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ON THE SPECTRAL STUDY OF SINGULAR STURM-LIOUVILLE PROBLEM WITH SIGN-VALUED WEIGHT

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Abstract. In this paper we study the asymptotic behavior of the eigenvalues, eigenfunctions and normalization numbers of some versions of singular Sturm-Liouville problem with sign valued weight. This paper is a part of series of spectral problems with turning point suggested in [9], also the results obtained help in the inverse problem investigation which will be discussed later. In fact this problem contains more analytical difficulties than the classical case, this difficulties comes from the introduction of the weight function as $\pm$ which forced us to treat the problem as two separated problems joint at x=a.

1. Introduction

Consider the following Sturm-Liouville problem

$$-y'' + q(x)\, y = \lambda\, \rho(x)\, y \; 0 \leq x \leq \pi \tag{1}$$

$$y(0) = 0 \;,\; y'(\pi) + H\, y(\pi) = 0, \tag{2}$$

where the non-negative real function $q(x)$ has a second piecewise integrable derivatives on $(0,\pi)$, H is positive number, λ is a spectral parameter and weighted function or the explosive factor $\rho(x)$ is of the form

$$\rho(x) = \begin{cases} 1 \;;\; 0 \leq x \leq a < \pi \\ -1 \;;\; a < x \leq \pi. \end{cases} \tag{3}$$

In the study of Sturm - Liouville problem many authors avoid the introduction of a negative weight function because of its extra analytical difficulties see[7]. It should be noted here that the introduction of negative weight function $\rho(x)$ in the form (3) gives rise to analytic difficulties although it is of less physical interest. The author had studied the factor $\rho(x) = \mp$ in [1-5]. In [1] the author studied the boundary value problem (1) subject to the Sturm-Liouville condition $y(0) = 0$, $y(\pi) = 0$ and also the author studied the same boundary value problem (1) subject to another separated boundary condition
$y'(0) - h\, y(0) = 0$, $y'(\pi) + H\, y(\pi) = 0$,

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where h, H are positive numbers.The present Sturm-Liouville condition (1) cannot be considered as a special case of that separated boundary condition.
In the following lemma we prove some helping properties related to the study of the eigenvalues and eigenfunctions of the boundary value problem (1)-(2)
lemma 1 The eigenvalues of Sturm-Liouville problem (1)-(2) are real.
Proof. Let $y_o(x)$ be the eigenfunction that corresponds to the eigenvalue of λ_o of the problem(1)-(2).Then

$$\begin{aligned} -y_o'' \ + \ q(x)\ y_o = \lambda_o\ \rho(x)\ y_o\ 0 \leq x \leq \pi \\ y_o(0)\ =\ 0\ ,\ y_o'(\pi) + H y_o(\pi) = 0. \end{aligned} \tag{4}$$

Multiplying both sides of (4) by $\bar{y}_o$ denotes the conjugate of y_o and then the integrate from 0 to π with respect to x, we have
$\bar{y}_o\ y_o'|_0^\pi + \int_0^\pi |y_o'|^2 dx + \int_0^\pi\ q(x)|y_o|^2 dx = \lambda_o \int_0^\pi \rho(x)|y_o|^2 dx.$
Using the boundary condition (4), we have

$$\lambda_o \ = \ \frac{\int_0^\pi\ (q(x)|y_o|^2\ +\ |y_o'|^2)dx\ +\ H\ |y_o(\pi)|^2}{\int_0^\pi \rho(x)|y_o|^2 dx.}$$

Where H is positive real number, from which it follows the reality of the eigenvalue λ_o .
lemma 2 Two eigenfunctions that corresponding to two different eigenvalues of the Sturm-Liouville problem (1.1)-(1.2) are orthogonal with weight $\rho(x)$.
Proof. Let $\lambda_1 \neq \lambda_2$ be two different eigenvalues of the Sturm-Liouville problem (1)-(2), and let $y_1(x), y_2(x)$ be the corresponding eigenfunctions. We show that $\int_o^\pi \rho(x)\ y_1(x)\ \bar{y}_2(x) dx\ =\ 0$, so that

$$\begin{aligned} -y_1'' \ + \ q(x)\ y_1 = \lambda_1\ \rho(x)\ y_1\ 0 \leq x \leq \pi \\ y_1(0)\ =\ 0\ ,\ y_1'(\pi) + H y_1(\pi) = 0. \end{aligned} \tag{5}$$

$$\begin{aligned} -y_2'' \ + \ q(x)\ y_2 = \lambda_2\ \rho(x)\ y_2\ 0 \leq x \leq \pi \\ y_2(0)\ =\ 0\ ,\ y_2'(\pi) + H y_2(\pi) = 0. \end{aligned} \tag{6}$$

Multiplying both sides of (5) by $\bar{y}_2$ and then integrating with respect to $x \in [0, \pi]$, we have

$$-\int_0^\pi\ y_1''\bar{y}_2 dx\ +\ \int_0^\pi\ q(x) y_1 \bar{y}_2 dx\ =\ \lambda_1 \int_0^\pi\ \rho(x)\ y_1 \bar{y}_2 dx, \tag{7}$$

by taking the complex conjugate of (6), and multiply it by y_1 and integrate the resulting expression with respect to $x \in [0, \pi]$, we have

$$-\int_0^\pi\ y_1 \bar{y}_2'' dx\ +\ \int_0^\pi\ q(x) y_1 \bar{y}_2 dx\ =\ \lambda_2 \int_0^\pi\ \rho(x)\ y_1 \bar{y}_2 dx, \tag{8}$$

subtracting (7) from (8) and using the boundary conditions of (5) and (6) we obtain $(\ \lambda_1\ -\ \lambda_2\ ) \displaystyle\int_0^\pi \rho(x)\ y_1(x)\ \bar{y}_2(x)\ dx\ =\ 0,\ \lambda_1 \neq \lambda_2$, which complete the proof.

Lemma 1 deals with the nature of the eigenvalues, now turn to the study of eigenvalues for the Sturm-Liouville problem (1)-(2). To investigate the eigenvalues we need to introduce its characteristic equation. Let $\varphi(x, \lambda)$,$\psi(x, \lambda)$, be two solutions of (1) which satisfy the conditions

$$\varphi(0,\lambda) = 0, \; \varphi'(0,\lambda) = 1 \tag{9}$$

$$\psi(\pi,\lambda) = 1, \; \psi'(\pi,\lambda) = -H. \tag{10}$$

Its clear that $\varphi(x,\lambda)$ satisfies the boundary condition (2) at $x = 0$ and $\psi(x,\lambda)$ satisfies the boundary conditions at $x = \pi$. Denoted the Wronskian of these two solutions by $W(\lambda)$ where

$$W(\lambda) = \varphi(x,\lambda)\,\psi'(x,\lambda) - \varphi'(x,\lambda)\,\psi(x,\lambda). \tag{11}$$

Notice that the Wornskain of these two solutions of the equation (1) are linearly independent of x and then $W(\lambda) \neq 0$.

lemma 3 The eigenvalues of the Sturm-Liouville problem (1)-(2) coincide with the roots of the equation $W(\lambda) = 0$.

Proof. We prove that $W(\lambda_o) = 0$ if and only if λ_o is an eigenvalue. Let $W(\lambda) = 0$ and let λ_o be its root i.e $W(\lambda_o) = 0$ so that the solutions $\varphi(x,\lambda_o)$, $\psi(x,\lambda_o)$ are linearly dependent, this means that $\varphi(x,\lambda_o)$ satisfies the boundary condition at $x = \pi$ so that, $\varphi(x,\lambda_o)$ is an eigenfunction of Sturm-Liouville problem (1)-(2) and then λ_o is an eigenvalue. Suppose now that λ_o is an eigenvalue of the Sturm-Liouville problem (1)-(2). We must show that $W(\lambda_o) = 0$, so that we assume the contrary i.e let $W(\lambda_o) \neq 0$, this means that an eigenfunction $y_o(x)$ is a linear combination of the linearly independent solutions $\varphi(x,\lambda_o), \psi(x,\lambda_o)$ so that

$$y_o(x) = c_1\varphi(x,\lambda_o) + c_2\psi(x,\lambda_o)$$

$y_o(x)$ must satisfy the boundary conditions (2), from which we have $c_2\psi(0,\lambda_o) = 0$ and $c_1\varphi'(\pi,\lambda_o) + H\,\varphi(\pi,\lambda_o) = 0$ consequently, the constants c_1, c_2 cannot be both zeros. Let $c_1 \neq 0$ and hence $\varphi'(\pi,\lambda_o) + H\,\varphi(\pi,\lambda_o) = 0$. From (11) at $x = \pi$ and the conditions (9)-(10), it follows that $W(\lambda_o) = 0$ which contradicts the assumption $W(\lambda_o) \neq 0$, and the proof is completed.

