 11 and Proposition 5. As for the particular case in 3), given $\lambda = f d\mu \in \mathcal{M}_a$ and given $E \in \Sigma$,

$$\begin{aligned} \left(\overline{N}_{\alpha g}(\lambda)\right)(E) &= \int_E N_{\alpha g}(f)\, d\mu = \int_E \alpha(x)\, g(x, f(x))\, d\mu \\ &\stackrel{\text{def}}{=} \left[\left(\alpha \overline{N}_g\right)(f d\mu)\right](E). \end{aligned}$$

Let us prove 4): By Definition 11, for μ-null sets $A, B \in \Sigma$, the function $u \to g_1(x, u)$ is continuous for each $x \in X \backslash A$ and the function $v \to g_2(x, v)$ is continuous for each $x \in X\backslash B$. So, the function $u \to g_2(x, g_1(x,u))$ is continuous for each $x \in X\backslash(A\bigcup B)$. Now, if we fix $u \in \mathbb{R}$, the function $x \to g_1(x,u)$ is Σ-measurable, so, according to Lemma 4, the function $x \to g_2(x, g_1(x,u))$ is Σ-measurable.

If we assume now that g_1 and g_2 are both non-negative N-functions, we can write

$$\begin{aligned}\left(\overline{N}_{g_2} \circ \overline{N}_{g_1}\right)(f d\mu) &= \overline{N}_{g_2}\left(N_{g_1}(f)\, d\mu\right) = \overline{N}_{g_2}\left(g_1(x, f(x)\, d\mu\right) \\ &= g_2\left(x, g_1\left(x, f(x)\right)\right) d\mu = \overline{N}_{g_2 \circ g_1}(f d\mu).\end{aligned}$$

Part 5) follows directly from the definition of $\overline{N}_g$ and the particular case (12) is an application of (4).

Finally, the proof of 6) is a direct application of (5) and the definition of $\overline{N}_g$.

This completes the proof of the theorem. □

In the next section we obtain an explicit representation of the operator $\overline{N}_g$ for N-functions generalizing the function $|u|$.

6. Extension of the Nemyckiĭ operator for piecewise linear N-functions

We begin with the following definition:

Definition 13. *An N-function $g : X\times\mathbb{R} \to \mathbb{R}$ is called piecewise linear if*

$$g(x,u) = \sum_{i=1}^{n} a_i(x)\left|d_i u - b_i(x)\right| + c(x)\, u, \tag{13}$$

where $d_i \in \mathbb{R}, a_i, b_i, c : X \to \mathbb{R}$ are Σ-measurable, bounded functions and b_i is μ-integrable.

If g is a piecewise linear N-function, the Nemyckiĭ operator N_g is well defined, bounded and continuous from $L^1(\mu)$ into itself. We consider now the space $\mathcal{M}_f$ of finite signed measures,

$$\mathcal{M}_f = \{\lambda : \Sigma \to \mathbb{R};\ \lambda \text{ signed measure}\}.$$

The properties listed in Proposition 6 suggest that given $\lambda \in \mathcal{M}_f$ we can write,

$$\overline{N}_g(\lambda) = \sum_{i=1}^{n} a_i(x)\left|d_i\lambda - b_i(x)\,\mu\right| + c(x)\,\lambda, \tag{14}$$

where $|d_i\lambda - b_i(x)\,\mu|$ means the total variation of the finite signed measure $d_i\lambda - b_i(x)\,\mu$ and $c(x)\,\lambda$ is the finite signed measure defined as

$$\left(c(x)\,\lambda\right)(E) = \int_E c(x)\, d\left(\lambda^+\right) - \int_E c(x)\, d\left(\lambda^-\right),$$

for $E \in \Sigma$. Since $|d_i\lambda - b_i(x)\,\mu|$ is a finite measure, $a_i(x)\,|d_i\lambda - b_i(x)\,\mu|$ is defined as is suggested in 3) of Proposition 6,

$$\left(a_i(x)\left|d_i\lambda - b_i(x)\,\mu\right|\right)(E) = \int_E a_i(x)\, d\left|d_i\lambda - b_i(x)\,\mu\right|,$$

for $E \in \Sigma$.

We claim that

$$|c(x)\,\lambda| = |c(x)|\,|\lambda|.$$

In fact, using the Jordan decomposition, we can write

$$|c(x)\lambda| = \left|c(x)\left(\lambda^+ - \lambda^-\right)\right| = \left|c(x)\lambda^+ - c(x)\lambda^-\right|.$$

. Since $\lambda^+$and λ^- are mutually singular, the signed measures $c(x)\lambda^+$ and $c(x)\lambda^-$ are mutually singular as well. According to Remark 3 and (5),

$$\begin{aligned}\left|c(x)\lambda^+ - c(x)\lambda^-\right| &= \left|c(x)\lambda^+\right| + \left|c(x)\lambda^-\right| = |c(x)|\lambda^+ + |c(x)|\lambda^- \\ &= |c(x)||\lambda|,\end{aligned}$$

where, once again, the context should make clear the distinction between the absolute value of a function and the total variation of a signed measure

