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COMMON FIXED POINTS FOR FOUR MAPS USING (α, η)-ADMISSIBLE FUNCTIONS IN PARTIAL METRIC SPACES

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Abstract. In this paper, we introduced (α, η)-admissible function associated with two pairs of partial$^{(*)}$ compatible maps and obtained a unique common fixed point theorem in partial metric spaces. We also given an example to illustrate our main theorem.

1. Introduction

There are many generalizations of the concept of metric spaces in the literature. One of them is a partial metric space introduced by Matthews [16] as a part of study of denotational semantics of data flow networks. After that fixed and common fixed point results in partial metric spaces were studied by many other authors,for example,[1, 3, 4, 5, 7, 8, 9, 10, 13, 15, 17, 18].

Throughout this paper, $\mathcal{R}^+$ and $\mathcal{N}$ denote the set of all non-negative real numbers and set of all positive integers respectively.
First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.
Definition 1 (See [16]) A partial metric on a nonempty set X is a function $p : X \times X \to \mathcal{R}^+$ such that for all $x, y, z \in X$:

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \leq p(x, y), p(y, y) \leq p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space (PMS).

If p is a partial metric on X, then the function $p^s : X \times X \to \mathcal{R}^+$ given by $p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X.
It is clear that (i) $p(x, y) = 0 \Rightarrow x = y$, (ii) $x \neq y \Rightarrow p(x, y) > 0$ and (iii) $p(x, x)$ may not be 0.
Example 2 (See e.g. [4, 16, 18]) Consider $X = \mathcal{R}^+$ with $p(x, y) = \max\{x, y\}$. Then (X, p) is a partial metric space. It is clear that p is not
a (usual) metric. Note that in this case $p^s(x, y) = |x - y|$.

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Example 3 (See [3]) Let $X = \{[a,b] : a,b \in \mathcal{R},\ a \leq b\}$ and define $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$. Then (X,p) is a partial metric space.

We now state some basic topological notions (such as convergence, completeness, continuity) on partial metric spaces (see e.g. [4, 5, 8, 16, 18]).

Definition 4

(i) A sequence $\{x_n\}$ in the PMS (X,p) converges to the limit x if and only if $p(x,x) = \lim\limits_{n\to\infty} p(x,x_n)$.

(ii) A sequence $\{x_n\}$ in the PMS (X,p) is called a Cauchy sequence if $\lim\limits_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.

(iii) A PMS (X,p) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim\limits_{n,m\to\infty} p(x_n,x_m)$.

(iv) A mapping $F : X \to X$ is said to be continuous at $x \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $F(B_p(x,\delta)) \subseteq B_p(Fx,\epsilon)$.

It is clear that if F is continuous at $x \in X$ then $\{Fx_n\}$ converges to Fx whenever the sequence $\{x_n\} \in X$ converges to x.

The following lemma is one of the basic results in PMS([4, 5, 8, 16, 18]).

Lemma 5

(i) A sequence $\{x_n\}$ is a Cauchy sequence in the PMS (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s).

(ii) A PMS (X,p) is complete if and only if the metric space (X,p^s) is complete. Moreover $\lim\limits_{n\to\infty} p^s(x,x_n) = 0 \Leftrightarrow p(x,x) = \lim\limits_{n\to\infty} p(x,x_n) = \lim\limits_{n,m\to\infty} p(x_n,x_m)$.

Next, we give a simple lemma which will be used in the proof of our main result. For the proof we refer to [18].

Lemma 6 Assume $x_n \to z$ as $n \to \infty$ in a PMS (X,p) such that $p(z,z) = 0$. Then $\lim\limits_{n\to\infty} p(x_n,y) = p(z,y)$ for every $y \in X$.