We notice that formula (11) is independent of x since $\rho(x)$ and consequently $\varphi(x,\lambda_o)$, $\psi(x,\lambda_o)$ depends on a, it is suitable to take in (11) $x = a$:

$$W(\lambda) = \varphi(a,\lambda)\,\psi'(a,\lambda) - \varphi'(a,\lambda)\,\psi(a,\lambda). \tag{12}$$

The function $W(\lambda)$ of (12) is an integral function for every $x \in [0,\pi]$. To study roots of $W(\lambda)$ it is necessary to know the asymptotic behavior of this function and consequently $\varphi(x,\lambda_o)\;\psi(x,\lambda_o)$ for $|\lambda| \to \infty$.

2. The asymptotic formula for the solution

We put $\lambda = s^2$ in (1), we notice that equation (1) is equivalent to

$$-y'' + q(x)\,y = s^2\,y \;\; 0 \le x \le a, \tag{13}$$

$$-y'' + q(x)\,y = -s^2\,y \;\; a \le x \le \pi. \tag{14}$$

We denote by $\varphi(x,\lambda)$ the solution of the equation (13), and $\psi(x,\lambda)$ the solution of the equation of(14), subject to the boundary conditions (9)-(10).

lemma 4 The solutions $\varphi(x,\lambda)$ of problem (13)-(9) and $\psi(x,\lambda)$ of problem (14)-(10)satisfy the integral equations

$$\varphi(x,\lambda) = \frac{\sin sx}{s} + \int_0^x \frac{\sin s(x-\tau)}{s}\,q(\tau)\,\varphi(\tau,s)\,d\tau \tag{15}$$

$$\psi(x,\lambda) \;=\; \cosh s(\pi-x) \;+\; \frac{H}{s}\sinh s(\pi-x) \;-\; \frac{1}{s}\int_x^{\pi} \;\frac{\sinh s(x-\tau)}{s}\; q(\tau)\;\psi(\tau,s)\;d\tau \tag{16}$$

Proof.
First we show that the integral representation (15) satisfies the problem (13)-(9). Let $\varphi(x,\lambda)$ be the solution of (13), so that

$$q(x)\;\varphi(x,\lambda) \;=\; \varphi''(x,\lambda) \;+\; s^2\;\varphi(x,\lambda).$$

We multiply both sides by $\dfrac{\sin s(x-\tau)}{s}$ and integrating with respect to τ from 0 to x we obtain

$$\int_0^x \frac{\sin s(x-\tau)}{s} q(\tau)\varphi(\tau,\lambda)d\tau \;=\int_0^x \frac{\sin s(x-\tau)}{s}\varphi''(\tau,\lambda)d\tau + s^2\int_0^x \frac{\sin s(x-\tau)}{s}\varphi(\tau,\lambda)d\tau, \tag{17}$$

integration by parts twice and using the condition (9), we have

$$\int_0^x \;\frac{\sin s(x-\tau)}{s}\;\varphi''(\tau,\lambda)\;d\tau \;=\varphi(x,\lambda) - \frac{\sin sx}{s} - s\int_0^x \sin s(x-\tau)\;\varphi(\tau,\lambda)\;d\tau. \tag{18}$$

By substituting from (18) into (17) we get the required formula (15). Moreover formula (15) can be obtained by the direct calculation through the variation of parameter. Let $q(x) \;= 0$ equation(13) becomes $-y'' \;=\; s^2\;y$, after varying the constants $c_1, andc_2$ the solution $\varphi(x,\lambda)$ becomes

$$\varphi(x,\lambda) \;=\; c_1(x,s)\;\cos sx \;+\; c_2(x,s)\;\sin sx \tag{19}$$

and the direct calculation of $c_1(x,s)$ and $c_2(x,s)$ from the variation of parameter method gives

$$\begin{aligned} c_1(x,s) &= \;-\int_0^x \frac{\sin s\tau}{s}\;q(\tau)\;\varphi(\tau,\lambda)\;d\tau,\\ c_2(x,s) &= \;\frac{1}{s} \;+\; \int_0^x \frac{\cos s\tau}{s}\;q(\tau)\;\varphi(\tau,\lambda)\;d\tau. \end{aligned} \tag{20}$$

Substituting from (20) into (19) equation (15) follows. The proof and derivation of the second integral equation (16) we can obtain it in similar way, when we apply the method of variation of parameter method of variation of parameter after varying the constants $m_1,\; m_2$. Indeed, for $q(x) \;= 0$ equation(14) becomes $y'' \;=\; s^2\;y$ with solution of the form

$$\psi(x,\lambda) \;=\; m_1(x,s)\;\cosh sx \;+\; m_2(x,s)\;\sinh sx \tag{21}$$

and the direct calculation of $m_1(x,s)$ and $m_2(x,s)$ from the variation of parameter method gives

$$\begin{aligned} m_1(x,s) &= \;\cosh s\pi \;+\; \frac{H}{s}\sinh s\pi \;+\; \frac{1}{s}\int_x^{\pi}\;q(\tau)\;\psi(\tau,\lambda)\;\sinh s\tau\;d\tau,\\ m_2(x,s) &= \;-\sinh s\pi \;-\; \frac{H}{s}\;\cosh s\pi \;-\; \frac{1}{s}\int_x^{\pi}\;q(\tau)\;\psi(\tau,\lambda)\;\cosh s\tau\;d\tau. \end{aligned} \tag{22}$$

The following lemma is concerned with the calculation of the asymptotic formulas of the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ of the Sturm-Liouville problem (1)-(2).
lemma 5 Let $\lambda = \;s^2,\; s \;=\; \sigma \;+\; it$. Then there exists s_o, such that $|s| > s_0$ the

following inequities for the solutions $\varphi(x,\lambda),\psi(x,\lambda)$ of Sturm-Liouville problem hold true.

$$\varphi(x,\lambda) \;=\; \frac{\sin sx}{s} \;+\; O\left(\frac{e^{|Ims|x}}{|s|^2}\right). \tag{23}$$

$$\psi(x,\lambda) \;=\; \cosh s(\pi - x) \;+\; O\left(\frac{e^{|Res|(\pi-x)}}{s}\right). \tag{24}$$

Proof.
We show first that

$$\varphi(x,\lambda) \;=\; O\left(\frac{e^{|t|x}}{s}\right),$$

where the inequality is uniformly with respect to $x \in \; [0,\pi]$. From the integral equation (15) we have

$$|\varphi(x,\lambda)| \;\leq\; \frac{e^{|t|x}}{s} \;+\; \frac{e^{|t|x}}{s}\int_0^x \; e^{|t|x}|q(\tau)||\varphi(\tau,\lambda)|\; d\tau. \tag{25}$$

By using the notation $\varphi(x,\lambda)\; e^{-|t|x}| \;=\; F(x,\lambda)$, equation (25) takes the form

$$|F(x,\lambda)| \;\leq\; \frac{1}{s} \;+\; \frac{1}{s}\int_0^\pi \; |q(\tau)||F(\tau,\lambda)|\; d\tau. \tag{26}$$

Let $\mu = \max\limits_{0\leq x\leq \pi} F(x,\lambda)$, so that from (26) it follow that

$$\mu \;\leq\; \frac{\frac{1}{s}}{1-\frac{1}{s}\int_0^\pi \; |q(\tau)|\; d\tau}$$

For $|s| > s_0 \;=\; \displaystyle\int_0^\pi \; |q(\tau)|\, d\tau$ it follows from the last inequality that $F(x,\lambda) \;\leq\; \frac{constant}{|s|}$ and this implies that

$$\varphi(x,\lambda) \;=\; O\left(\frac{e^{|t|x}}{s}\right). \tag{27}$$

By the aid of (26) we find that

$$\int_0^x \; \frac{\sin s(x-\tau)}{s}\; q(\tau)\; \varphi(\tau,\lambda)\; d\tau \;=\; O\left(\frac{e^{|Ims|x}}{|s|^2}\right). \tag{28}$$

