Electronic Journal of Mathematical Analysis and Applications
Vol. 5(2) July 2017, pp. 116-128.
ISSN: 2090-729X(online)
http://fcag-egypt.com/Journals/EJMAA/

CENTRAL MEAN OSCILLATION AND RECTANGULARLY DEFINED SPACES

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Abstract. We define a rectangular version of the space $\dot{A}^p(\mathbb{R}^2)$ studied by García-Cuerva, Chen and Lau and construct its dual. We also define the atomic Hardy space associated to this space and identify its dual with the space $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}(\mathbb{R}^2)$ of functions with bounded central rectangular mean oscillation. Finally, we obtain continuity on $L^p(\mathbb{R}^2)$ for the commutator of the rectangular Hardy operator and $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2)$ functions.

1. Introduction

Recently, the theory of Herz spaces has been developed in order to study continuity of classical operators in harmonic analysis, as well as the Hardy spaces associated to the former spaces. This theory has its origin in the work of N. Wiener [15], A. Beurling [2] and C. Herz [13].

According to the classical definition, a measurable function f belongs to the homogeneous Herz space $\dot{K}^\alpha_{p,q}(\mathbb{R}^n)$, $1 \le p, q < \infty$, $\alpha \in \mathbb{R}$ if

$$\|f\|_{\dot{K}^\alpha_{p,q}} := \left(\sum_{k=-\infty}^{\infty} 2^{nk\alpha q} \|f\chi_{E_k}\|_p^q \right)^{1/q} < \infty, \tag{1}$$

and for $q = \infty$

$$\|f\|_{\dot{K}^\alpha_{p,\infty}} := \sup_{k\in\mathbb{Z}} \left(2^{nk\alpha} \|f\chi_{E_k}\|_p\right) < \infty. \tag{2}$$

Here $E_k = \{x \in \mathbb{R}^n : 2^{k-1} < |x| \le 2^k\}$ for $k \in \mathbb{Z}$.

Taking $q = 1$ and $\alpha = 1/p'$ in (1) or $\alpha = -1/p$ in (2) we obtain homogeneous versions of the spaces $A^p(\mathbb{R}^n)$ and $B^p(\mathbb{R}^n)$, studied first by Chen and Lau [6] and later by J. García-Cuerva [12]. The second author obtained several characterizations of $HA^p(\mathbb{R}^n)$, the Hardy space associated to $A^p(\mathbb{R}^n)$, and it is by means of the atomic characterization that the dual space $(HA^p(\mathbb{R}^n))^*$ is identified with $CMO^{p'}(\mathbb{R}^n)$.

2010 *Mathematics Subject Classification.* 42B35, 26D15, 26D10.

Key words and phrases. Central rectangular mean oscillation, Herz spaces, Hardy operator, commutator.

Submitted Nov. 3, 2016.

This work was financially supported by CONACYT, grant 386114.

In this work we consider homogeneous versions of $A^p(\mathbb{R}^n)$, $B^p(\mathbb{R}^n)$ and $CMO^{p'}(\mathbb{R}^n)$ in the simplest product space, $\mathbb{R}\times\mathbb{R}$. These spaces are denoted as $\dot{\mathcal{A}}^p(\mathbb{R}^2)$, $\dot{\mathcal{B}}^p(\mathbb{R}^2)$ and $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2)$. In order to define an atomic Hardy space associated to $\dot{\mathcal{A}}^p(\mathbb{R}^n)$, we could have used the atoms defined by S.-Y. A. Chang and R. Fefferman (see [4] and [5]), which give the right atomic decomposition for $H^p(\mathbb{R}\times\mathbb{R})$, but those atoms are complicated to handle. For that reason, we choose a significantly simpler way of defining our atoms, by assumming that they are supported on rectangles centered at the origin, instead of being supported on arbitrary open sets. Following this idea, our definition for the product $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2)$ relates better to the space *bmo* studied by M. Cotlar, S. Ferguson and D. C. Chang and C. Sadosky in [7], [10], [3] and [14], than to the classical product version of BMO. Even though we obtain a space smaller than $C\dot{M}O^p(\mathbb{R}^2)$, $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2)$ still is a useful class of functions that let us, for instance, define continuous operators on $L^p(\mathbb{R}^2)$. Indeed, we studied the boundedness on $L^p(\mathbb{R}^2)$ of the commutator of the rectangular Hardy operator defined in [9] with functions in a particular $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^q(\mathbb{R}^2)$. More general versions of this operator, in the radial context, are considered in [11].

In the first section we introduce the space $\dot{\mathcal{A}}^p(\mathbb{R}^2)$ and its dual $\dot{\mathcal{B}}^p(\mathbb{R}^2)$, and prove some basic properties of these spaces. In the second section we define the atomic Hardy space associated to $\dot{\mathcal{A}}^p(\mathbb{R}^2)$ whose dual is identified with $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}(\mathbb{R}^2)$. Finally, the third section is devoted to prove continuity for commutators of the rectangular Hardy operator with $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2)$ functions.

We will use standard notation along this paper and we will adopt the convention to denote by C a constant that could be changing line by line.

2. Rectangular Herz spaces

For $j_1, j_2 \in \mathbb{Z}$ consider the following subsets in $\mathbb{R}^2$:

$$C_{j_1,j_2} = C_{j_1} \times C_{j_2},$$

where $C_j = \{x \in \mathbb{R} : 2^{j-1} < |x| \le 2^j\}$, and denote by χ_{j_1,j_2} the characteristic function of the set C_{j_1,j_2}.