From (10), we can see that the operator $\overline{N}_g$ defined by (14) makes the following diagram commutative:

$$\begin{array}{ccc} L^1(\mu) & \overset{N_g}{\to} & L^1(\mu) \\ \Lambda\downarrow & & \downarrow\Lambda \\ \mathcal{M}_f & \overset{\overline{N}_g}{\to} & \mathcal{M}_f \end{array} \tag{15}$$

In fact, if $\lambda = \Lambda(f)$, for $f \in L^1(\mu)$,

$$\begin{aligned}\overline{N}_g(\lambda) = \left(\overline{N}_g \circ \Lambda\right)(f) &= \sum_{i=1}^n a_i |d_i f d\mu - b_i\mu| + cf d\mu \\ &= \left(\sum_{i=1}^n a_i |c_i f - b_i| + cf\right) d\mu = (\Lambda \circ N_g)(f).\end{aligned}$$

That is to say, the definition given by (14) is a natural extension of $\overline{N}_g$ to the whole space $\mathcal{M}_f$. Observe that Λ is an isometric isomorphism between $L^1(\mu)$ and the proper closed subspace $\mathcal{M}_{fa}$ of $\mathcal{M}_f$ consisting of those signed measures of the form $fd\mu$, for $f \in L^1(\mu)$. As such, the restriction of $\overline{N}_g$ to $\mathcal{M}_{fa}$ is the only operator that makes the following diagram conmutative:

$$\begin{array}{ccc} L^1(\mu) & \overset{N_g}{\to} & L^1(\mu) \\ \Lambda\downarrow & & \downarrow\Lambda \\ \mathcal{M}_{fa} & \overset{\overline{N}_g}{\to} & \mathcal{M}_{fa} \end{array}$$

We can write (14) in a more explicit form. In fact, let,

$$l^+(g)(x) = \lim_{u\to+\infty} \frac{g(x,u)}{u} = \sum_{i=1}^n a_i(x)|d_i| + c(x) \tag{16}$$

and

$$l^-(g)(x) = \lim_{u\to-\infty} \frac{g(x,u)}{u} = -\sum_{i=1}^n a_i(x)|d_i| + c(x). \tag{17}$$

We observe that $l^+(g)(x)$ and $l^-(g)(x)$ exist for each $x \in X$ and are Σ-measurable, bounded functions. Then we have

Proposition 7. *Given $\lambda \in \mathcal{M}_f$, we can write,*

$$\overline{N}_g(\lambda)(E) = \int_E N_g(f)\, d\mu + \left(l^+(g)(\lambda_s)^+\right)(E) - \left(l^-(g)(\lambda_s)^-\right)(E),$$

where, using Theorem 2 and Theorem 3, $\lambda = fd\mu + \lambda_s$.

Proof. According to (14),

$$\overline{N}_g(\lambda) = \sum_{i=1}^{n} a_i |d_i f d\mu + d_i \lambda_s - b_i \mu| + c\left(f d\mu + \lambda_s\right).$$

We observe that $d_i f d\mu - b_i \mu$ and $d_i \lambda_s$ are mutually singular. Therefore, using Lemma 3,

$$\begin{aligned}\overline{N}_g(\lambda) &= \sum_{i=1}^{n} a_i \left|d_i f d\mu - b_i \mu\right| + c f d\mu + \sum_{i=1}^{n} a_i \left|d_i\right| \left|\lambda_s\right| + c\lambda_s \\ &= \overline{N}_g(f d\mu) + \sum_{i=1}^{n} a_i \left|d_i\right| \left|\lambda_s\right| + c\lambda_s.\end{aligned}$$

Using (16) and (17), we obtain, for each $E \in \Sigma$,

$$\overline{N}_g(\lambda)(E) = \int_E N_g\left(f\right) d\mu + \left(\frac{l^+(g) - l^-(g)}{2} |\lambda_s|\right)(E) + \left(\frac{l^+(g) + l^-(g)}{2} \lambda_s\right)(E).$$

Since

$$\lambda_s = \left(\lambda_s\right)^+ - \left(\lambda_s\right)^-$$

and

$$|\lambda_s| = \left(\lambda_s\right)^+ + \left(\lambda_s\right)^-,$$

after some cancellations, we can write

$$\overline{N}_g(\lambda)(E) = \int_E N_g\left(f\right) d\mu + \left(l^+(g)\left(\lambda_s\right)^+\right)(E) - \left(l^-(g)\left(\lambda_s\right)^-\right)(E).$$

This completes the proof of the proposition. □

We are now ready to consider an initial value problem associated with the operator $\overline{N}_g$, where g is the N-function given by (13).

7. An initial value problem associated with a piecewise linear N-function

We first introduce some preliminary definitions and results, starting with the following proposition:

Proposition 8. *The space $\mathcal{M}_f$ becomes a Banach space if we define*

$$\left\|\lambda\right\|_{\mathcal{M}_f} = \left|\lambda\right|(X). \tag{18}$$

For a proof of this result see [11], Theorem 1, p. 226.

Given $0 < T < +\infty$ fixed, we denote with $C\left[0, T; \mathcal{M}_f\right]$ the space of continuous functions $\lambda : [0, T] \to \mathcal{M}_f$. The space $C\left[0, T; \mathcal{M}_f\right]$ becomes a Banach space with the norm

$$\left\|\lambda\right\| = \sup_{0 \le t \le T} \left\|\lambda\left(t\right)\right\|_{\mathcal{M}_f}. \tag{19}$$

Likewise, the space $C^1\left[0, T; \mathcal{M}_f\right]$ of continuously differentiable functions $\lambda : [0, T] \to \mathcal{M}_f$, becomes a Banach space with the norm $\left\|\lambda\right\| + \left\|\lambda'\right\|$.