Definition 7([1]) Let (X,p) be a partial metric space and $F,g : X \to X$.Then the pair (F,g) is said to be partial compatible if the following conditions hold:
(i)$p(x,x) = 0 \Rightarrow p(gx,gx) = 0$ whenever $x \in X$,
(ii) $\lim\limits_{n\to\infty} p(Fgx_n, gFx_n) = 0$ whenever there exists a sequence $\{x_n\}$ in X such that $Fx_n \to t$ and $gx_n \to t$ for some $t \in X$.

We observe that the Definition 7 seems to be insufficient. Hence we modify it as

Definition 8 Let (X,p) be a partial metric space and $F,g : X \to X$.Then the pair (F,g) is said to be partial$^{(*)}$ compatible if the following conditions hold:
(i)$p(x,x) = 0 \Rightarrow p(gx,gx) = 0$ whenever $x \in X$,
(ii) $\lim\limits_{n\to\infty} p(Fgx_n, gFx_n) = 0$ whenever there exists a sequence $\{x_n\}$ in X such that $Fx_n \to t$ and $gx_n \to t$ for some $t \in X$ with $p(t,t) = 0$.

Now we give an example in which the pair (F,g) is partial$^{(*)}$ compatible,but not partial compatible.

Example 9 Let $X = [0,1]$ and $p(x,y) = \max\{x,y\}$. Let $F,g : X \to X$ as $Fx = \frac{x^2}{2}$ and $gx = x^2$.

Clearly $p(x,x) = 0 \Rightarrow x = 0$. Hence $p(gx,gx) = 0$. Let $\{x_n\}$ be any sequence in X such that $Fx_n \to t$ and $gx_n \to t$ as $n \to \infty$ for some $t \in X$ with $p(t,t) = 0$.Then clearly $t = 0$ and $x_n \to 0$ as $n \to \infty$. Hence $\lim\limits_{n\to\infty} p(Fgx_n, gFx_n) = 0$. Thus the

pair (F, g) is partial$^{(*)}$ compatible.
If we take $\{x_n\} = \{1\}$ and $t = 2$, then $Fx_n \to t$ and $gx_n \to t$ as $n \to \infty$. But $\lim_{n\to\infty} p(Fgx_n, gFx_n) = \frac{1}{2} \neq 0$. Hence the pair (F, g) is not partial compatible.

Samet et al. [2] introduced the notion of α- admissible mappings as follows

Definition 10 ([2]) Let X be a non empty set, $T : X \to X$ and $\alpha : X \times X \to \mathcal{R}^+$ be mappings. Then T is called α- admissible if for all $x, y \in X$, we have $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$.

Some interesting examples of such mappings are given in ([2]).

Later Salimi et al.([14]) modified the concept of α-admissible mappings as follows

Definition 11 ([14]) Let T be a self mapping on a metric space (X, d) and $\alpha, \eta : X \times X \to \mathcal{R}^+$ be two functions. Then T is said to be α-admissible mapping with respect to η if $\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty)$ for all $x, y \in X$.

Very recently Babu et al.([6]) extended the above definition for Jungck type maps as follows

Definition 12 ([6]) Let f and g be two self mappings on a metric space (X, d) and $\alpha, \eta : X \times X \to \mathcal{R}^+$ be two functions. Then f is said to be (α, g)-admissible mapping with respect to η if $\alpha(gx, gy) \geq \eta(gx, gy) \Rightarrow \alpha(fx, fy) \geq \eta(fx, fy)$ for all $x, y \in X$.

Now we define (α, η)-admissible condition for two pairs of maps as follows

Definition 13 Let X be a non empty set and $f, g, S, T : X \to X$ be mappings and $\alpha, \eta : X \times X \to \mathcal{R}^+$ be functions . We say that the pair(f, g) satisfies (α, η)-admissible condition with respect to the pair (S, T) if $\alpha(Sx, Ty) \geq \eta(Sx, Ty)$ implies $\alpha(fx, gy) \geq \eta(fx, gy)$ and $\alpha(Tx, Sy) \geq \eta(Tx, Sy)$ implies $\alpha(gx, fy) \geq \eta(gx, fy)$ for all $x, y \in X$.