Definition 1. Let $1 < p < \infty$ and denote by p' the conjugate exponent of p. We shall call $\dot{\mathcal{A}}^p(\mathbb{R}^2)$ the space consisting of those functions $f \in L^p_{loc}(\mathbb{R}^2)$ for which

$$\|f\|_{\dot{\mathcal{A}}^p} = \sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty} 2^{(j_1+j_2)/p'} \|f\chi_{j_1,j_2}\|_p < \infty. \tag{3}$$

It is not difficult to prove that $(\dot{\mathcal{A}}^p(\mathbb{R}^2), \|\cdot\|_{\dot{\mathcal{A}}^p})$ is a Banach space and that $\dot{A}^p(\mathbb{R}^2) \subset \dot{\mathcal{A}}^p(\mathbb{R}^2)$ continuously. Moreover, for $1 < p_1 \le p_2 < \infty$ we have the inclusions $\dot{\mathcal{A}}^{p_2}(\mathbb{R}^2) \subset \dot{\mathcal{A}}^{p_1}(\mathbb{R}^2)$ and $\dot{\mathcal{A}}^p(\mathbb{R}^2) \subset L^1(\mathbb{R}^2)$ for all $1 < p < \infty$.

Definition 2. For $1 < p < \infty$, the space $\dot{\mathcal{B}}^p(\mathbb{R}^2)$ consists of all those functions $f \in L^p_{loc}(\mathbb{R}^2)$ such that

$$\|f\|_{\dot{\mathcal{B}}^p} = \sup_{\substack{R_j>0\\ j=1,2}} \left[\frac{1}{4R_1R_2}\int_{[-R_1,R_1]\times[-R_2,R_2]} |f(x_1,x_2)|^p dx_1dx_2\right]^{1/p} < \infty. \tag{4}$$

There is an alternative way to describe $\dot{\mathcal{B}}^p(\mathbb{R}^2)$ in terms of the behavior of the functions in the subsets C_{j_1,j_2}.

Proposition 2.1. *A function f belongs to the space $\dot{\mathcal{B}}^p(\mathbb{R}^2)$ if and only if the following quantity is finite:*

$$\sup_{j_1,j_2\in\mathbb{Z}} 2^{-(j_1+j_2)/p}\|f\chi_{j_1,j_2}\|_p. \tag{5}$$

Furthermore, the quantities in (5) and (4) are comparable.

Proof. Let $f \in \dot{\mathcal{B}}^p(\mathbb{R}^2)$. For any pair of integers j_1 and j_2

$$\begin{aligned}\|f\chi_{j_1,j_2}\|_p^p &= \int_{C_{j_1,j_2}} |f(x_1,x_2)|^p dx_1dx_2 \\ &\leq \int_{[-2^{j_1},2^{j_1}]\times[-2^{j_2},2^{j_2}]} |f(x_1,x_2)|^p dx_1dx_2 \\ &\leq C2^{j_1+j_2}\|f\|_{\dot{\mathcal{B}}^p}^p.\end{aligned}$$

Therefore

$$\sup_{j_1,j_2\in\mathbb{Z}} 2^{-(j_1+j_2)/p}\|f\chi_{j_1,j_2}\|_p \leq C\|f\|_{\dot{\mathcal{B}}^p}.$$

For the converse, suppose the supremum in (5) is finite and denote it by S. Given $R_1 > 0$ and $R_2 > 0$, take integers k_1 and k_2 such that $2^{k_i-1} < R_i \leq 2^{k_i}$ for $i = 1, 2$. Then

$$\begin{aligned}\int_{[-R_1,R_1]\times[-R_2,R_2]} |f(x_1x_2)|^p dx_1dx_2 &\leq \sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2}\int_{C_{j_1,j_2}} |f(x_1,x_2)|^p dx_1dx_2 \\ &\leq \sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2} 2^{j_1+j_2}S^p \\ &\leq 2^{k_1+k_2}C\cdot S^p \\ &\leq (4R_1R_2)CS^p.\end{aligned}$$

Consequently, $\|f\|_{\dot{\mathcal{B}}^p} \leq C\cdot S$. □

After a simple calculation we verify that (5) induces a norm in $\dot{\mathcal{B}}^p(\mathbb{R}^2)$ that makes it a Banach space, and by the previous proposition the same is true for $\|\cdot\|_{\dot{\mathcal{B}}^p}$. We will denote the norm induced by (5) as $\|\cdot\|_{\dot{\mathcal{B}}^p}^*$. In addition, since

$$E_j \subset \left(\bigcup_{j_1=-\infty}^{j}\bigcup_{j_2=-\infty}^{j} C_{j_1,j_2}\right)\bigcap\left(\bigcup_{j_1=-\infty}^{j-2}\bigcup_{j_2=-\infty}^{j-2} C_{j_1,j_2}\right)^c$$

for every integer j, it is easy to show that $\dot{\mathcal{B}}^p(\mathbb{R}^2) \subset \dot{B}^p(\mathbb{R}^2)$ continuously. Indeed, in [8] we proved that $\dot{\mathcal{B}}^p(\mathbb{R}^2)$ is a proper subspace of $\dot{B}_p(\mathbb{R}^2)$.