We now consider the function $G : [0, T] \times X \times \mathbb{R} \to \mathbb{R}$ defined as

$$G\left(t, x, u\right) = \sum_{i=1}^{n} a_i\left(x\right) \left|t d_i u - b_i\left(x\right)\right| + c\left(x\right) t u,$$

where, as in the previous section, $d_i \in \mathbb{R}, a_i, b_i, c : X \to \mathbb{R}$ are Σ-measurable, bounded functions and b_i is μ-integrable.

Now, we define the operator $\mathcal{A} = \mathcal{A}(t)$ on $C[0,T;\mathcal{M}_f]$ as

$$\mathcal{A}(\lambda)(t) = \sum_{i=1}^{n} a_i(x) |td_i\lambda - b_i(x)\mu| + c(x)t\lambda. \tag{20}$$

That is to say, if we define

$$G_t(x,u) = G(t,x,u) = \sum_{i=1}^{n} a_i(x) |td_iu - b_i(x)| + c(x)tu,$$

the operator $\mathcal{A}$, for $0 \le t \le T$ fixed, is

$$\mathcal{A}(\lambda)(t) = \overline{N}_{G_t}(\lambda(t)),$$

for $\lambda \in C[0,T;\mathcal{M}_f]$, where $\overline{N}_{G_t}$ is defined by (14).

Proposition 9. *The operator $\mathcal{A}$ satisfies the following conditions:*

(1) *Given $\lambda \in C[0,T;\mathcal{M}_f]$, $\mathcal{A}(\lambda)(t) \in \mathcal{M}_f$, for each $0 \le t \le T$.*
(2) *$|\mathcal{A}(\lambda)(t)| \le A|\lambda(t)| + B(x)\mu$, for $0 \le t \le T$, $x \in X$ and $\lambda \in C[0,T;\mathcal{M}_f]$, where A is a non-negative real number and $B \in L^1(\mu)$.*
(3) *$|\mathcal{A}(\lambda_1)(t) - \mathcal{A}(\lambda_2)(t)| \le A|\lambda_1(t) - \lambda_2(t)|$ for all $0 \le t \le T$, $x \in X$ and $\lambda_1, \lambda_2 \in C[0,T;\mathcal{M}_f]$, where the constant A is the same one as in 2).*
(4) *$\mathcal{A}$ is continuous and bounded from $C[0,T;\mathcal{M}_f]$ to itself.*

Proof. To simplify the notation, and without loss of generality, we will work with one term in (20). That is to say, we will assume for this proof that

$$\mathcal{A}(t)(\lambda) = a(x)|td\lambda - b(x)\mu| + c(x)t\lambda,$$

where $d_i \in \mathbb{R}, a, b, c : X \to \mathbb{R}$ are Σ-measurable, bounded functions and $b \in L^1(\mu)$.

1) follows directly from the definition of $\mathcal{A}$.

Proof of 2): By the properties of the total variation, we can write

$$\begin{aligned}|\mathcal{A}(\lambda)(t)| &\le |a(x)|(t|d||\lambda| + |b(x)|\mu) + t|c(x)||\lambda| \\ &= t(|a(x)||d| + |c(x)|)|\lambda| + |a(x)||b(x)|\mu,\end{aligned}$$

So, the estimate follows if we take

$$\begin{aligned}A &= T\sup_{x\in X}(|a(x)||d| + |c(x)|), \\ B(x) &= |a(x)||b(x)|.\end{aligned}$$

Proof of 3):

$$\begin{aligned}&|\mathcal{A}(\lambda_1)(t) - \mathcal{A}(\lambda_2)(t)| \\ &= |a(x)|td\lambda_1(t) - b(x)\mu| + c(x)t\lambda_1(t) - a(x)|td\lambda_2(t) - b(x)\mu| - c(x)t\lambda_2(t)| \\ &\le |a(x)|(||td\lambda_1(t) - b(x)\mu| - |td\lambda_2(t) - b(x)\mu||) \\ &+ |c(x)|t|\lambda_1(t) - \lambda_2(t)| \\ &\le |a(x)||d|t|\lambda_1(t) - \lambda_2(t)| + |c(x)|t|\lambda_1(t) - \lambda_2(t)| \\ &\le t(|a(x)||d| + |c(x)|)|\lambda_1(t) - \lambda_2(t)|,\end{aligned}$$

from which the estimate follows.