From (15) and (28) it follows that, $\varphi(x,\lambda)$ has the asymptotic formula (23). In a similar way the asymptotic formula (24) can be proved where $\psi(x,\lambda) \;=\; O\left(\frac{e^{|Res|(\pi-x)}}{s}\right)$. The following theorem gives the asymptotic formula for the solution of the Sturm-Liouville problem (1)-(2) by making the asymptotic formulas of lemma 5 more precise.

theorem 1 Let $\lambda \;=\; s^2,\; s \;=\; \sigma \;+\; it$ and suppose that $q(x)$ has a second order piecewise differentiable derivatives on $[0,\pi]$. Then the solutions $\varphi(x,\lambda),\; \psi(x,\lambda)$ of Sturm-Liouville problem (1)-(2) have the following formulas:

$$\begin{aligned}
\varphi(x,\lambda) &= \frac{\sin sx}{s} \; - \; \frac{\alpha_1(x)}{s^2}\cos sx \; + \; \frac{\alpha_2(x)}{s^3}\sin sx \; + \; O\left(\frac{e^{|Ims|x}}{|s|^4}\right), \\
\varphi'(x,\lambda) &= \cos sx \; + \; \frac{\alpha_1(x)}{s}\sin sx \; + \; \frac{\beta_1(x)}{s^2}\cos sx \; + \; \frac{\beta_2(x)}{s^3}\sin sx \; + \; O\left(\frac{e^{|Ims|x}}{|s|^4}\right), \\
\psi(x,\lambda) &= \cosh s(\pi-x) \; + \; \frac{\gamma_2(x)}{s}\sinh s(\pi-x) \; + \; \frac{\gamma_3(x)}{s^2}\cosh s(\pi-x) \; + \; O\left(\frac{e^{|Res|x}}{|s|^3}\right), \\
\psi'(x,\lambda) &= -s\sinh s(\pi-x) \; - \; \gamma_2(x)\ \cosh s(\pi-x) \; - \; \frac{\delta_1(x)}{s}\sinh s(\pi-x) \\
&\quad - \; \frac{\delta_2(x)}{s^2}\cosh s(\pi-x) \; + \; O\left(\frac{e^{|Res|x}}{|s|^3}\right),
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\alpha_1(x) &= \frac{1}{2}\int_0^x q(t)\ dt, \\
\alpha_2(x) &= \frac{1}{4}(\int_0^x q(t)\ dt)^2 \; + \; \frac{1}{4}[q(x) \; + \; q(0)], \\
\beta_1(x) &= \frac{-1}{4}(\int_0^x q(t)\ dt)^2 \; + \; \frac{1}{4}[q(x) \; - \; q(0)], \\
\beta_2(x) &= \frac{1}{8}(\int_0^x q(t)\ dt)^3 \; + \; \frac{1}{8}[q(0)\int_0^x q(t)\ dt \; + \; q'(0) \; + \; q'(x)], \\
\gamma_1(x) &= \frac{1}{2}\int_x^\pi q(t)\ dt, \\
\gamma_2(x) &= H \; + \; \frac{1}{2}\int_x^\pi q(t)\ dt, \\
\gamma_3(x) &= \frac{1}{4}(\int_x^\pi q(t)\ dt)^2 \; + \; \frac{1}{4}[2H\int_x^\pi q(t)\ dt \; + \; q(\pi) \; - \; q(x)], \\
\delta_1(x) &= \frac{-1}{4}(\int_x^\pi q(t)\ dt)^2 \; - \; \frac{1}{4}[2H\int_x^\pi q(t)\ dt \; + \; q(\pi) \; + \; q(x)], \\
\delta_2(x) &= \frac{1}{8}(\int_x^\pi q(t)\ dt)^3 \; + \; \frac{H}{4}[(\int_x^\pi q(t)\ dt)^2 \; + \; q(x) \; - \; q(\pi)] \\
&\quad + \; \frac{1}{8}[\int_x^\pi q(t)\ dt(q(x))] \; + \; \frac{1}{8}(q'(x) \; - \; q'(\pi)).
\end{aligned} \tag{30}$$

Proof. By substituting from (23) into the integral (15), we have

$$\varphi(x,\lambda) \; = \; \frac{\sin sx}{s} + \frac{\cos sx}{2s^2}\int_o^x q(t)\,dt - \frac{1}{2s^2}\int_0^x \cos s(x-2t)\,q(t)\,dt + O\left(\frac{e^{|Ims|x}}{|s|^3}\right) \tag{31}$$

Integrating the last integration of (31) by parts and noticing that there exists $q'(x)$ such that $q'(x) \in L_1[0,\pi]$

$$\begin{aligned}\int_0^x \cos s(x-2t)q(t)dt &= \frac{\sin sx}{2s}[q(x)-q(0)] + \frac{1}{2s}\int_0^x \sin s(x-2t)q'(t)dt \\ &= O\left(\frac{e^{|Ims|x}}{|s|}\right). \end{aligned} \tag{32}$$

Substituting from (32) into (31), we get

$$\varphi(x,\lambda) = \frac{\sin sx}{s} + \frac{\alpha_1(x)}{s^2}\cos sx + O\left(\frac{e^{|Ims|x}}{|s|^3}\right) \tag{33}$$

where $\alpha_1(x)$ is defined by (30). In order to make $\varphi(x,\lambda)$ more precise we repeat this procedure again by substituting from the last result (2.21) into the same integral equation (15), we have

$$\begin{aligned}\varphi(x,\lambda) = \frac{\sin sx}{s} &+ \int_0^x \frac{\sin s(x-t)\sin st}{s^2}\ q(t)\ dt \\ &+ \int_0^x \frac{\sin s(x-t)\cos st}{s^2}\ q(t)\ \alpha_1(t)\ dt \\ &+ \int_0^x \frac{\sin s(x-t)}{s} q(t)\ O\left(\frac{e^{|Ims|x}}{|s|^3}\right). \end{aligned} \tag{34}$$

Now we estimate each term in (34). Integrating by parts twice the first term of(34), and noticing that $q''(x) \in L_1[0.\pi]$, we have

$$\int_0^x \frac{\sin s(x-t)\sin st}{s^2}\ q(t)\ dt = \frac{-\alpha_1(x)}{s^2}\cos sx + \frac{\frac{1}{4}(q(x)+q(0))}{s^3}\sin sx + O\left(\frac{e^{|Ims|x}}{|s|^4}\right). \tag{35}$$

Similarly, we have

$$\int_0^x \frac{\sin s(x-t)\cos st}{s^3}\ \alpha_1(t)\ q(t)\ dt = \frac{\frac{1}{2}\int_0^x \alpha_1(t)q(t)dt}{s^3}\sin sx + O\left(\frac{e^{|Ims|x}}{|s|^4}\right). \tag{36}$$

Substituting from (35) and (36) into (32) we obtain

$$\varphi(x,\lambda) = \frac{\sin sx}{s} - \frac{\alpha_1(x)}{s^2}\cos sx + \frac{\alpha_2(x)}{s^3}\sin sx + O\left(\frac{e^{|Ims|x}}{|s|^4}\right). \tag{37}$$

Where $\alpha_1(x)$, $\alpha_2(x)$ are defined in (30) to prove the asymptotic formula for $\varphi'(x,\lambda)$, we differentiate the integral equation (15) with respect to x and then use (37)

$$\begin{aligned}\varphi'(x,\lambda) = \cos sx &+ \frac{1}{s}\int_0^x \cos s(x-t)\ \sin st\ q(t)dt - \frac{1}{s^2}\int_0^x \cos s(x-t)\cos st\ q(t)\ \alpha_1(t)\ dt \\ &+ \frac{1}{s^3}\int_0^x \cos s(x-t)\sin st\ q(t)\alpha_2(t)dt + \int_0^x \cos s(x-t)\ q(t)\ O\left(\frac{e^{|Ims|t}}{|s|^4}\right)\ dt. \end{aligned} \tag{38}$$