Proposition 2.2. *The space of those C^∞ functions having compact support in $\mathbb{R}^2$ is dense in $\dot{\mathcal{A}}^p(\mathbb{R}^2)$ for every $1 < p < \infty$.*

Proof. Consider $f \in \dot{\mathcal{A}}^p(\mathbb{R}^2)$ and take $\epsilon > 0$. Since

$$\|f\|_{\dot{\mathcal{A}}^p} = \lim_{\substack{k_1\to\infty\\k_2\to\infty}}\sum_{j_1=-k_1}^{k_1}\sum_{j_2=-k_2}^{k_2} 2^{(j_1+j_2)/p'}\|f\chi_{j_1,j_2,}\|_p,$$

we can choose natural numbers K_1 and K_2 such that

$$\left| \sum_{j_1=-K_1}^{K_1} \sum_{j_2=-K_2}^{K_2} 2^{(j_1+j_2)/p'} \|f\chi_{j_1,j_2,}\|_p - \|f\|_{\dot{\mathcal{A}}^p} \right| < \frac{\epsilon}{5}.$$

Therefore

$$S_1^- = \sum_{j_1=-\infty}^{-(K_1+1)} \sum_{j_2=-\infty}^{\infty} 2^{(j_1+j_2)/p'} \|f\chi_{j_1,j_2,}\|_p < \frac{\epsilon}{5}$$

$$S_1^+ = \sum_{j_1=K_1+1}^{\infty} \sum_{j_2=-\infty}^{\infty} 2^{(j_1+j_2)/p'} \|f\chi_{j_1,j_2,}\|_p < \frac{\epsilon}{5}$$

$$S_2^- = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-(K_2+1)}^{\infty} 2^{(j_1+j_2)/p'} \|f\chi_{j_1,j_2,}\|_p < \frac{\epsilon}{5}$$

$$S_2^+ = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=K_2+1}^{\infty} 2^{(j_1+j_2)/p'} \|f\chi_{j_1,j_2,}\|_p < \frac{\epsilon}{5}.$$

Since $f\chi_{j_1,j_2} \in L^p(C_{j_1,j_2})$, for each pair of indexes j_1 and j_2 we can take a $C^\infty(C_{j_1,j_2})$ function g_{j_1,j_2} supported in C_{j_1,j_2} such that

$$\|f\chi_{j_1,j_2} - g_{j_1,j_2}\|_p < \frac{\epsilon}{5 \cdot 2^{(j_1+j_2)/p'}(2K_1+1)(2K_2+1)}$$

for every $-K_i \leq j_i \leq K_i$, $j = 1, 2$. If we define

$$g = \sum_{j_1=-K_1}^{K_1} \sum_{j_2=-K_2}^{K_2} g_{j_1,j_2}$$

is clear that g is a smooth function with compact support and that

$$\begin{aligned} \|f - g\|_{\dot{\mathcal{A}}^p} &= S_1^+ + S_1^- + S_2^+ + S_2^- + \sum_{j_1=-K_1}^{K_1} \sum_{j_2=-K_2}^{K_2} 2^{(j_2+j_2)/p'} \|f\chi_{j_1,j_2} - g_{j_1,j_2}\|_p \\ &< \frac{4\epsilon}{5} + \frac{\epsilon}{5}. \end{aligned}$$

□

Using this density result, we will be able to prove the next duality theorem.

Theorem 2.3. *Let $1 < p < \infty$. Then $(\dot{\mathcal{A}}^p(\mathbb{R}^2))^* = \dot{\mathcal{B}}^{p'}(\mathbb{R}^2)$ in the following sense: For every $g \in \dot{\mathcal{B}}^{p'}(\mathbb{R}^2)$, the functional Λ_g defined by*

$$\Lambda_g(f) = \int_{\mathbb{R}^2} f(x_1, x_2) g(x_1, x_2) dx_1 dx_2,$$

is continuous on $\dot{\mathcal{A}}^p(\mathbb{R}^2)$ and its norm in $(\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$ satisfies $\|\Lambda_g\| \leq \|g\|_{\dot{\mathcal{B}}^p}$. Conversely, given $\Lambda \in (\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$, there is a unique $g \in \dot{\mathcal{B}}^{p'}(\mathbb{R}^2)$ such that $\Lambda = \Lambda_g$. Furthermore, $\|g\|_{\dot{\mathcal{B}}^p} \leq \|\Lambda\|$.

Proof. Given $f \in \dot{\mathcal{A}}^p(\mathbb{R}^2)$, a smooth compactly supported function, let k_1 and k_2 be the smallest integers satisfying that $supp(f) \subset [-2^{k_1}, 2^{k_1}] \times [-2^{k_2}, 2^{k_2}]$. Then

$$|\Lambda_g(f)| = \left| \int_{[-2^{k_1},2^{k_1}]\times[-2^{k_2},2^{k_2}]} f(x_1, x_2) g(x_1, x_2) dx_1, dx_2 \right|$$

$$\leq \sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} \int_{C_{j_1,j_2}} |f(x_1,x_2)||g(x_1,x_2)|dx_1, dx_2$$
$$\leq \sum_{j_1=-\infty}^{k_1} \sum_{j_2=-\infty}^{k_2} 2^{-(j_1+j_2)/p'} \|g\chi_{j_1,j_2}\|_{p'} 2^{j_1+j_2/p'} \|f\chi_{j_1,j_2}\|_p$$
$$\leq \|g\|_{\dot{\mathcal{B}}^{p'}} \|f\|_{\dot{\mathcal{A}}^p}.$$

By Proposition 2.2, the class of compactly supported C^∞ functions is dense in $\dot{\mathcal{A}}^p(\mathbb{R}^2)$, and as a consequence, Λ_g extends to a unique continuous linear functional $\Lambda_g \in (\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$ for which $\|\Lambda_g\| \leq \|g\|_{\dot{\mathcal{B}}^p}$ holds.