Proof of 4): According to 1), we know that given $\lambda \in C\left[0,T;\mathcal{M}_f\right]$, $\mathcal{A}\left(\lambda\right)\left(t\right)$ maps $[0,T]$ to $\mathcal{M}_f$. Let us prove that this function is continuous. If $\{t_j\}$ converges to t,

$$\begin{aligned}
&\left|\mathcal{A}\left(\lambda\right)\left(t_j\right)-\mathcal{A}\left(\lambda\right)\left(t\right)\right| \\
&= \left|a\left(x\right)\left|t_j d\lambda\left(t_j\right)-b\left(x\right)\mu\right|+c\left(x\right)t_j\lambda\left(t_j\right)-a\left(x\right)\left|td\lambda\left(t\right)-b\left(x\right)\mu\right|-c\left(x\right)t\lambda\left(t\right)\right| \\
&\leq \left(\left|a\left(x\right)\right|\left|d\right|+\left|c\left(x\right)\right|\right)\left|t_j d\lambda\left(t_j\right)-td\lambda\left(t\right)\right| \\
&\leq A\left(\left|t_j-t\right|\left|\lambda\left(t_j\right)\right|+t\left|\lambda\left(t_j\right)-\lambda\left(t\right)\right|\right).
\end{aligned}$$

So,

$$\left\|\mathcal{A}\left(\lambda\right)\left(t_j\right)-\mathcal{A}\left(\lambda\right)\left(t\right)\right\|_{\mathcal{M}_f} \leq A\left(\left|t_j-t\right|\left\|\lambda\right\|+T\left|\lambda\left(t_j\right)-\lambda\left(t\right)\right|\right) \underset{j\to\infty}{\to} 0.$$

We now prove that $\mathcal{A}$ is continuous from $C\left[0,T;\mathcal{M}_f\right]$ to itself. If $\{\lambda_j\}$ converges to λ in $C\left[0,T;\mathcal{M}_f\right]$, using 3) we can write

$$\sup_{0\leq t\leq T}\left|\mathcal{A}\left(\lambda_j\right)\left(t\right)-\mathcal{A}\left(\lambda\right)\left(t\right)\right|\left(X\right) \leq AT\sup_{0\leq t\leq T}\left|\lambda_j\left(t\right)-\lambda\left(t\right)\right|\left(X\right) \underset{j\to\infty}{\to} 0.$$

Finally, if $\mathcal{B}$ is a bounded set in $C\left[0,T;\mathcal{M}_f\right]$, using 2) we write

$$\sup_{\lambda\in\mathcal{B}}\left\|\mathcal{A}\left(\lambda\right)\right\| \leq AT\sup_{\lambda\in\mathcal{B}}\left\|\lambda\right\|+\int_X B\left(x\right)d\mu<\infty.$$

It is clear that the same proof works for the full operator given by (20), with constants

$$\begin{aligned}
A &= T\sup_{x\in X}\left(\sum_{i=1}^{n}\left|a_i\left(x\right)\right|\left|d_i\right|+\left|c\left(x\right)\right|\right), \\
B\left(x\right) &= \sum_{i=1}^{n}\left|a_i\left(x\right)\right|\left|b_i\left(x\right)\right|.
\end{aligned}$$

This completes the proof of the proposition. □

Given $\lambda_0 \in \mathcal{M}_f$, we consider the initial value problem

$$\left\{\begin{array}{rcl}\frac{d\lambda}{dt}-\mathcal{A}(\lambda)\left(t\right) &=& 0 \text{ for } 0<t<T \\ \lambda(0) &=& \lambda_0\end{array}\right., \tag{21}$$

where $\mathcal{A}=\mathcal{A}\left(t\right)$ is defined by (20).

We recall the following well known extension of the Banach fixed point theorem:

Proposition 10. *Let (S,d) be a complete metric space and consider a map $f:S\to S$. If there exists $k\in\{1,2,...\}$ such that the composite map $f^{(k)}$ is a contraction, then the map f has a unique fixed point.*

Theorem 11. *The initial value problem (21) has one and only one solution in $C^1\left[0,T;\mathcal{M}_f\right]$.*

Proof. We begin by observing that (21) has the same solutions in $C^1\left[0,T;\mathcal{M}_f\right]$ as the integral equation

$$\lambda(t)=\lambda_0+\int_0^t\mathcal{A}\left(\lambda\right)\left(s\right)ds. \tag{22}$$

Thus, to prove the theorem, it suffices to show that (22) has one and only one solution in $C^1[0,T;\mathcal{M}_f]$, by proving that the operator $\mathcal{T}$ defined on $C[0,T;\mathcal{M}_f]$ as

$$\mathcal{T}(\lambda)(t) = \lambda_0 + \int_0^t \mathcal{A}(\lambda)(s)\,ds \tag{23}$$

has a unique fixed point. According to Proposition 10, it is enough to show that the composite operator $\mathcal{T}^{(k)}$, where $\mathcal{T}$ is given by (23), is a contraction on $C[0,T;\mathcal{M}_f]$ for some $k \in \{1,2,...\}$.

We claim that $\mathcal{T}^{(k)}$ satisfies the following estimate:

$$\left\|\mathcal{T}^{(k)}(\lambda_1)(t) - \mathcal{T}^{(k)}(\lambda_2)(t)\right\|_{\mathcal{M}_f} \leq \frac{A^k t^k}{k!}\|\lambda_1 - \lambda_2\|. \tag{24}$$

We will prove this claim by induction on k.