We estimate the integrals of (38). Arguing as before in treating the integral (35), we have

$$\frac{1}{s}\int_0^x \cos s(x-t)\ \sin stq(t)dt = \frac{\alpha_1(x)}{s}\sin sx + \frac{\frac{1}{4}q(x)-q(0)}{s^2}\cos sx + \frac{\frac{1}{8}(q'(x)+q'(0))}{s^3}\sin sx + O\left(\frac{e^{|Ims|t}}{|s|^4}\right)\ dt, \tag{39}$$

Further,

$$-\frac{1}{s^2}\int_0^x \cos s(x-t)\cos stq(t)\ \alpha_1(t)dt = \frac{\frac{-1}{2}\int_0^x\ q(t)\alpha_1(t)\ dt}{s^2}\ \cos sx - \frac{\frac{1}{4}q(x)\alpha_1(x)}{s^3}\sin sx + O\left(\frac{e^{|Ims|t}}{|s|^4}\right), \tag{40}$$

and

$$\frac{1}{s^3}\int_0^x \cos s(x-t)\ \sin st\ q(t)\ \alpha_2(t)\ dt = \frac{\frac{1}{2}\int_0^x\ q(t)\ \alpha_2(t)\ dt}{s^3}\ \sin sx + O\left(\frac{e^{|Ims|x}}{|s|^4}\right). \tag{41}$$

Substituting from (39)-(41) into (38) we get the asymptotic relation for $\varphi'(x,\lambda)$ in the form

$$\varphi'(x,\lambda) = \cos sx + \frac{\alpha_1(x)}{s}\sin sx + \frac{\beta_1(x)}{s^2}\cos sx + \frac{\beta_2(x)}{s^3}\sin sx + O\left(\frac{e^{|Ims|x}}{|s|^4}\right), \tag{42}$$

Where the coefficients $\alpha_1(x), \beta_1(x), \beta_2(x)$ are defined by (30). Now we calculate the asymptotic formula for the solution $\psi(x,\lambda)$. Substituting from (24) into the integral equation (16), we have

$$\psi(x,\lambda) = \cosh s(\pi-x) + \frac{H}{s}\sinh s(\pi-x) \tag{43}$$

$$-\frac{1}{s}\int_x^\pi \sinh s(x-t)\ \cosh s(\pi-t)\ q(t)\ dt - \frac{1}{s}\int_x^\pi \sinh s(x-t)\ q(t)O\left(\frac{e^{|Res|(\pi-t)}}{|s|}\right)dt$$

,
which is equivalent to

$$\psi(x,\lambda) = \cosh s(\pi-x) + \frac{H}{s}\sinh s(\pi-x) \tag{44}$$

$$-\frac{1}{2s}\int_x^\pi \sinh s(\pi+x-2t)\ q(t)dt + \frac{1}{2s}\int_x^\pi q(t)dt\ \sinh s(\pi-x) + O\left(\frac{e^{|Res|(\pi-t)}}{|s|^2}\right)$$

. We can estimate the third term of (44)as follows:

$$-\frac{1}{2s}\int_x^\pi \sinh s(\pi+x-2t)\ q(t)\ dt$$
$$= \frac{q(\pi)-q(x)}{4s^2}\cosh(\pi-x) - \frac{1}{4s^2}\int_x^\pi q'(t)\cosh s(\pi+x-2t)dt = O\left(\frac{e^{|Res|(\pi-t)}}{|s|^2}\right). \tag{45}$$

So that, the estimation of $\psi(x,\lambda)$

$$\psi(x,\lambda) = \cosh s(\pi-x) + \frac{\gamma_2(x)}{s}\sinh s(\pi-x) + O\left(\frac{e^{|Res|(\pi-t)}}{|s|^2}\right). \tag{46}$$

Where $\gamma_2(x) = H + \dfrac{1}{2}\displaystyle\int_x^\pi q(t)dt.$

We continue making $\psi(x,\lambda)$ more precise by substituting the last estimation (46), again, into(16), we have

$$\begin{aligned}\psi(x,\lambda) = {} & \cosh s(\pi-x) + \frac{H}{s}\sinh s(\pi-x)\\ & - \frac{1}{s}\int_x^\pi \sinh s(x-t)\ \cosh s(\pi-t)\ q(t)\ dt\\ & - \frac{1}{s^2}\int_x^\pi \sinh s(x-t)\sinh s(\pi-t)\ \gamma_2(t)q(t)dt + O\left(\frac{e^{|Res|(\pi-t)}}{|s|^3}\right).\end{aligned} \tag{47}$$

By estimating the two integrals of (47) we obtain

$$\begin{aligned}-\frac{1}{s}\int_x^\pi \sinh s(x-t)\ \cosh s(\pi-t)\ q(t)\ dt = {} & \frac{\gamma_1(x)}{s}\sinh s(\pi-x)\\ & + \frac{\frac{1}{4}(q(\pi)-q(x))}{s^2}\cosh s(\pi-x) + O\left(\frac{e^{|Res|(\pi-t)}}{|s|^3}\right)\end{aligned} \tag{48}$$

.

$$-\frac{1}{s^2}\int_x^\pi \sinh s(x-t)\sinh s(\pi-t)\ \gamma_2(t)q(t)dt = \frac{\frac{1}{2}\int_x^\pi q(t)dt}{s^2}\cosh s(\pi-x) + O\left(\frac{e^{|Res|(\pi-t)}}{|s|^3}\right). \tag{49}$$

From (48), (49) and (47), we have

$$\psi(x,\lambda) = \cosh s(\pi-x) + \frac{\gamma_2(x)}{s}\sinh s(\pi-x) + \frac{\gamma_3(x)}{s^2}\cosh s(\pi-x) + O\left(\frac{e^{|Res|(\pi-t)}}{|s|^3}\right). \tag{50}$$

Where $\gamma_1(x)$, $\gamma_2(x)$ are defined in (30). To evaluate the asymptotic formula for $\psi'(x,\lambda)$ and use (50)

$$\begin{aligned}\psi;(x,\lambda) = {} & -s\sinh s(\pi-x) - H\cosh s(\pi-x) - \int_x^\pi \cosh s(x-t)\cosh s(\pi-t)q(t)dt\\ & - \frac{1}{s}\int_x^\pi \cosh s(x-t)\sinh s(\pi-t)\gamma_2(t)q(t)dt\\ & - \frac{1}{s^2}\int_x^\pi \cosh s(x-t)\cosh s(\pi-t)\gamma_3(t)q(t)dt\\ & - \int_x^\pi \cosh s(x-t)q(t)O\left(\frac{e^{|Res|(\pi-t)}}{|s|^3}\right)dt.\end{aligned} \tag{51}$$

Estimating each integral of (51), as before we obtain

$$\begin{aligned}-\int_x^\pi \cosh s(x-t)\cosh s(\pi-t)q(t)dt = {} & -\gamma_1(x)\cosh s(\pi-x) - \frac{\frac{1}{4}(q(\pi)-q(x))}{s}\sinh s(\pi-x)\\ & + \frac{\frac{1}{8}(q'(\pi-q'(x))}{s^2}\cosh s(\pi-x) + O\left(\frac{e^{|Res|(\pi-x)}}{|s|^3}\right).\end{aligned} \tag{52}$$

$$-\frac{1}{s}\int_x^\pi \cosh s(x-t)\ \sinh s(\pi-t)\ \gamma_2(t)\ q(t)dt = \frac{\frac{-1}{2}\int_x^\pi \gamma_2(t)\ q(t)\ dt}{s}\sinh s(\pi-x) \tag{53}$$

$$+\frac{\frac{1}{4}(Hq(\pi-\gamma_2(x)q(x))}{s^2}\cosh s(\pi-x)+\ O\left(\frac{e^{|Res|(\pi-x)}}{|s|^3}\right).$$

Further,

$$\frac{1}{s^2}\int_x^\pi \cosh s(x-t)\cosh s(\pi-t)\gamma_3(t)q(t)dt=\frac{\frac{1}{2}\int_x^\pi\gamma_3(t)q(t)dt}{s^2}\cosh s(\pi-x)$$
$$+\ O\left(\frac{e^{|Res|(\pi-x)}}{|s|^3}\right). \tag{54}$$

Finally from (51)-(54), we have

$$\psi'(x,\lambda)=-s\sinh s(\pi-x)-\gamma_2(x)\cosh s(\pi-x)-\frac{\delta_1(x)}{s}\sinh s(\pi-x)$$
$$-\frac{\delta_2(x)}{s^2}\cosh s(\pi-x)\ +\ O\left(\frac{e^{|Res|(\pi-x)}}{|s|^3}\right). \tag{55}$$

It should be noted here that, in lemma 5 and theorem 1 during the calculation of the asymptotic formulas of the solutions we make use of the integral equation of the solution and urging as in [7] while the treating of such point in [1] the author made use of a certain formula from [5] to construct the solution this is because of the new conditions i.e. the Sturm-Liouville condition.