For the converse, first note that for each pair of integers j_1 and j_2, $L^p(C_{j_1,j_2})$ is continuously contained in $\dot{\mathcal{A}}^p(\mathbb{R}^2)$ with $2^{(j_1+j_2)/p'}\|\cdot\|_{L^p(C_{j_1,j_2})} = \|\cdot\|_{\dot{\mathcal{A}}^p}$. In this sense, any $\Lambda \in (\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$ induces a continuous linear functional on $L^p(C_{j_1,j_2})$ whose $(L^p(C_{j_1,j_2}))^*$-norm is not greater than $2^{(j_1+j_2)/p'}\|\Lambda\|$. This fact, together with the duality of $L^p(C_{j_1,j_2})$ and $L^{p'}(C_{j_1,j_2})$ gives a function $g_{j_1,j_2} \in L^{p'}(C_{j_1,j_2})$ with norm not greater than $2^{(j_1+j_2)/p'}\|\Lambda\|$, such that

$$\Lambda(f) = \int_{C_{j_1,j_2}} f(x_1,x_2) g_{j_1,j_2}(x_1 x_2) dx_1 dx_2$$

for every $f \in L^p(C_{j_1,j_2})$. Let us define

$$g = \sum_{j_1=-\infty}^{\infty} \sum_{j_2=-\infty}^{\infty} g_{j_1,j_2}\chi_{j_1,j_2}.$$

Then g belongs to $\dot{\mathcal{B}}^{p'}(\mathbb{R}^2)$ and $\|g\|_{\dot{\mathcal{B}}^{p'}} \leq \|\Lambda\|$. Also, a simple calculation shows that for every smooth function f with compact support $\Lambda(f) = \Lambda_g(f)$, so that $\Lambda = \Lambda_g$. □

Corollary 2.4. *Let $f \in L^p_{loc}(\mathbb{R}^2)$. Then $f \in \dot{\mathcal{A}}^p(\mathbb{R}^2)$ if and only if*

$$\left| \int_{\mathbb{R}^2} f(x_1,x_2) g(x_1,x_2) dx_1 dx_2 \right| < \infty \tag{6}$$

for every $g \in \dot{\mathcal{B}}^{p'}(\mathbb{R}^2)$. If this is the case,

$$\|f\|_{\dot{\mathcal{A}}^p} = \sup\left\{ \left| \int_{\mathbb{R}^2} f(x_1,x_2) g(x_1,x_2) dx_1 dx_2 \right| : \|g\|_{\dot{\mathcal{B}}^{p'}} \leq 1 \right\}.$$

Proof. When $f \in \dot{\mathcal{A}}^p(\mathbb{R}^2)$, the result follows easily from the previous theorem by using the Hahn-Banach theorem.

For the converse take $f \in L^p_{loc}(\mathbb{R}^2)$ such that (6) holds whenever $g \in \dot{\mathcal{B}}^{p'}(\mathbb{R}^2)$. Without loss of generality we can assume that $f \geq 0$. For $n \in \mathbb{N}$ define Λ_n in $\dot{\mathcal{B}}^{p'}(\mathbb{R}^2)$ as

$$\Lambda_n(g) = \sum_{j_1=-n}^{n} \sum_{j_2=-n}^{n} \int_{C_{j_1,j_2}} f(x_1,x_2) g(x_1,x_2) dx_1 dx_2.$$

It is not difficult to see that $\Lambda_n \in (\dot{\mathcal{B}}^{p'}(\mathbb{R}^2))^*$ with

$$\|\Lambda_n\| = \sum_{j_1=-n}^{n} \sum_{j_2=-n}^{n} 2^{(j_1+j_2)/p'} \|f\chi_{j_1,j_2}\|_p.$$

Also for every $n \in \mathbb{N}$ and $g \in \dot{\mathcal{B}}^{p'}(\mathbb{R}^2)$

$$\begin{aligned}|\Lambda_n(g)| &= |\Lambda_n(g^+) - \Lambda_n(g^-)| \\ &\leq \int_{\mathbb{R}^2} f(x_1,x_2)g^+(x_1,x_2)dx_1dx_2 + \int_{\mathbb{R}^2} f(x_1,x_2)g^-(x_1,x_2)dx_1dx_2 \\ &< \infty,\end{aligned}$$

where $g^+(x_1,x_2) = \max\{g(x_1,x_2),0\}$ and $g^-(x_1,x_2) = \max\{-g(x_1,x_2),0\}$. The above inequalities guarantee that the family of continuous linear functionals $\{\Lambda_n\}_{n\in\mathbb{N}}$ is pointwise bounded. By the Banach-Steinhaus theorem $\sup\{\|\Lambda_n\| : n \in \mathbb{N}\}$ is finite. Thus

$$\sup_{n\in\mathbb{N}} \sum_{j_1=-n}^{n} \sum_{j_2=-n}^{n} 2^{(j_1+j_2)/p'} \|f\chi_{j_1,j_2}\|_p < \infty,$$

so $f \in \dot{\mathcal{A}}^p(\mathbb{R}^2)$. □

3. Atoms and central rectangular mean oscillation

Our goal in this section is to prove a duality result concerning to an atomic space related to $\dot{\mathcal{A}}^p(\mathbb{R}^2)$ and to the space of functions with bounded central rectangular mean oscillation. For this purpose we introduce the notion of a central rectangular $(1,p)$-atom and we define the space $H\dot{\mathcal{A}}^p(\mathbb{R}^2)$.

Definition 3. For $1 < p < \infty$, a central rectangular $(1,p)$-atom is a function a, with support contained in a rectangle $[-R_1,R_1]\times[-R_2,R_2]$, that satisfies

i) $\left[\dfrac{1}{4R_1R_2}\displaystyle\int_{[-R_1,R_1]\times[-R_2,R_2]} |a(x_1,x_2)|^p dx_1dx_2\right]^{1/p} \leq \dfrac{1}{4R_1R_2}$.

ii) $\int_{\mathbb{R}^2} a(x_1,x_2)dx_1dx_2 = 0$.