For $k=1$, we can write, using (23) and Part 3) of Proposition 9,

$$\begin{aligned}
&\|\mathcal{T}(\lambda_1)(t) - \mathcal{T}(\lambda_2)(t)\|_{\mathcal{M}_f} \\
&\leq \int_0^t \|\mathcal{A}(\lambda_1)(s) - \mathcal{A}(\lambda_2)(s)\|_{\mathcal{M}_f}\,ds \\
&\leq A\int_0^t \|\lambda_1(s) - \lambda_2(s)\|_{\mathcal{M}_f}\,ds \\
&\leq At\,\|\lambda_1 - \lambda_2\|,
\end{aligned}$$

which gives us (24) for $k=1$. Assuming that (24) holds for $k=n$, let us prove it for $k=n+1$.

$$\begin{aligned}
&\|\mathcal{T}^{(n+1)}(\lambda_1)(t) - \mathcal{T}^{(n+1)}(\lambda_2)(t)\|_{\mathcal{M}_f} \\
&\leq \int_0^t \left\|\mathcal{A}\left(\mathcal{T}^{(n)}(\lambda_1)\right)(s) - \mathcal{A}\left(\mathcal{T}^{(n)}(\lambda_2)\right)(s)\right\|_{\mathcal{M}_f}\,ds \\
&\leq A\int_0^t \|\mathcal{T}^{(n)}(\lambda_1)(s) - \mathcal{T}^{(n)}(\lambda_2)(s)\|_{\mathcal{M}_f}ds \\
&\leq A\int_0^t \frac{A^n s^n}{n!}\|\lambda_1(s) - \lambda_2(s)\|_{\mathcal{M}_f}\,ds \leq \frac{A^{n+1}t^{n+1}}{(n+1)!}\|\lambda_1 - \lambda_2\|,
\end{aligned}$$

so the estimate (24) holds. Finally,

$$\begin{aligned}
\left\|\mathcal{T}^{(k)}(\lambda_1) - \mathcal{T}^{(k)}(\lambda_2)\right\| &= \sup_{0\leq t\leq T}\left\|\mathcal{T}^{(k)}(\lambda_1)(t) - \mathcal{T}^{(k)}(\lambda_2)(t)\right\|_{\mathcal{M}_f} \\
&\leq \frac{A^k T^k}{k!}\|\lambda_1 - \lambda_2\|.
\end{aligned}$$

Since $\frac{(AT)^k}{k!} \to 0$ as $k \to \infty$, we can conclude that the operator $\mathcal{T}^{(k)}$ will be a contraction on $C[0,T;\mathcal{M}_f]$, for k large enough.

This concludes the proof of the theorem. □

Remark 6. *From the proof of Theorem 11, we can see that the estimate of $\mathcal{T}^{(k)}$ for $k=1$, that is to say the estimate of $\mathcal{T}$, will just give a solution to the equation (22) for T sufficiently small. However, using an observation that appears in* [3] *we can obtain a unique global solution of (22) without using iterations. The idea is to* *replace the norm on* $C\left[0,T;\mathcal{M}_f\right]$ *with the norm*

$$\|\lambda\|_* = \sup_{0\le t\le T}\left(e^{-At}\|\lambda(t)\|_{\mathcal{M}_f}\right), \tag{25}$$

where A *is the constant appearing in Part 3 of Proposition 9. From the inequalities*

$$e^{-AT}\|\lambda(t)\|_{\mathcal{M}_f} \le e^{-At}\|\lambda(t)\|_{\mathcal{M}_f} \le \|\lambda(t)\|_{\mathcal{M}_f}$$

we can see that the norm given by (25) is equivalent to the original norm given by (19). Now, we claim that the operator $\mathcal{T}$ *satisfies the inequality*

$$\|\mathcal{T}(\lambda_1)-\mathcal{T}(\lambda_2)\|_* \le \left(1-e^{-AT}\right)\|\lambda_1-\lambda_2\|_*, \tag{26}$$

that is, it is a contraction with respecto to the new norm on $C\left[0,T;\mathcal{M}_f\right]$*. In fact,*

$$\begin{aligned}
&\|\mathcal{T}(\lambda_1)(t)-\mathcal{T}(\lambda_2)(t)\|_{\mathcal{M}_f}\\
&\le \int_0^t \|\mathcal{A}(\lambda_1)(s)-\mathcal{A}(\lambda_2)(s)\|_{\mathcal{M}_f}\,ds\\
&\le A\int_0^t \|\lambda_1(s)-\lambda_2(s)\|_{\mathcal{M}_f}\,ds\\
&= A\int_0^t e^{sA}e^{-sA}\|\lambda_1(s)-\lambda_2(s)\|_{\mathcal{M}_f}\,ds\\
&\le \|\lambda_1-\lambda_2\|_*\int_0^t Ae^{sA}ds = \left(e^{tA}-1\right)\|\lambda_1-\lambda_2\|_*,
\end{aligned}$$

from which (26) follows.