3. The asymptotic formula for the eigenvalues and the normalization numbers

So long as, according to lemma 3, the eigenvalues of Sturm-Liouville problem(1)-(2) coincide with roots of the function $W(\lambda)$, we use this function to evaluate the asymptotic formula of the eigenvalues. First we calculate the asymptotic formula of the eigenvalues. We use the asymptotic formulas of the solution defined in theorem 1. From (11) and theorem 1, after elementary calculation, $W(\lambda)$ can be represented in the form

$$W(\lambda)=Z_0(s)+\frac{Z_1(s)}{s}+\frac{Z_2(s)}{s^2}+\frac{Z_3(s)}{s^3}+\ O\left(\frac{e^{|Ims|a+|Res|(\pi-a)}}{|s|^4}\right). \tag{56}$$

Where

$$\begin{aligned}
Z_0(s) &= -\sin sa\sinh s(\pi-a)\ -\cos sa\cosh s(\pi-a)\\
Z_1(s) &= -p_1\sin sa\cosh s(\pi-a)-p_2\cos sa\sinh s(\pi-a)\\
Z_2(s) &= -Q_1\sin sa\sinh s(\pi-a)-Q_2\cos sa\cosh s(\pi-a)\\
Z_3(s) &= -R_1\sin sa\cosh s(\pi-a)-R_2\cos sa\sinh s(\pi-a)
\end{aligned} \tag{57}$$

and

$$\begin{aligned}
p_1 &= \alpha_1(a)+\gamma_2(a),\\
p_2 &= \gamma_2(a)-\alpha_1(a),\\
Q_1 &= \alpha_2(a)+\delta_(a)+\alpha_1(a)\gamma_2(a),\\
Q_2 &= \beta_1(a)+\gamma_3(a)-\alpha_1(a)\gamma_2(a),\\
R_1 &= \beta_2(a)+\delta_2(a)+\alpha_1(a)\gamma_2(a)+\alpha_1(a)\gamma_3(a),\\
R_2 &= -\alpha_1(a)\delta_1(a)+\beta_1(a)\gamma_2(a).
\end{aligned} \tag{58}$$

Where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \delta_1$ and δ_2 are given by (30). We set $\Psi(s) = W(\lambda)$ then (56) is written as

$$\Psi(s) \;=\; Z_0(s) + \frac{Z_1(s)}{s} + \frac{Z_2(s)}{s^2} + \frac{Z_3(s)}{s^3} + \; O\left(\frac{e^{|Ims|a+|Res|(\pi-a)}}{|s|^4}\right). \tag{59}$$

According to lemma 1 the eigenvalues are real, this means that the roots of the function $W(\lambda)$ are also real, so that the roots of the function $\Psi(s)$ are only real or pure imaginary and this means that we study the behavior of $\Psi(s)$ for $s \to \; +\infty$ or $s = i\mu, \mu \to \infty$ and consequently the eigenvalues λ_n will be composed of non - negative eigenvalues λ_n^+ and negative eigenvalues λ_n^-. The following theorem gives the asymptotic for the eigenvalues $\lambda_n^\pm$.

Theorem 2 Let λ_n^+ and λ_n^-, n=0,1,2,... denote the non-negative and the negative eigenvalues of the Sturm-Liouville problem (1)-(2) then the following asymptotic formula take place:

$$\begin{aligned}
\lambda_n^+ &= \frac{\pi^2}{a^2}\left(n-\frac{1}{4}\right)^2 + \frac{2k_o\pi}{a} + \left(\frac{2k_1\pi}{a}\right)\frac{1}{n} + \frac{2\pi}{a}\left(\frac{a}{2\pi}k_o^2 + \frac{1}{4}k_1 + k_2\right)\frac{1}{n^2} + O\left(\frac{1}{n^3}\right),\\
\lambda_n^- &= -\frac{(\pi^2}{(\pi-a)^2}\left(n+\frac{1}{4}\right)^2 - \frac{2h_o\pi}{\pi-a} - \left(\frac{(2h_1+\frac{1}{2}h_o)\pi}{\pi-a}\right)\frac{1}{n} - \frac{2\pi}{\pi-a}\left(\frac{\pi-a}{2\pi}h_o^2 - \frac{1}{4}h_1 + h_2\right)\frac{1}{n^2}\\
&\quad + O\left(\frac{1}{n^3}\right).
\end{aligned} \tag{60}$$

Where

$$\begin{aligned}
k_o &= \frac{1}{2\pi}\int_0^a q(t)dt\\
k_1 &= \frac{a}{2\pi^2}\left[\alpha_2(a) - \delta_1(a) - \beta_1(a) - \gamma_3(a)\right]\\
k_2 &= \frac{a^2}{2\pi^3}[\alpha_1(a)\left((\frac{\alpha_1^2(a)}{3} - \frac{2\alpha_1(a)}{a}) + \delta_1(a) + \gamma_2(a) + \gamma_3(a) - 2\alpha_2(a)\right)\\
&+ \beta_2(a) - \delta_2(a) - \beta_1(a)\gamma_2(a)]\\
h_o &= \frac{1}{2\pi}\left[H + \int_a^\pi q(t)dt\right]\\
h_1 &= \frac{\pi-a}{a}k_1\\
h_2 &= \frac{(\pi-a)^2}{2\pi^3}\left[\gamma_2(a^2)(\frac{1}{3}\gamma_2(a) - 2\alpha_1(a)(\pi-a) + 2\alpha_1(a)) - \alpha_1(a)(\gamma_2(a)+\gamma_3(a)-\delta_1(a))\right]\\
&- \frac{(\pi-a)^2}{2\pi^3}\left[\beta_2(a) - \beta_1(a)\gamma_2(a) - \delta_2(a) - \frac{2\gamma_2(a)}{(\pi-a)}(\gamma_2(a)-\delta_1(a)-\alpha_2(a))\right]
\end{aligned} \tag{61}$$

Proof. Let $s \neq 0$ be a root of the equation $\Psi(s) = 0$ i.e the equation

$$Z_0(s) + \frac{Z_1(s)}{s} + \frac{Z_2(s)}{s^2} + \frac{Z_3(s)}{s^3} + O\left(\frac{e^{|Ims|a+|Res|(\pi-a)}}{|s|^4}\right) = 0, \tag{62}$$

where $Z_0(s), Z_1(S), Z_2(s)$ and $Z_3(s)$ are defined by (3.2). For $|s| \to \infty$ equation (3.7)is equivalent to the equation

$$\sin sa + \cos sa + \frac{p_1 \sin sa + p_2 \cos sa}{s} + \frac{Q_1 \sin sa + Q_2 \cos sa}{s^2} + \frac{R_1 \sin sa + R_2 \cos sa}{s^3} + O\left(\frac{e^{|Ims|a}}{|s|^4}\right). \tag{63}$$

Dividing (63) by $\cos sa$ we obtain

$$1 + \tan sa + \frac{p_2}{s} + \frac{p_1}{s} \tan sa + \frac{Q_2}{s^2} + \frac{Q_1}{s^2} \tan sa + \frac{R_2}{s^3} + \frac{R_1}{s^3} \tan sa + O\left(\frac{1}{|s|^4}\right) = 0. \tag{64}$$