The first thing we notice is that if condition $i)$ holds for some rectangle R containing the support of a, then it holds for any rectangle $\tilde{R} \subset R$ such that $\tilde{R}$ contains the support of a. For that reason we can consider the smallest rectangle containing the support of a. We also observe that every central rectangular $(1,p)$-atom belongs to a closed ball in $\dot{\mathcal{A}}^p(\mathbb{R}^2)$: suppose $supp(a) \subset [-R_1,R_1]\times[-R_2,R_2]$ and take k_1 and k_2 such that $2^{k_1-1} \leq R_1 \leq 2^{k_1}$ and $2^{k_2-1} \leq R_1 \leq 2^{k_2}$. Then

$$\begin{aligned}\|a\|_{\dot{\mathcal{A}}^p} &= \sum_{j_1=-\infty}^{\infty}\sum_{j_2=-\infty}^{\infty} 2^{(j_1+j_2)/p'}\|a\chi_{j_1,j_2}\|_p \\ &= \sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2} 2^{(j_1+j_2)/p'}\|a\chi_{j_1,j_2}\|_p \\ &\leq \left(\int_{[-R_1,R_1]\times[-R_2,R_2]} |a(x_1,x_2)|^p dx_1dx_2\right)^{1/p} \sum_{j_1=-\infty}^{k_1}\sum_{j_2=-\infty}^{k_2} 2^{(j_1+j_2)/p'} \\ &\leq C(4R_1R_2)^{\frac{1}{p}-1}2^{(k_1+k_2)/p'} \\ &\leq C(2^{k_1+k_2})^{\frac{1}{p}-1}2^{(k_1+k_2)/p'} \\ &\leq C\end{aligned}$$

with C a constant that depends on p but not on the particular atom.

Definition 4. Let $f \in L^p_{loc}(\mathbb{R}^2)$. We say that f belongs to $H\dot{\mathcal{A}}^p(\mathbb{R}^2)$ if $f = \sum \lambda_j a_j$, where the a_j are central rectangular $(1,p)$-atoms and $\sum |\lambda_j| < \infty$.

For $f \in H\dot{\mathcal{A}}^p(\mathbb{R}^2)$ we define

$$\|f\|_{H\dot{\mathcal{A}}^p} = \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : f = \sum \lambda_j a_j \right\}.$$

Any atomic space constructed in this way and endowed with the atomic norm becomes a Banach space (see [1]), and since for a function f in $H\dot{\mathcal{A}}^p(\mathbb{R}^2)$ with $f = \sum \lambda_j a_j$ we have $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, we also have that $\|f\|_{\dot{\mathcal{A}}^p} \le C\|f\|_{H\dot{\mathcal{A}}^p}$.

Next we introduce a space of functions with bounded central rectangular mean oscillation and we discuss its relation with $H\dot{\mathcal{A}}^p(\mathbb{R}^2)$.

Definition 5. For $1 \le p < \infty$ we define

$$\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2) = \{f \in L^p_{loc}(\mathbb{R}^2) : \|f\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p} < \infty\},$$

where

$$\|f\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p} = \sup_{\substack{R_j>0\\ j=1,2}} \left[\frac{1}{4R_1R_2} \int_{[-R_1,R_1]\times[-R_2,R_2]} |f(x_1,x_2) - f_{R_1,R_2}|^p dx_1 dx_2 \right]^{1/p},$$

and f_{R_1,R_2} is the average of f on $[-R_1,R_1]\times[-R_2,R_2]$.

It is not difficult to prove that $(\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2), \|\cdot\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p})$ is a Banach space after identifying functions that differ by a constant almost everywhere in $\mathbb{R}^2$. Also it can be verified that a function f belongs to $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2)$ if and only if

$$\sup_{\substack{R_j>0\\ j=1,2}} \inf_{a\in\mathbb{R}} \left[\frac{1}{4R_1R_2} \int_{[-R_1,R_1]\times[-R_2,R_2]} |f(x_1,x_2) - a|^p dx_1 dx_2 \right]^{1/p} \tag{7}$$

is finite. Actually, the supremum in (7) defines a norm that is equivalent to $\|\cdot\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p}$. Clearly $\dot{\mathcal{B}}^p(\mathbb{R}^2) \subset \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2)$ for $1 < p < \infty$, while $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^2)$ is a subspace of the classical $C\dot{M}O^p(\mathbb{R}^2)$ studied in [6] and [12]. For $1 \le p_1 < p_2 < \infty$ we also have the inclusion $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_2}(\mathbb{R}^2) \subset \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p_1}(\mathbb{R}^2)$.

Theorem 3.1. *Let $1 < p < \infty$. Given $g \in \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}(\mathbb{R}^2)$, the functional Λ_g defined over compactly supported functions by*

$$\Lambda_g(f) = \int_{\mathbb{R}^2} f(x_1,x_2)g(x_1,x_2)dx_1dx_2$$

extends in a unique way to a continuous linear functional $\Lambda_g \in (H\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$ whose $(H\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$-norm satisfies $\|\Lambda_g\| \le C\|g\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}}$.

Conversely, given $\Lambda \in (H\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$ there is a unique, up to a constant, $g \in \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}(\mathbb{R}^2)$ such that $\Lambda = \Lambda_g$. Moreover, $\|g\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}} \le C\|\Lambda_g\|$.