Example 14 Let $X = [0, 2]$ with the usual metric.Define $f, g, S, T : X \to X$ and $\alpha, \eta : X \times X \to \mathcal{R}^+$ by

$$f(x) = \begin{cases} \frac{x}{18}, \ if\ x \in [0,1] \\ x, \ otherwise \end{cases}, \ g(x) = \begin{cases} \frac{x^2}{27}, \ if\ x \in [0,1] \\ \frac{x+1}{2}, \ otherwise \end{cases},$$

$$S(x) = \begin{cases} \frac{x}{2}, \ if\ x \in [0,1] \\ x, \ otherwise \end{cases}, \ T(x) = \begin{cases} \frac{x^2}{3}, \ if\ x \in [0,1] \\ \frac{x+1}{2}, \ otherwise \end{cases},$$

$$\alpha(x, y) = \begin{cases} 2, \ if\ x, y \in [0,1] \\ 0, \ otherwise \end{cases} \text{ and } \eta(x, y) = \begin{cases} 1, \ if\ x, y \in [0,1] \\ 2, \ otherwise \end{cases}.$$

We will verify Definition 13 as follows.
Case(i): $x, y \in [0, 1]$.
Then $Sx, Sy \in [0, \frac{1}{2}]; Tx, Ty \in [0, \frac{1}{3}]; fx, fy \in [0, \frac{1}{18}]; gx, gy \in [0, \frac{1}{27}]$.
Clearly $\alpha(Sx, Ty) > \eta(Sx, Ty) \Rightarrow \alpha(fx, gy) > \eta(fx, gy)$ and
$\alpha(Tx, Sy) > \eta(Tx, Sy) \Rightarrow \alpha(gx, fy) > \eta(gx, fy)$.
Case(ii): $x, y \in (1, 2]$
Then $\alpha(Sx, Ty) < \eta(Sx, Ty)$ and $\alpha(Tx, Sy) < \eta(Tx, Sy)$.
Case(iii): $x \in [0, 1], y \in (1, 2]$ or $x \in (1, 2], y \in [0, 1]$.
Then also $\alpha(Sx, Ty) < \eta(Sx, Ty)$ and $\alpha(Tx, Sy) < \eta(Tx, Sy)$.
Thus the pair (f, g) satisfies (α, η)-admissible condition with respect to the pair (S, T).

Let Ψ denote the class of all functions $\psi : \mathcal{R}^+ \to \mathcal{R}^+$, where ψ is continuous,monotonically increasing and $\sum \psi^n(t) < \infty$ for each $t > 0$.It is clear that $\psi(t) < t$ for every $t > 0$.

Now we prove our main result.

2. Main Result

Theorem 2.1 Let (X,p) be a complete partial metric space and $\alpha,\eta : X\times X \to \mathcal{R}^+$ be two functions. Let f,g,S and T be self mappings on X satisfying

(2.1.1) $f(X)\subseteq T(X)$, $g(X)\subseteq S(X)$,

(2.1.2) $\alpha(Sx,fx)\ \alpha(Ty,gy)\geq \eta(Sx,fx)\ \eta(Ty,gy)$ for all $x,y\in X$ implies $p(fx,gy)\leq \psi(M(x,y))$ for all $x,y\in X$ where $\psi\in\Psi$ and
$M(x,y)=\max\left\{\begin{array}{c} p(Sx,Ty),p(Sx,fx),p(Ty,gy),\\ \frac{1}{2}\left[p(Sx,gy)+p(Ty,fx)\right]\end{array}\right\}$,

(2.1.3) $\alpha(Sx_1,fx_1)\geq\eta(Sx_1,fx_1)$ for some $x_1\in X$,

(2.1.4) pair (f,g) satisfies (α,η)-admissible condition with respect to the pair (S,T),

(2.1.5) the pairs (f,S) and (g,T) are partial$^{(*)}$ compatible and S and T are continuous on X,

(2.1.6) Assume that $\alpha(Sy_{2n},fy_{2n})\geq\eta(Sy_{2n},fy_{2n})$, $\alpha(z,fz)\geq\eta(z,fz)$, $\alpha(Ty_{2n+1},gy_{2n+1})\geq\eta(Ty_{2n+1},gy_{2n+1})$ and $\alpha(z,gz)\geq\eta(z,gz)$ whenever there exists a sequence $\{y_n\}$ in X such hat $\alpha(y_n,y_{n+1})\geq\eta(y_n,y_{n+1})$ for $n=1,2,3,\cdots$ and $y_n\to z$ for some $z\in X$.