The principle part of (64) is $1 + \tan sa$ which has the root

$$s_n^o = \frac{\pi}{a}(n - \frac{1}{4}), \ n = 0, \pm 1, \pm 2, ...$$

We estimate s_n^0 by letting

$$s_n = s_n^o + \varepsilon_n, \ n = 0, 1, 2, ... \tag{65}$$

We notice, by Rouche's theorem , that $\varepsilon_n \to 0 \ as \ n \to \infty$. By the aid of (65),equation (64) becomes

$$2\varepsilon_n a - 2\varepsilon_n^2 a^2 + \frac{5}{3}\varepsilon_n^3 a^3 + O\left(\frac{1}{|n|^4}\right) = -\frac{p_2}{s_n} - \frac{p_1}{s_n} \tan s_n a \tag{66}$$

$$-\frac{Q_2}{s_n^2} - \frac{Q_1}{s_n^2} \tan s_n a - \frac{R_2}{s_n^3} - \frac{R_1}{s_n^3} \tan s_n a + + O\left(\frac{1}{|n|^4}\right).$$

From (65) and (66) it is easy to see that $\varepsilon_n = O\left(\frac{1}{n}\right)$, so that equation (65) becomes

$$s_n = s_n^o + + O\left(\frac{1}{n}\right). \tag{67}$$

We make ε_n more accurate by using (67) to get

$$-\frac{p_2}{s_n} = -\frac{p_2}{s_n^o} + O\left(\frac{1}{n^2}\right), \tag{68}$$

$$-\frac{p_1}{s_n} \tan s_n a = \frac{p_1}{s_n^o} + O\left(\frac{1}{n^2}\right).$$

From (66) and (68) we obtain

$$\varepsilon_n = \frac{p_1 - p_2}{2as_n^o} + O\left(\frac{1}{n^2}\right). \tag{69}$$

By virtue of (69) s_n has the form

$$s_n = s_n^o + \frac{p_1 - p_2}{2as_n^o} + O\left(\frac{1}{n^2}\right). \tag{70}$$

We continue making s_n more precise, from(70) we can see that

$$-\frac{p_2}{s_n} = -\frac{p_2}{s_n^o} + \frac{p_2(p_1 - p_2)}{2a{s_n^o}^3} + O\left(\frac{1}{n^4}\right),$$

$$-\frac{p_1}{s_n} \tan s_n a = \frac{p_1}{s_n^o} + \frac{p_1(p_2 - p_1)}{{s_n^o}^2} + \frac{p_1(p_2 - p_1) + ap_1(p_2 - p_1)^2}{2a{s_n^o}^3} + O\left(\frac{1}{n^4}\right),$$

$$-\frac{Q_2}{s_n^2} = -\frac{Q_2}{{s_n^o}^2} + O\left(\frac{1}{n^4}\right),$$

$$\begin{aligned}
-\frac{Q_1}{s_n^2}\tan s_n a &= \frac{Q_1}{{s_n^o}^2}+\frac{Q_1(p_2-p_1)}{{s_n^o}^3},\\
-\frac{R_2}{s_n^3} &= -\frac{R_2}{{s_n^o}^3}+O\left(\frac{1}{n^4}\right),\\
-\frac{R_1}{s_n^3}\tan s_n a &= \frac{R_1}{{s_n^o}^3}+O\left(\frac{1}{n^4}\right),\\
a^2\varepsilon^2 &= \frac{(p_1-p_2)^2}{4{s_n^o}^2}+O\left(\frac{1}{n^4}\right),\\
a^3\varepsilon^3 &= \frac{(p_1-p_2)^3}{8{s_n^o}^3}+O\left(\frac{1}{n^4}\right).
\end{aligned} \tag{71}$$

Using inequalities (71) and (66) we obtain

$$\varepsilon_n = \frac{p_1-p_2}{2as_n^o}+\frac{\frac{1}{2}(p_1-p_2)^2-p_1(p_1-p_2)+Q_1-Q_2}{2a{s_n^o}^2} \tag{72}$$
$$+\frac{\frac{1}{2}(p_1-p_2)^2(p_1-\frac{1}{a}-Q_1(P_1-P_2)-\frac{5}{24}(P_1-P_2)^3+R_1-R_2}{2a{s_n^o}^3}+O\left(\frac{1}{n^4}\right)$$

From (72) and (65), we have

$$s_n = \frac{\pi}{a}(n-\frac{1}{4})+\frac{k_o}{n}+\frac{\frac{1}{4}k_o+k_1}{n^2}+\frac{\frac{1}{16}k_o+\frac{1}{2}k_1+k_2}{n^3}+O\left(\frac{1}{n^4}\right). \tag{73}$$

Where the constants k_o, k_1 andk_2 are given by (61). Since $\lambda_n^+ = s_n^2$, it follows from (73) that

$$\lambda_n^+ = \frac{\pi^2}{a^2}(n-\frac{1}{4})^2+\frac{2k_o\pi}{a}+\frac{2k_1\pi}{a}\frac{1}{n}+\frac{2\pi(\frac{1}{4}k_1+k_2+\frac{a}{2\pi}k_o^2)}{a}\frac{1}{n^2}+O\left(\frac{1}{n^3}\right). \tag{74}$$

Now we calculate the corresponding asymptotic formula for the negative eigenvalues λ_n^-. For this propose we set $\lambda = -\mu^2$ i.e we substitute $s = i\mu$ in $\Psi(s)$ and urging as before to find the asymptotic formula of μ_n. Under the substitution $s = i\mu$, $Im\mu = 0$, (62) takes the form

$$Z_0(\mu)+\frac{Z_1(\mu)}{\mu}+\frac{Z_2(\mu)}{\mu^2}+\frac{Z_3(\mu)}{\mu^3}+O\left(\frac{e^{|\mu|a}}{|s|^4}\right). \tag{75}$$

Where

$$\begin{aligned}
Z_0(\mu) &= \sinh\mu a\ \cos\mu(\pi-a)\ -\cosh\mu a\ \cos\mu(\pi-a),\\
Z_1(\mu) &= -p_1\sinh\mu a\ \cos\mu(\pi-a)-p_2\cosh\mu a\ \sin\mu(\pi-a),\\
Z_2(\mu) &= -Q_1\sinh\mu a\ \sin\mu(\pi-a)+Q_2\cosh\mu a\ \cos\mu(\pi-a),\\
Z_3(\mu) &= R_1\sinh\mu a\ \cos\mu(\pi-a)+\ R_2\cosh\mu a\ \sin\mu(\pi-a).
\end{aligned} \tag{76}$$

Where $p_1, p_2, Q_1, Q_2,\ R_1 and\ R_2$ are defined by (58) for $\mu\to\infty$ and divided (75) by $cos\mu(\pi-a)$ we get

$$-1+\tan\mu(\pi-a)-\frac{p_1}{\mu}-\frac{p_2}{\mu}\tan\mu(\pi-a)+\frac{Q_2}{\mu^2}-\frac{Q_1}{\mu^2}\tan\mu(\pi-a)+\frac{R_1}{\mu^3}+\frac{R_2}{\mu^3}\tan\mu(\pi-a)+O\left(\frac{1}{|\mu|^4}\right)=0. \tag{77}$$

In similar way as for estimating λ_n^+ we can see that the principle part for the equation (3.22) is $-1+\tan\mu(\pi-a)$ which has the root

$$\mu_n^o = \frac{\pi}{\pi-a}(n+\frac{1}{4}),\ n=0,\pm1,\pm2,.....$$

. We estimate s_n^o by letting

$$\mu_n \ = \ \mu_n^o \ + \ \varepsilon_n^*, \ n = 0, 1, 2, ... \ . \tag{78}$$

We notice, by Rouche's theorem , that $\varepsilon_n^* \to \ 0 \, as \, n \to \infty$.By the aid of (65), equation (64) becomes

$$2\varepsilon_n^* a \ + 2{\varepsilon_n^*}^2 a^2 + \frac{5}{3}\varepsilon_n^3 a^3 + \ O\left(\frac{1}{|n|^4}\right) = -\frac{p_1}{\mu_n} + \frac{p_2}{\mu_n}\tan\mu_n(\pi - a) \tag{79}$$

$$-\frac{Q_2}{\mu_n^2} + \frac{Q_1}{\mu_n^2}\tan\mu_n(\pi - a) - \frac{R_1}{\mu_n^3} - \frac{R_2}{\mu_n^3}\tan\mu_n(\pi - a) + + O\left(\frac{1}{|n|^4}\right).$$