8. An initial value problem associated with a class of non-negative N-functions

As before, we fix a complete σ-finite measure space (X,Σ,μ). We denote

$$\mathcal{M}_{fa}(\mu) = \{\lambda:\Sigma\to\mathbb{R} \text{ signed measure; } \lambda << \mu\}$$

and we consider on $\mathcal{M}_{fa}(\mu)$ the same norm we used on $\mathcal{M}_f$,

$$\|\lambda\|_{\mathcal{M}_{fa}(\mu)} = |\lambda|(X).$$

Remark 7. *With this norm,* $\mathcal{M}_{fa}(\mu)$ *is a closed subspace of* $\mathcal{M}_f$ *and, thus, it is a Banach space. In fact, let* $\{\lambda_j\}_{j\ge1}$ *be a sequence in* $\mathcal{M}_{fa}(\mu)$ *converging to* λ *in* $\mathcal{M}_f$*. Then for each* j*,* $\lambda_j = f_j d\mu$*, with* $f_j\in L^1(\mu)$*. We claim that* $\{f_j\}_{j\ge1}$ *is a Cauchy sequence in* $L^1(\mu)$*.*

$$\begin{aligned}
\|f_j-f_k\|_{L^1(\mu)} &= \int_X |f_j(x)-f_k(x)|\,d\mu = |\lambda_j-\lambda_k|(X)\\
&= \|\lambda_j-\lambda_k\|_{\mathcal{M}_f} \underset{j,k\to\infty}{\to} 0.
\end{aligned}$$

Thus, $\{f_j\}_{j\ge1}$ *converges in* $L^1(\mu)$ *to some function* f*. We claim that* $\lambda = fd\mu$*.*

$$\begin{aligned}
&\|\lambda-fd\mu\|_{\mathcal{M}_f}\\
&\le \|\lambda-\lambda_j\|_{\mathcal{M}_f} + \|\lambda_j-fd\mu\|_{\mathcal{M}_f}\\
&= \|\lambda-\lambda_j\|_{\mathcal{M}_f} + \|(f_j-f)\,d\mu\|_{\mathcal{M}_f}\\
&= \|\lambda-\lambda_j\|_{\mathcal{M}_f} + \|f_j-f\|_{L^1(\mu)} \underset{j\to\infty}{\to} 0.
\end{aligned}$$

Notice that, in particular, we have really proved that $\mathcal{M}_{fa}(\mu)$ *is isometrically isomorphic to* $L^1(\mu)$*, as we mentioned before. Indeed,*

$$\|f d\mu\|_{\mathcal{M}_{fa}(\mu)} = |f d\mu|(X) = \int_X |f(x)|\, d\mu = \|f\|_{L^1(\mu)},$$

([11]*, p. 94).*

For $0 < T < +\infty$ fixed, $C[0,T;\mathcal{M}_{fa}(\mu)]$ is the space of continuous functions $\lambda : [0,T] \to \mathcal{M}_{fa}(\mu)$. It becomes a Banach space with the norm

$$\|\lambda\| = \sup_{0\le t\le T} \|\lambda(t)\|_{\mathcal{M}_{fa}(\mu)}.$$

Likewise, the space $C^1[0,T;\mathcal{M}_{fa}(\mu)]$ of continuously differentiable functions $\lambda : [0,T] \to \mathcal{M}_{fa}(\mu)$, becomes a Banach space with the norm $\|\lambda\| + \|\lambda'\|$, while $C\left[0,T;L^1(\mu)\right]$ is the space of continuous functions $f : [0,T] \to L^1(\mu)$, which becomes a Banach space with the norm

$$\|f\| = \sup_{0\le t\le T} \|f(t)\|_{L^1(\mu)}.$$

The spaces $C[0,T;\mathcal{M}_{fa}(\mu)]$ and $C\left[0,T;L^1(\mu)\right]$ are isometrically isomorphic. According to all we have said before, it suffices to prove that $C[0,T;\mathcal{M}_{fa}(\mu)]$ is isometrically isomorphic to the subspace $\mathcal{C}(\mu)$ of $C[0,T;\mathcal{M}_f]$ defined as

$$\mathcal{C}(\mu) = \left\{\lambda \in C[0,T;\mathcal{M}_f] : \lambda(t) = f(t)\, d\mu \text{ for } f \in C\left[0,T;L^1(\mu)\right]\right\}.$$

We first prove that $\mathcal{C}(\mu) = C[0,T;\mathcal{M}_{fa}(\mu)]$ as sets. It is clear that $\mathcal{C}(\mu) \subseteq C[0,T;\mathcal{M}_{fa}(\mu)]$. Conversely, if $\lambda \in C[0,T;\mathcal{M}_{fa}(\mu)]$, then for each $0 \le t \le T$, there is a function $f_t \in L^1(\mu)$ so that $\lambda(t) = f_t d\mu$. As for the continuity of the function $t \to f_t$, if $\{t_j\}_{j\ge1}$ converges to t,

$$\left\|f_{t_j} - f_t\right\|_{L^1(\mu)} = \int_X \left|f_{t_j} - f_t\right| d\mu = \|\lambda(t_j) - \lambda(t)\|_{\mathcal{M}_{fa}(\mu)} \underset{j\to\infty}{\to} 0.$$

A similar estimate will show that $C[0,T;\mathcal{M}_{fa}(\mu)]$ and $\mathcal{C}(\mu)$ are isometrically isomorphic. Consequently, each of these spaces is isometrically isomorphic to $C\left[0,T;L^1(\mu)\right]$.

Given $f_0 \in L^1(\mu)$, we formulate the initial value problem

$$\left\{\begin{array}{rcl} \frac{df}{dt} - \mathcal{A}(f)(t) & = & 0 \text{ for } 0 < t < T \\ f(0) & = & f_0 \end{array}\right., \tag{27}$$

where

$$\mathcal{A}(f)(t) = G(t,x,f), \tag{28}$$

for some function $G : [0,T] \times X \times \mathbb{R} \to \mathbb{R}$ to be defined later.