Proof. Fix $g \in \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}(\mathbb{R}^2)$. For a central rectangular $(1,p)$-atom supported in $[-R_1,R_1]\times[-R_2,R_2]$ we have

$$|\Lambda_g(a)| = \left| \int_{[-R_1,R_1]\times[-R_2,R_2]} a(x_1,x_2)g(x_1,x_2)dx_1dx_2 \right|$$

$$
\begin{aligned}
&= \left| \int_{[-R_1,R_1]\times[-R_2,R_2]} a(x_1,x_2)[g(x_1,x_2) - g_{R_1,R_2}]dx_1dx_2 \right| \\
&\leq 4R_1R_2 \left[\frac{1}{4R_1R_2} \int_{[-R_1,R_1]\times[-R_2,R_2]} |a(x_1,x_2)|^p dx_1dx_2 \right]^{1/p} \\
&\quad \times \left[\frac{1}{4R_1R_2} \int_{[-R_1,R_1]\times[-R_2,R_2]} |g(x_1,x_2) - g_{R_1,R_2}|^{p'} dx_1dx_2 \right]^{1/p'} \\
&\leq \|g\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}}.
\end{aligned}
$$

Now, if $f \in H\dot{\mathcal{A}}^p(\mathbb{R}^2)$ is compactly supported, we can write

$$f = \sum_{j=1}^{\infty} \lambda_j a_j,$$

where the functions a_j are central rectangular $(1,p)$-atoms, all supported on a fixed rectangle $[-R_1,R_1]\times[-R_2,R_2]$ and

$$\sum_{j=1}^{\infty} |\lambda_j| \leq C\|f\|_{H\dot{\mathcal{A}}^p}.$$

The series converges in $\dot{\mathcal{A}}^p(\mathbb{R}^2)$ (because it is absolutely convergent), and consequently in L^p. Since $g \in L^{p'}([-R_1,R_1]\times[-R_2,R_2])$, we have

$$\Lambda_g(f) = \sum_{j=1}^{\infty} \lambda_j \Lambda_g(a_j),$$

and as a consequence we obtain

$$|\Lambda_g(f)| \leq C\|g\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}} \|f\|_{H\dot{\mathcal{A}}^p}.$$

Recall that the class of compactly supported functions in $H\dot{\mathcal{A}}^p(\mathbb{R}^2)$ include the subspace of finite linear combinations of central rectangular $(1,p)$-atoms and the last one is dense in $H\dot{\mathcal{A}}^p(\mathbb{R}^2)$. It follows that Λ_g extends in a unique way to a continuous linear functional $\Lambda_g \in (H\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$ that satisfies $\|\Lambda_g\| \leq C\|g\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}}$.

For the converse, fix $\Lambda \in (H\dot{\mathcal{A}}^p(\mathbb{R}^2))^*$ and for $R_1, R_2 > 0$ consider the space $L^p_0([-R_1,R_1]\times[-R_2,R_2])$ defined as

$$\left\{ f \in L^p\left([-R_1,R_1]\times[-R_2,R_2]\right) : \int_{[-R_1,R_1]\times[-R_2,R_2]} f(x_1,x_2)dx_1dx_2 = 0 \right\}.$$

Clearly, $L^p_0([-R_1,R_1]\times[-R_2,R_2])$ is continuously included in $H\dot{\mathcal{A}}^p(\mathbb{R}^2)$ with $\|\cdot\|_{H\dot{\mathcal{A}}^p} \leq (4R_1R_2)^{1/p'}\|\cdot\|_p$. From the duality between L^p and $L^{p'}$ and the previous comment, we obtain a function g locally in $L^{p'}$ that allows us to represent Λ over compactly supported functions h having average 0 as

$$\Lambda(h) = \Lambda_g(h) = \int_{\mathbb{R}^2} g(x_1,x_2)h(x_1,x_2)dx_1dx_2.$$

Let us prove that g belongs to $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}(\mathbb{R}^2)$. For any $R_1, R_2 > 0$, the integral

$$\left[\int_{[-R_1,R_1]\times[-R_2,R_2]} |g(x_1,x_2) - g_{R_1,R_2}|^{p'} dx_1 dx_2\right]^{1/p'}$$

is equal to

$$\sup\left\{\left|\int_{\mathbb{R}^2} (g(x_1,x_2) - g_{R_1,R_2}) h(x_1,x_2) dx_1 dx_2\right| : \|h\|_{L^p([-R_1,R_1]\times[-R_2,R_2])} = 1\right\}.$$

But

$$\begin{aligned}
&\int_{\mathbb{R}^2} (g(x_1,x_2) - g_{R_1,R_2}) h(x_1,x_2) dx_1 dx_2 \\
&= \int_{\mathbb{R}^2} g(x_1,x_2)(h(x_1,x_2) - h_{R_1,R_2}) dx_1 dx_2 \\
&= \Lambda_g(h - h_{R_1,R_2}) \\
&= \Lambda(h - h_{R_1,R_2}),
\end{aligned}$$

and given the inequality $\|h - h_{R_1,R_2}\|_{H\dot{\mathcal{A}}^p} \le C(4R_1R_2)^{1/p'}$, it is easy to see that

$$\begin{aligned}
&\left[\int_{[-R_1,R_1]\times[-R_2,R_2]} |g(x_1,x_2) - g_{R_1,R_2}|^{p'} dx_1 dx_2\right]^{1/p'} \\
&\le \sup\left\{|\Lambda(h)| : \|h\|_{L^p([-R_1,R_1]\times[-R_2,R_2])} \le C\right\} \\
&\le \|\Lambda\|(4R_1R_2)^{1/p'}.
\end{aligned}$$