Then f,g,S and T have a common fixed point.

(2.1.7) Further if we assume that $\alpha(u,u)\geq\eta(u,u)$ whenever u is a common fixed point of f,g,S and T then f,g,S and T have a unique common fixed point in X.

Proof. From (2.1.3), we have $\alpha(Sx_1,fx_1)\geq\eta(Sx_1,fx_1)$ for some $x_1\in X$. From (2.1.1), define the sequences $\{x_n\}$ and $\{y_n\}$ as follows:
$y_1=fx_1=Tx_2$, $y_2=gx_2=Sx_3$, $y_3=fx_3=Tx_4$, $y_4=gx_4=Sx_5$, $\cdots$
$y_{2n+1}=fx_{2n+1}=Tx_{2n+2}$, $y_{2n+2}=gx_{2n+2}=Sx_{2n+3}$, $n=0,1,2,\cdots$.
Now

$$\begin{array}{l}
\alpha(Sx_1,Tx_2)\geq\eta(Sx_1,Tx_2),\ \text{ from (2.1.3)}\\
\Rightarrow\alpha(fx_1,gx_2)\geq\eta(fx_1,gx_2),\ \text{ from (2.1.4), } i.e\ \alpha(y_1,y_2)\geq\eta(y_1,y_2)\\
\Rightarrow\alpha(Tx_2,Sx_3)\geq\eta(Tx_2,Sx_3),\ \text{ from } definition\ of\ \{y_n\}\\
\Rightarrow\alpha(gx_2,fx_3)\geq\eta(gx_2,fx_3),\ \text{ from (2.1.4), } i.e\ \alpha(y_2,y_3)\geq\eta(y_2,y_3)\\
\Rightarrow\alpha(Sx_3,Tx_4)\geq\eta(Sx_3,Tx_4),\ \text{ from } definition\ of\ \{y_n\}\\
\Rightarrow\alpha(fx_3,gx_4)\geq\eta(fx_3,gx_4),\ \text{ from (2.1.4), } i.e\ \alpha(y_3,y_4)\geq\eta(y_3,y_4)
\end{array}$$

Continuing in this way, we have

$$\alpha(y_n,y_{n+1})\geq\eta(y_n,y_{n+1})\ \text{ for } n=1,2,3,\cdots. \tag{1}$$

Case (a): Suppose $y_{2m}=y_{2m+1}$ for some m.

$$\begin{array}{ll}
\alpha(Sx_{2m+1},fx_{2m+1})\ \alpha(Tx_{2m+2},gx_{2m+2}) & =\alpha(y_{2m},y_{2m+1})\ \alpha(y_{2m+1},y_{2m+2})\\
& \geq\eta(y_{2m},y_{2m+1})\ \eta(y_{2m+1},y_{2m+2}), from (1)\\
& =\eta(Sx_{2m+1},fx_{2m+1})\ \eta(Tx_{2m+2},gx_{2m+2}).
\end{array}$$

Hence from (2.1.2), we have

$$\begin{array}{ll}
p(y_{2m+1},y_{2m+2}) & =p(fx_{2m+1},gx_{2m+2})\\
& \leq\psi(M(x_{2m+1},x_{2m+2})),
\end{array}$$

where

$$M(x_{2m+1}, x_{2m+2}) = \max \left\{ \begin{array}{c} p(y_{2m}, y_{2m+1}), p(y_{2m}, y_{2m+1}), p(y_{2m+1}, y_{2m+2}), \\ \frac{1}{2}\left[p(y_{2m}, y_{2m+2}) + p(y_{2m+1}, y_{2m+1})\right] \end{array} \right\}.$$