From (78) and (79) it is easy to see that $\varepsilon_n^* \ = O\left(\frac{1}{n}\right)$, so that equation (78)becomes

$$\mu_n \ = \ \mu_n^o \ + \ + O\left(\frac{1}{n}\right). \tag{80}$$

We make ε_n^* more accurate by using (80) to get

$$\frac{p_1}{\mu_n} = \frac{p_1}{\mu_n^o} + O\left(\frac{1}{n^2}\right), \tag{81}$$

$$\frac{p_2}{\mu_n}\tan\mu_n(\pi - a) = \frac{p_2}{\mu_n^o} + O\left(\frac{1}{n^2}\right).$$

From (79) and (81) we obtain

$$\varepsilon_n^* \ = \ \frac{p_1 + p_2}{2(\pi - a)\mu_n^o} + O\left(\frac{1}{n^2}\right). \tag{82}$$

By virtue of (82) μ_n has the form

$$\mu_n \ = \ \mu_n^o + \frac{p_1 + p_2}{2(\pi - a)\mu_n^o} + O\left(\frac{1}{n^2}\right). \tag{83}$$

We continue making μ_n more precise, from (83) we can see that

$$\begin{aligned}
\frac{p_1}{\mu_n} \ &= \ \frac{p_1}{\mu_n^o} - \frac{p_1(p_1 + p_2)}{2(\pi - a){\mu_n^o}^3} + O\left(\frac{1}{n^4}\right), \\
\frac{p_2}{\mu_n}\tan\mu_n(\pi - a) \ &= \ \frac{p_2}{\mu_n^o} + \frac{p_2(p_1 + p_2)}{{\mu_n^o}^2} + \frac{(\pi - a)p_2(p_1 + p_2)^2 - p_2(p_1 + p_2)}{2(\pi - a){\mu_n^o}^3} + O\left(\frac{1}{n^4}\right), \\
-\frac{Q_2}{\mu_n^2} \ &= \ -\frac{Q_2}{{\mu_n^o}^2} + O\left(\frac{1}{n^4}\right), \\
\frac{Q_1}{\mu_n^2}\tan\mu_n(\pi - a) \ &= \ \frac{Q_1}{{\mu_n^o}^2} + \frac{Q_1(p_1 + p_2)}{(\pi - a){\mu_n^o}^3}, \\
-\frac{R_1}{\mu_n^3} \ &= \ -\frac{R_1}{{\mu_n^o}^3} + O\left(\frac{1}{n^4}\right), \\
-\frac{R_2}{\mu_n^3}\tan\mu_n(\pi - a) \ &= \ -\frac{R_2}{{\mu_n^o}^3} + O\left(\frac{1}{n^4}\right), \qquad (84) \\
(\pi - a)^2{\varepsilon^*}^2 \ &= \ \frac{(p_1 + p_2)^2}{4{\mu_n^o}^2} + O\left(\frac{1}{n^4}\right), \\
(\pi - a)^3{\varepsilon^*}^3 \ &= \ \frac{(p_1 + p_2)^3}{8{\mu_n^o}^3} + O\left(\frac{1}{n^4}\right).
\end{aligned}$$

Using inequalities (84) and (78) we obtain

$$\varepsilon_n^* = \frac{p_1+p_2}{2(\pi-a)\mu_n^o} + \frac{p_2(p_1+p_2)-\frac{1}{2}(p_1+p_2)^2+Q_1-Q_2}{2(\pi-a){\mu_n^o}^2} \tag{85}$$

$$+\frac{\frac{1}{2}(p_1+p_2)^2(p_2(\pi-a)-1)+Q_1(P_1+P_2)-\frac{5}{24}(P_1+P_2)^3(\pi-a)-(\pi-a)(R_1+R_2)}{2(\pi-a){\mu_n^o}^3}+O\left(\frac{1}{n^4}\right)$$

From (80) and (85), we have

$$\mu_n = \frac{\pi}{(\pi-a)}(n+\frac{1}{4})+\frac{h_o}{n}+\frac{h_1-\frac{1}{4}h_o}{n^2}+\frac{\frac{1}{16}h_o-\frac{1}{2}h_1+h_2}{n^3}+O\left(\frac{1}{n^4}\right). \tag{86}$$

Where the constants h_o, h_1 and h_2 are given by (61). Since $\lambda_n^- = -\mu_n^2$, it follows from (86) that

$$\lambda_n^- = -\frac{\pi^2}{(\pi-a)^2}(n+\frac{1}{4})^2-\frac{2h_0\pi}{(\pi-a)}-\frac{2\pi(h_1+\frac{1}{4}h_0)}{(\pi-a)}\frac{1}{n}-\frac{2\pi(\frac{\pi-a}{2\pi}h_0+h_2-\frac{1}{4}h_1)}{(\pi-a)}\frac{1}{n^2}+O\left(\frac{1}{n^3}\right). \tag{87}$$

Corresponding to the eigenvalues $\lambda_n^{\pm},\ n = 0,1,2,...$ of the Sturm-Liouville problem (1)-(2) there associated the eigenfunctions $y_n^{\pm},\ n=0,1,2,...$, which are defined by number, λ is a spectral parameter and weighted function or the explosive factor $\rho(x)$ is of the form

$$y_n^+(x) = \begin{cases} \varphi(x,\lambda_n^+),\ 0\le x\le a<\pi, \\ c_n^+\psi(x,\lambda_n^+),\ a<x\le\pi, \end{cases} \tag{88}$$

$$y_n^-(x) = \begin{cases} \varphi(x,\lambda_n^-),\ 0\le x\le a<\pi, \\ c_n^-\psi(x,\lambda_n^-),\ a<x\le\pi, \end{cases}$$

where $c_n^{\pm}$ are calculated such that $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are continuous at $x=a$

$$c_n^+ = \frac{\varphi(a,\lambda_n^+)}{\psi(a,\lambda_n^+)},\ c_n^- = \frac{\varphi(a,\lambda_n^-)}{\psi(a,\lambda_n^-)},\ n= 0,1,2,... \tag{89}$$

definition For every $n=0,1,2,......$,the numbers

$$a_n^+ = \int_0^\pi \rho(x)\ [y_n^+]^2dx = \int_0^a \varphi^2(x,\lambda_n^+)\ dx-(c_n^+)^2\int_a^\pi \psi^2(x,\lambda_n^+)\ dx,$$

$$a_n^- = \int_0^\pi \rho(x)\ [y_n^-]^2dx = \int_0^a \varphi^2(x,\lambda_n^-)\ dx-(c_n^-)^2\int_a^\pi \psi^2(x,\lambda_n^-)\ dx \tag{90}$$

are called the normalization numbers of the Sturm-Liouville problem (1)-(2). The following theorem studies the asymptotic behavior of these numbers.

Theorem 3 Under the conditions and results of theorem **??**, the normalization numbers $a_n^{\pm}$ of the eigenfunctions of Sturm-Liouville problem (1)-(2) have the following asymptotic formula:

$$\begin{aligned} a_n^+ &= \frac{b_1}{n^2}+\frac{b_2}{n^3}+O\left(\frac{1}{n^4}\right), \\ a_n^- &= -\frac{(\pi-a)^2}{16\pi^3}\exp\{\frac{2n\pi a}{\pi-a}\}\exp\{\frac{\pi a}{2(\pi-a)}\}\left[\frac{b_3}{n^2}+\frac{b_3b_4}{n^3}+O\left(\frac{1}{n^3}\right)\right], \end{aligned} \tag{91}$$

where

$$\begin{aligned}
b_1 &= \frac{a^3}{2\pi^2}\ ,\ b_2 = \frac{a^3}{4\pi^2},\\
b_3 &= \frac{1}{2\pi}[H + \int_a^{\pi} q(t)dt],\\
b_4 &= -\frac{(\pi - a)}{\pi}\left[\frac{1}{4\pi}(\int_0^a q(t)dt)^2\ + \frac{1}{2}\int_0^a q(t)dt\ - \int_a^{\pi} q(t)dt\ + \frac{1}{4\pi}q(a)\right.\\
&\quad \left. + \frac{1}{4\pi}q(0)\ + \frac{\pi}{2(\pi - a)} - 3H\right].
\end{aligned} \tag{92}$$