Likewise, given $\lambda_0 \in \mathcal{M}_{fa}(\mu)$, we consider the initial value problem

$$\left\{\begin{array}{rcl} \frac{d\lambda}{dt} - \overline{\mathcal{A}}(\lambda)(t) & = & 0 \text{ for } 0 < t < T \\ \lambda(0) & = & \lambda_0 \end{array}\right., \tag{29}$$

where

$$\overline{\mathcal{A}}(\lambda)(t) = G(t,x,f)\, d\mu.$$

The solutions in $C^1[0,T;\mathcal{M}_{fa}(\mu)]$ of (29) are exactly the measures of the form $\lambda = f d\mu$, where the function f runs throught all the solutions in $C^1\left[0,T;L^1(\mu)\right]$ of the problem (27). The proof of this claim follows from the relationship we have shown between continuous functions with values in $\mathcal{M}_{fa}(\mu)$ and continuous functions with values in $L^1(\mu)$. So, to prove that (29) has one and only one solution in $C^1[0,T;\mathcal{M}_{fa}(\mu)]$, it suffices to prove that (27) has one and only one solution in $C^1\left[0,T;L^1(\mu)\right]$.

We now formulate conditions on the function G, so that (27) has one and only one solution in $C^1\left[0,T;L^1(\mu)\right]$. We begin with the following lemma:

Lemma 12. *Let $G:[0,T]\times X\times\mathbb{R}\to\mathbb{R}$ be a function satisfying the conditions:*

(1) $|G(t,x,u)|\le a(x)+b|u|$, *for some $a\in L^1(\mu)$ and $b\ge 0$, for μ-a.e. $x\in X$, $0\le t\le T$ and $u\in\mathbb{R}$.*
(2) *The function $x\to G(t,x,u)$ is Σ-measurable for each $0\le t\le T$ and $u\in\mathbb{R}$.*
(3) *There exists $C>0$ such that $|G(t,x,u_1)-G(t,x,u_2)|\le C|u_1-u_2|$, for $0\le t\le T$, $u_1,u_2\in\mathbb{R}$ and for μ-a.e. $x\in X$.*
(4) *There exists $C>0$ such that $|G(t_1,x,u)-G(t_2,x,u)|\le C|u||t_1-t_2|$, for $0\le t_1,t_2\le T$, $u\in\mathbb{R}$ and for μ-a.e. $x\in X$.*

Then, the following properties hold:

a): *For each $0\le t\le T$, the function $G_t:X\times\mathbb{R}\to\mathbb{R}$ defined as $G_t(x,u)=G(t,x,u)$ is an N-function.*
b): *For each $0\le t\le T$, the Nemyckiĭ operator N_{G_t} maps $L^1(\mu)$ to itself.*
c): *The function $f\to N_{G_t}(f)$ maps $C[0,T;L^1(\mu)]$ continuously into itself.*

Proof. The proof of a) is a direct application of 2) and 3), while the proof of b) follows from 1). To prove c) we begin by observing that given $f\in C[0,T;L^1(\mu)]$, the function $N_{G_t}(f)(x)$ belongs to $L^1(\mu)$ for each $0\le t\le T$, as a consequence of b). Moreover, $N_{G_t}(f)$ belongs to $C[0,T;L^1(\mu)]$ because of 4. Finally, if $\{f_j\}_{j\ge1}$ converges to f in $C[0,T;L^1(\mu)]$, we use 3) to write

$$\begin{aligned}\|N_{G_t}(f_j)(t)-N_{G_t}(f)(t)\|_{L^1(\mu)}&\le C\|f_j(t)-f(t)\|_{L^1(\mu)}\\&\le C\|f_j-f\|.\end{aligned}$$

Thus,

$$\sup_{0\le t\le T}\|N_{G_t}(f_j)(t)-N_{G_t}(f)(t)\|_{L^1(\mu)}\underset{j\to\infty}{\to}0.$$

This completes the proof of the lemma. □

Theorem 13. *The initial value problem (27) has one and only one solution in $C^1\left[0,T;L^1(\mu)\right]$ if we assume that the operator $\mathcal{A}$ is given by (28) and the function G satisfies the conditions stated in Lemma 12.*

Proof. The proof goes along the same lines as the proof of Theorem 11. The initial value problem (27) has the same solutions in $C^1[0,T;L^1(\mu)]$ as the integral equation

$$f(t)=f_0+\int_0^t\mathcal{A}(f)(s)\,ds. \tag{30}$$

We show that (30) has one and only one solution in $C^1[0,T;L^1(\mu)]$ by proving that the operator $\mathcal{T}$ defined on $C[0,T;L^1(\mu)]$ as

$$\mathcal{T}(f)(t)=f_0+\int_0^t\mathcal{A}(f)(s)\,ds$$

has a unique fixed point. According to Proposition 10, it suffices to show that $\mathcal{T}^{(k)}$ is a contraction in $C[0,T;L^1(\mu)]$ for some $k\in\{1,2,...\}$, for which it is enough to prove that for $f_1, f_2 \in C[0,T; L^1(\mu)]$ and $k \in \{1,2,...\}$,

$$\|\mathcal{T}^{(k)}(f_1)(t) - \mathcal{T}^{(k)}(f_2)(t)\|_{L^1(\mu)} \leq C^k \frac{t^k}{k!}\|f_1 - f_2\|. \tag{31}$$