But $p(y_{2m}, y_{2m+1}) = p(y_{2m+1}, y_{2m+1}) \leq p(y_{2m+1}, y_{2m+2})$, from (p_2).

$$\begin{aligned} \frac{1}{2}\left[p(y_{2m}, y_{2m+2}) + p(y_{2m+1}, y_{2m+1})\right] &\leq \frac{1}{2}\left[p(y_{2m}, y_{2m+1}) + p(y_{2m+1}, y_{2m+2})\right], from\ (p_4) \\ &\leq p(y_{2m+1}, y_{2m+2}). \end{aligned}$$

Hence $M(x_{2m+1}, x_{2m+2}) = p(y_{2m+1}, y_{2m+2})$.
Thus $p(y_{2m+1}, y_{2m+2}) \leq \psi(p(y_{2m+1}, y_{2m+2}))$
which in turn yields that $p(y_{2m+1}, y_{2m+2}) = 0$ so that $y_{2m+1} = y_{2m+2}$.
Continuing in this way we get $y_{2m} = y_{2m+1} = y_{2m+2} = \cdots$.
Hence $\{y_n\}$ is Cauchy.
Case (b): Suppose that $y_n \neq y_{n+1}$ for all n.
Then

$$\begin{aligned} \alpha(Sx_{2n+1}, fx_{2n+1})\ \alpha(Tx_{2n+2}, gx_{2n+2}) &= \alpha(y_{2n}, y_{2n+1})\ \alpha(y_{2n+1}, y_{2n+2}) \\ &\geq \eta(y_{2n}, y_{2n+1})\ \eta(y_{2n+1}, y_{2n+2}), from (1) \\ &= \eta(Sx_{2n+1}, fx_{2n+1})\ \eta(Tx_{2n+2}, gx_{2n+2}). \end{aligned}$$

As in Case (a), we have

$$p(y_{2n+1}, y_{2n+2})) \leq \psi(M(x_{2n+1}, x_{2n+2})) \tag{2}$$

where $M(x_{2n+1}, x_{2n+2}) = \max\{p(y_{2n}, y_{2n+1}), p(y_{2n+1}, y_{2n+2})\}$.
If $M(x_{2n+1}, x_{2n+2}) = p(y_{2n+1}, y_{2n+2})$ then we get

$$\begin{aligned} p(y_{2n+1}, y_{2n+2})) &\leq \psi(p(y_{2n+1}, y_{2n+2})) \\ &< p(y_{2n+1}, y_{2n+2}). \end{aligned}$$

It is a contradiction. Hence

$$p(y_{2n+1}, y_{2n+2}) \quad \leq \psi(p(y_{2n}, y_{2n+1})).$$

Similarly we can show that $p(y_{2n}, y_{2n+1}) \leq \psi(p(y_{2n-1}, y_{2n}))$.
Thus $p(y_{n+1}, y_{n+2}) \leq \psi(p(y_n, y_{n+1}))$ for $n = 1, 2, 3, ...$

$$\begin{aligned} p(y_{n+1}, y_{n+2}) &\leq \psi(p(y_n, y_{n+1})) \\ &\leq \psi^2(p(y_{n-1}, y_n)) \\ &\leq \psi^3(p(y_{n-2}, y_{n-1})) \\ &. \\ &. \\ &. \\ &\leq \psi^n(p(y_1, y_2)) \\ &\to 0\ as\ n\ \to \infty. \end{aligned} \tag{3}$$