Since the above is true for every $R_1 > 0$ and $R_2 > 0$, we get $\|g\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}} \le C\|\Lambda\|$. □

4. Commutators of the rectangular 2-dimensional Hardy operator

The classical 2-dimensional Hardy operator H_2 is defined for $x \in \mathbb{R}^2 \setminus \{0\}$ as

$$\begin{aligned}
H_2 f(x) &= \frac{1}{|x|^2}\int_{|y|<|x|} f(y) dy \\
&= \int_{B_1(0)} f(t|x|) dt.
\end{aligned}$$

Instead of the radial operator we can consider an operator acting on each coordinate separately,

$$H_2^R f(x) = \frac{1}{|x_1||x_2|}\int_{\{y:|y_j|<|x_j|, j=1,2\}} f(y) dy,$$

where $x_j \neq 0$ for $j = 1, 2$. In [9] we proved the continuity of the n-dimensional Hardy operator H_n^R on $\dot{\mathcal{B}}^p(\mathbb{R}^n)$ and $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p(\mathbb{R}^n)$. In this section we consider the commutator of the Hardy operator H_2^R, defined as

$$H_b^R f = bH_2^R f - H_2^R(bf),$$

where b is a locally integrable function. Our aim is to prove that when b belongs to $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^q(\mathbb{R}^2)$, for certain value of q depending on p, the commutator operator H_b^R is bounded on $L^p(\mathbb{R}^2)$.

The next lemma will be used to obtain the continuity of the commutator defined above.

Lemma 4.1. *For a function b in $\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1(\mathbb{R}^2)$, there is a constant C such that*

$$|b(t_1,t_2) - b_{2^k,2^j}| \le |b(t_1,t_2) - b_{2^s,2^l}| + C(|k-s| + |j-l|)\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1}.$$

Proof. First notice that when $k < s$

$$|b(t_1,t_2) - b_{2^k,2^j}| \le |b(t_1,t_2) - b_{2^s,2^j}| + \sum_{i=k}^{s-1} |b_{2^i,2^j} - b_{2^{i+1},2^j}|.$$

Analogously when $s < k$

$$|b(t_1,t_2) - b_{2^k,2^j}| \le |b(t_1,t_2) - b_{2^s,2^j}| + \sum_{i=s}^{k-1} |b_{2^i,2^j} - b_{2^{i+1},2^j}|.$$

Additionally for every i and j

$$\begin{aligned} |b_{2^i,2^j} - b_{2^{i+1},2^j}| &= \left| \frac{1}{4\cdot 2^{i+j}} \int_{R_{2^i,2^j}} (b(x_1,x_2) - b_{2^{i+1},2^j}) dx_1 dx_2 \right| \\ &\le \frac{2}{4\cdot 2^{i+1+j}} \int_{R_{2^{i+1},2^j}} |b(x_1,x_2) - b_{2^{i+1},2^j}| dx_1 dx_2 \\ &\le 2\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1}. \end{aligned}$$

Thus

$$|b(t_1,t_2) - b_{2^k,2^j}| \le |b(t_1,t_2) - b_{2^s,2^j}| + C|k-s|\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1}.$$

In a similar way, we see

$$|b(t_1,t_2) - b_{2^s,2^j}| \le |b(t_1,t_2) - b_{2^s,2^l}| + C|l-j|\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1},$$

and finally we obtain

$$\begin{aligned} |b(t_1,t_2) - b_{2^k,2^j}| &\le |b(t_1,t_2) - b_{2^s,2^j}| + C|k-s|\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1} \\ &\le |b(t_1,t_2) - b_{2^s,2^l}| + C(|k-s| + |l-j|)\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1}. \end{aligned}$$

□

Let us now introduce some notation. For $k \in \mathbb{Z}$, denote by S_k the square $[-2^k, 2^k]^2$ and define $C_k = S_k \setminus S_{k-1}$. Observe that $C_k \cap C_j = \emptyset$ when $k \ne j$ and that $\mathbb{R}^2 = \cup_{k\in\mathbb{Z}} C_k$.

Then for a function f in $L^p(\mathbb{R}^2)$ we can write

$$\|f\|_p^p = \sum_{k=-\infty}^{\infty} \|f_k\|_p^p,$$

where $f_k = f\chi_{C_k}$.

Now we state our result.

Theorem 4.2. *Let $1 < p < \infty$, $b \in \mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{\max\{p,p'\}}(\mathbb{R}^2)$. Then H_b^R is bounded on $L^p(\mathbb{R}^2)$ with norm*

$$\|H_b^R\| \le C\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{\max\{p,p'\}}}.$$

Proof. First let us examine $\|(H_b^R f)_k\|_p^p$.

$$\|(H_b^R f)_k\|_p^p = \int_{C_k} \left| \frac{1}{|x_1||x_2|} \int_{[-|x_1|,|x_1|]\times[-|x_2|,|x_2|]} f(t_1,t_2) \right.$$