Proof. To calculate the asymptotic formulas for $a_n^{\pm}$ we must, according to the definition of the normalization numbers (91), calculate the asymptotic formulas for the integrations $\int_0^a \phi^2(x, \lambda_n^{\pm})\, dx$ and $\int_a^{\pi} \psi^2(x, \lambda_n^{\pm})\, dx$ and the quantities $c_n^{\pm}$. From (29), we have

$$\begin{aligned}
\varphi^2(x, \lambda_n^+) &= \frac{sin^2 s_n x}{s_n^2}\ - \frac{\alpha_1(x)}{s_n^3}\sin s_n x \cos s_n x\ + O\left(\frac{e^{|Ims_n|a}}{|s|^4}\right),\\
\psi^2(x, \lambda_n^+) &= \cosh^2 s_n(\pi - x) + \frac{2\gamma_2(x)}{s_n}\cosh s_n(\pi - x)\sinh s_n(\pi - x)\\
&\quad + \frac{\gamma_2^2(x)\sinh^2 s_n(\pi - x) + 2\gamma_3(x)\cosh^2 s_n(\pi - x)}{s_n^2}\\
&\quad + \frac{2\gamma_2(x)\gamma_3(x)}{s_n^3}\sinh s_n(\pi - x)\cosh s_n(\pi - x) + O\left(\frac{e^{|Res_n|a}}{|s|^4}\right).
\end{aligned} \tag{93}$$

By using(73) it can be easily seen that

$$\begin{aligned}
\frac{1}{s_n} &= \frac{a}{\pi n} + \frac{a}{4\pi n^2} + (\frac{a}{16\pi} - \frac{a^2 k_o}{\pi^2})\frac{1}{n^3} + O\left(\frac{1}{n^4}\right),\\
\frac{1}{s_n^2} &= \frac{a^2}{\pi^2 n^2} + \frac{a^2}{2\pi^2 n^3} + O\left(\frac{1}{n^4}\right),\\
\frac{1}{s_n^3} &= \frac{a^3}{\pi^3 n^3} + O\left(\frac{1}{n^4}\right).
\end{aligned} \tag{94}$$

From (93) and (94), after some calculation, we have

$$\int_0^a \varphi^2(x, \varphi_n^+) = \frac{a^3}{2\pi^2 n^2} + (\frac{a^3}{4\pi^2} + \frac{a^3}{4\pi^3})\frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \tag{95}$$

Also using (93) and (94), we have

$$\begin{aligned}
\varphi^2(a, \lambda_n^+) &= \frac{a^2}{2\pi^2 n^2} + \left(\frac{a^3\alpha_1(a)}{2\pi^3} + \frac{a^3 k_0}{\pi^2} + \frac{a^2}{4\pi^2}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right),\\
\psi^2(a, \lambda_n^+) &= \frac{e^{2s_n(\pi - a)}}{4}\left[1 + \frac{2a\gamma_2(a)}{n\pi} + \left(\frac{2a^2\gamma_3(a)}{\pi^2} + \frac{a^2\gamma_2^2(a)}{\pi^2} + \frac{a\gamma_2(a)}{2\pi}\right)\frac{1}{n^2}\right.\\
&\quad \left. + \left(\frac{\gamma_2^2(a)}{\pi^2} + \frac{2\gamma_3(a)}{\pi^2} - \frac{2a^2 k_o\gamma_2(a)}{\pi^2} + \frac{a\gamma_2(a)}{8\pi} + \frac{2a^3\gamma_2(a)\gamma_3(a)}{\pi^3}\right)\frac{1}{n^3}\right] + O\left(\frac{1}{n^4}\right).
\end{aligned} \tag{96}$$

Further, from (89) and (95), we obtain

$$(c_n^+)^2 = 4e^{-2s_n(\pi-a)}\left[\frac{a^2}{2\pi^2}\frac{1}{n^2} + (\frac{a^3\alpha_1(a)}{2\pi^3} - \frac{a^3\gamma_2(a)}{\pi^3} + \frac{a^3k_0}{\pi^2} + \frac{a^2}{4\pi^2})\frac{1}{n^3} + O\left(\frac{1}{n^4}\right)\right]. \tag{97}$$

From (93) and (94), we can get a similar expression like (95), for integration of $\psi^2(x,\lambda_n^+)$

$$\int_a^\pi \psi^2(x,\lambda_n^+)\,dx = \frac{e^{2s_n(\pi-a)}}{4}\left[\frac{a}{2\pi n} + \frac{a}{8\pi n^2} + (\frac{a}{32\pi^2} - \frac{a^2k_o}{2\pi^2})\frac{1}{n^3} + O\left(\frac{1}{n^4}\right)\right]. \tag{98}$$

Substituting from (95), (97) and (98) into (90) we get the required formula for a_n^+ which is given by (91), (92). Now we prove the expression for a_n^- . Substituting $s = i\mu$ into (93), we have

$$\begin{aligned}
\varphi^2(x,\lambda_n^-) &= \frac{\sinh^2\mu_n x}{\mu_n^2} + \frac{\alpha_1(x)}{\mu_n^3}\sinh\mu_n x\cosh\mu_n x + O\left(\frac{e^{2\mu_n x}}{|\mu_n|^4}\right),\\
\psi^2(x,\lambda_n^-) &= \cos^2\mu_n(\pi-x) + \frac{2\gamma_2(x)}{\mu_n}\cos\mu_n(\pi-x)\sin\mu_n(\pi-x)\\
&\quad + \frac{\gamma_2^2(x)\sin^2\mu_n(\pi-x) - 2\gamma_3(x)\cos^2\mu_n(\pi-x)}{\mu_n^2}\\
&\quad - \frac{2\gamma_2(x)\gamma_3(x)}{\mu_n^3}\sin\mu_n(\pi-x)\cos\mu_n(\pi-x) + O\left(\frac{1}{|\mu|^4}\right).
\end{aligned} \tag{99}$$

From (86), we have

$$\begin{aligned}
\frac{1}{\mu_n} &= \frac{\pi-a}{n\pi} - \frac{(\pi-a)}{4\pi n^2} + [\frac{(\pi-a)}{16\pi} - \frac{(\pi-a)^2h_0}{\pi^2}]\frac{1}{n^3} + O\left(\frac{1}{n^4}\right),\\
\frac{1}{\mu_n^2} &= \frac{(\pi-a)^2}{n^2\pi^2} - \frac{(\pi-a)^2}{2\pi^2n^3} + O\left(\frac{1}{n^4}\right),\\
\frac{1}{\mu_n^3} &= \frac{(\pi-a)^3}{n^3\pi^3} + O\left(\frac{1}{n^4}\right).
\end{aligned} \tag{100}$$

From (99), by integration and using (100), we have

$$\int_0^a \varphi^2(x,\lambda_n^-)dx = e^{2s_na}\left[\frac{(\pi-a)^3}{8n^3\pi^3} + O\left(\frac{1}{n^4}\right)\right], \tag{101}$$

and

$$\int_a^\pi \psi^2(x,\lambda_n^-)\,dx = \frac{\pi-a}{4n\pi} - \frac{\pi-a}{16\pi n^2} + \left(\frac{\pi-a}{64\pi} - \frac{(\pi-a)^2h_0}{4\pi^2} - \frac{(\pi-a)^3h_0^2}{2\pi}\right)\frac{1}{n^3} + O\left(\frac{1}{n^4}\right). \tag{102}$$

Again from (98) and (99), after some calculation, we obtain

$$\begin{aligned}
(c_n^-)^2 &= \frac{\phi^2(a,\lambda_n^-)}{\psi^2(a,\lambda_n^-)}\\
&= 2e^{2s_na}\left[\frac{(\pi-a)}{4\pi^2h_o}\frac{1}{n} - \left(\frac{\pi-a}{16\pi^2h_o} + \frac{h_1(\pi-a)}{4\pi^2h_o} + \frac{(\alpha_1(a)+4\gamma_2(a))(\pi-a)^2}{4\pi^3h_o}\right)\frac{1}{n^2} + O\left(\frac{1}{n^3}\right)\right].
\end{aligned} \tag{103}$$

Finally by substituting from (101), (102) and (103) into (89), we get the required asymptotic formula (90) and (91) for a_n^-.

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