When $k = 1$,

$$\begin{aligned}
&\|\mathcal{T}(f_1)(t) - \mathcal{T}(f_2)(t)\|_{L^1(\mu)} \\
&\leq \int_0^t \|\mathcal{A}(f_1)(s) - \mathcal{A}(f_2)(s)\|_{L^1(\mu)}\, ds \\
&= \int_0^t \|G(t,x,f_1(s)) - G(t,x,f_2(s))\|_{L^1(\mu)} \\
&\leq Ct\,\|f_1 - f_2\|.
\end{aligned}$$

We prove now that (31) holds for $k = n+1$, assuming that it holds for $k = n$.

$$\begin{aligned}
&\|\mathcal{T}^{(n+1)}(f_1)(t) - \mathcal{T}^{(n+1)}(f_2)(t)\|_{L^1(\mu)} \\
&= \|\int_0^t (\mathcal{A}(\mathcal{T}^{(n)}(f_1))(s) - \mathcal{A}(\mathcal{T}^{(n)}(f_2))(s))ds\|_{L^1(\mu)} \\
&\leq \int_0^t \|G(s,\cdot,\mathcal{T}^{(n)}(f_1)(s)) - G(s,\cdot,\mathcal{T}^{(n)}(f_2)(s))\|_{L^1(\mu)}ds \\
&\leq C\int_0^t \|\mathcal{T}^{(n)}(f_1)(s) - \mathcal{T}^{(n)}(f_2)(s)\|_{L^1(\mu)}ds \\
&\leq C\int_0^t \frac{C^n s^n}{n!}\|f_1 - f_2\|ds = \frac{C^{n+1}t^{n+1}}{(n+1)!}\|f_1 - f_2\|.
\end{aligned}$$

As in Theorem 11, (31) implies the estimate

$$\|\mathcal{T}^{(k)}(f_1) - \mathcal{T}^{(k)}(f_2)\| \leq C^k \frac{t^k}{k!}\|f_1 - f_2\|,$$

from which the result follows. This completes the proof of the theorem. □

Remark 8. *As an illustration, we present now an example of an operator $\mathcal{A}$ as considered in Theorem 13. With this purpose, we construct first a function G that satisfies the hypothesis of Lemma 12. We fix a function $H : \mathbb{R} \to \mathbb{R}$ satisfying the two conditions:*

1. There exists $C_1 > 0$ such that $|H(r)| \leq C_1 |r|$ for all $r \in \mathbb{R}$.

2. H is a Lipschitz function; that is to say, there exists $C_2 > 0$ such that $|H(r_1) - H(r_2)| \leq C_2 |r_1 - r_2|$ for all $r_1, r_2 \in \mathbb{R}$.

Then, given $a \in L^1(\mu)$, we define $G : [0,T] \times X \times \mathbb{R} \to \mathbb{R}$ as $G(t,x,u) = H(a(x) + tu)$.

We claim that G satisfies conditions 1) through 4) in Lemma 12.

In fact, $|G(t,x,u)| = |H(a(x)+tu)| \leq C_1 |a(x)+tu| \leq C_1 (|a(x)| + T|u|)$, so condition 1) is satisfied.

If we fix $0 \leq t \leq T$, $u \in \mathbb{R}$, the function $x \to H(a(x)+tu)$ is Σ-measurable, because it is the composition, in the appropriate order, of the Σ-measurable function $x \to a(x)$ and the continuous function $r \to H(r+tu)$. So condition 2) holds.

We can write

$$\begin{aligned} &|G(t,x,u_1) - G(t,x,u_2)| \\ &= |H(a(x) + tu_1) - H(a(x) + tu_2)| \\ &\leq C_2 t\,|u_1 - u_2| \leq C_2 T\,|u_1 - u_2|\,. \end{aligned}$$

Therefore, condition 3) is satisfied.

Finally, if we fix $0 \leq t_1, t_2 \leq T$, $x \in X$, $u \in \mathbb{R}$, *we have that*

$$\begin{aligned} &|G(t_1,x,u) - G(t_2,x,u)| \\ &= |H(a(x) + t_1 u) - H(a(x) + t_2 u)| \leq C_2\,|u|\,|t_1 - t_2|\,. \end{aligned}$$

Thus, condition 4) is satisfied as well.

According to Lemma 12, for each $0 \leq t \leq T$, *the function* $G_t : X \times \mathbb{R} \to \mathbb{R}$ *defined as* $G_t(x,u) = H(a(x) + tu)$ *is an* N*-function.*

If $f \in C[0,T; L^1(\mu)]$, *we can define the operator*

$$\mathcal{A} : C[0,T; L^1(\mu)] \to C[0,T; L^1(\mu)]$$

as

$$\mathcal{A}(f)(t) = N_{G_t}(f(t)) = H(a(\cdot) + tf(t,\cdot)).$$

Remark 9. *According to Remark 5, Theorem 13 applies, with the obvious changes in notation, to a system of* m *equations in* n *unknowns.*

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