$$\times (b(x_1,x_2)-b(t_1,t_2))dt_1dt_2\Big|^p dx_1dx_2$$
$$\leq \int_{C_k} \frac{1}{|x_1|^p|x_2|^p}\left(\int_{S_k} |f(t_1,t_2)||b(x_1,x_2)-b(t_1,t_2)|dt_1dt_2\right)^p dx_1dx_2$$
$$\leq C2^{-2kp}\int_{C_k}\left(\sum_{i=-\infty}^{k}\int_{C_i}|f(t_1,t_2)||b(x_1,x_2)-b(t_1,t_2)|dt_1dt_2\right)^p dx_1dx_2$$
$$\leq C2^{-2kp}\int_{C_k}\left(\sum_{i=-\infty}^{k}\int_{C_i}|f(t_1,t_2)||b(x_1,x_2)-b_{2^k,2^k}|dt_1dt_2\right)^p dx_1dx_2$$
$$+ C2^{-2kp}\int_{C_k}\left(\sum_{i=-\infty}^{k}\int_{C_i}|f(t_1,t_2)||b(t_1,t_2)-b_{2^k,2^k}|dt_1dt_2\right)^p dx_1dx_2$$
$$= C2^{-2kp}\left(\int_{C_k}|b(x_1,x_2)-b_{2^k,2^k}|^p dx_1dx_2\right)\left(\sum_{i=-\infty}^{k}\int_{C_i}|f(t_1,t_2)|dt_1dt_2\right)^p$$
$$+ C2^{-2kp}|C_k|\left(\sum_{i=-\infty}^{k}\int_{C_i}|f(t_1,t_2)||b(t_1,t_2)-b_{2^k,2^k}|dt_1dt_2\right)^p$$
$$= I + J.$$

For I observe that

$$I \leq C2^{-2kp}\left(\int_{S_k}|b(x_1,x_2)-b_{2^k,2^k}|^p dx_1dx_2\right)$$
$$\times\left(\sum_{i=-\infty}^{k}\left(\int_{C_i}|f(t_1,t_2)|^p dt_1dt_2\right)^{1/p}|C_i|^{1/p'}\right)^p$$
$$\leq C2^{-2kp/p'}\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p}\left(\sum_{i=-\infty}^{k}2^{2i/p'}\|f_i\|_p\right)^p$$
$$= C\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p}\left(\sum_{i=-\infty}^{k}2^{2(i-k)/p'}\|f_i\|_p\right)^p.$$

Now to estimate J we use Lemma 4.1.

$$J = C\left(2^{-2k/p'}\sum_{i=-\infty}^{k}\int_{C_i}|f(t_1,t_2)||b(t_1,t_2)-b_{2^k,2^k}|dt_1dt_2\right)^p$$
$$\leq C\left(2^{-2k/p'}\sum_{i=-\infty}^{k}\int_{C_i}|f(t_1,t_2)||b(t_1,t_2)-b_{2^i,2^i}|dt_1dt_2\right)^p$$
$$+ C\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1}\left(2^{-2k/p'}\sum_{-\infty}^{k}(k-i)\int_{C_i}|f(t_1,t_2)|dt_1dt_2\right)^p = J_1 + J_2.$$

For the first term we have

$$J_1 \leq C\left(2^{-2k/p'}\sum_{i=-\infty}^{k}\left[\int_{C_i}|f(t_1,t_2)|^p dt_1dt_2\right]^{1/p}\right.$$

$$\times \left[\int_{C_i} |b(t_1,t_2) - b_{2^i,2^i}|^{p'} dt_1 dt_2\right]^{1/p'}\Bigg)^p$$

$$\leq C \left(\sum_{i=-\infty}^{k} 2^{2(i-k)/p'} \|f_i\|_p \left[\frac{1}{|S_i|}\int_{S_i} |b(t_1,t_2) - b_{2^i,2^i}|^{p'} dt_1 dt_2\right]^{1/p'}\right)^p$$

$$\leq C\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}} \left(\sum_{i=-\infty}^{k} 2^{2(i-k)/p'} \|f_i\|_p\right)^p.$$

For the second term we can use the Hölder inequality to obtain

$$J_2 \leq C\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1} \left(2^{-2k/p'} \sum_{i=-\infty}^{k} (k-i)\|f_i\|_p |C_i|^{1/p'}\right)^p$$

$$\leq C\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1} \left(\sum_{i=-\infty}^{k} 2^{2(i-k)/p'} (k-i)\|f_i\|_p\right)^p.$$

From all the calculations above we get

$$\sum_{k=-\infty}^{\infty} \|(H_b^R f)_k\|_p^p \leq C(\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^p} + \|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{p'}}) \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{k} 2^{2(i-k)/p'} \|f_i\|_p\right)^p$$

$$+ C\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^1} \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{k} 2^{2(i-k)/p'} (k-i)\|f_i\|_p\right)^p$$

$$\leq C\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{\max\{p,p'\}}} \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{k} 2^{2(i-k)/p'} (k-i)\|f_i\|_p\right)^p$$

$$\leq C\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{\max\{p,p'\}}} \sum_{k=-\infty}^{\infty} \left(\sum_{i=-\infty}^{k} 2^{i-k} (k-i)^{p'}\right)^{p/p'} \left(\sum_{i=-\infty}^{k} 2^{(i-k)p/p'} \|f_i\|_p^p\right).$$

Since the series $\sum_{i=-\infty}^{k} 2^{i-k}(k-i)^{p'}$ converges, to the same value, for every integer k, we can use Tonelli's theorem to obtain

$$\|H_b^R f\|_p^p = \sum_{k=-\infty}^{\infty} \|(H_b^R f)_k\|_p^p$$

$$\leq C\|b\|^p_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{\max\{p,p'\}}} \sum_{i=-\infty}^{\infty} \|f_i\|_p^p \left(\sum_{k=i}^{\infty} 2^{(i-k)p/p'}\right).$$

But again, for every integer number i, the series $\sum_{k=i}^{\infty} 2^{(i-k)p/p'}$ converges to the same value. Thus we finally get

$$\|H_b^R f\|_p \leq C\|b\|_{\mathcal{C}\dot{\mathcal{M}}\mathcal{O}^{\max\{p,p'\}}} \|f\|_p.$$

□

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