For $n > m$,consider

$$\begin{aligned} p(y_m, y_n) &\leq p(y_m, y_{m+1}) + p(y_{m+1}, y_{m+2}) + ... + p(y_{n-1}, y_n) \\ &\leq \psi^{m-1}(p(y_1, y_2)) + \psi^m(p(y_1, y_2)) + ... + \psi^{n-2}(p(y_1, y_2)) \\ &\leq \sum_{k=m-1}^{\infty} \psi^k(p(y_1, y_2)) \to 0 \end{aligned} \tag{4}$$

Thus $\{y_n\}$ is a Cauchy sequence in (X, p). Since (X, p) is a complete partial metric space, there exists $z \in X$ such that $p(z, z) = \lim_{n\to\infty} p(y_n, y_m)$.
From (4), we have

$$p(z, z) = 0 \tag{5}$$

Hence

$$\begin{aligned} p(z,z) &= \lim_{n\to\infty} p(fx_{2n+1},z) = \lim_{n\to\infty} p(gx_{2n+2},z) \\ &= \lim_{n\to\infty} p(Sx_{2n+1},z) = \lim_{n\to\infty} p(Tx_{2n+2},z) = 0 \end{aligned} \tag{6}$$

Since the pair (f,S) is partial$^{(*)}$ compatible, from (5), we have $p(Sz,Sz)=0$. and

$$\lim_{n\to\infty} p(fSx_{2n+1},Sfx_{2n+1}) = 0 \tag{7}$$

Since S is continuous at z, we have

$$\lim_{n\to\infty} p(SSx_{2n+1},Sz) = p(Sz,Sz) = 0 \tag{8}$$

and

$$\lim_{n\to\infty} p(Sfx_{2n+1},Sz) = p(Sz,Sz) = 0 \tag{9}$$

Also $p(fSx_{2n+1},Sz) \le p(fSx_{2n+1},Sfx_{2n+1}) + p(Sfx_{2n+1},Sz)$.
Now by using (7) and (9), we have $\lim_{n\to\infty} p(fSx_{2n+1},Sz) \le 0$.
Hence

$$\lim_{n\to\infty} p(fSx_{2n+1},Sz) = 0 \tag{10}$$

Now $p(fSx_{2n+1},SSx_{2n+1}) \le p(fSx_{2n+1},Sz) + p(Sz,SSx_{2n+1})$.

$$\lim_{n\to\infty} p(fSx_{2n+1},SSx_{2n+1}) \le 0 \ from \ (10) \ and \ (8).$$

Hence

$$\lim_{n\to\infty} p(fSx_{2n+1},SSx_{2n+1}) = 0 \tag{11}$$

Letting $n\to\infty$ and using (10) and (6) in
$|p(fSx_{2n+1},gx_{2n}) - p(z,Sz)| \le p(fSx_{2n+1},Sz) + p(z,gx_{2n})$, we get

$$\lim_{n\to\infty} p(fSx_{2n+1},gx_{2n}) = p(Sz,z) \tag{12}$$

Letting $n\to\infty$ and using (8)and(6) in
$|p(SSx_{2n+1},Tx_{2n}) - p(Sz,z)| \le p(SSx_{2n+1},Sz) + p(z,Tx_{2n})$, we get

$$\lim_{n\to\infty} p(SSx_{2n+1},Tx_{2n}) = p(Sz,z) \tag{13}$$

Letting $n\to\infty$ and using (8)and(6) in
$|p(SSx_{2n+1},gx_{2n}) - p(Sz,z)| \le p(SSx_{2n+1},Sz) + p(z,gx_{2n})$, we get

$$\lim_{n\to\infty} p(SSx_{2n+1},gx_{2n}) = p(Sz,z) \tag{14}$$

Letting $n\to\infty$ and using (10)and(6) in
$|p(Tx_{2n},fSx_{2n+1}) - p(z,Sz)| \le p(fSx_{2n+1},Sz) + p(z,Tx_{2n})$, we get

$$\lim_{n\to\infty} p(Tx_{2n},fSx_{2n}) = p(Sz,z) \tag{15}$$

Now Consider

$$\begin{aligned} \alpha(SSx_{2n+1},fSx_{2n+1})\ \alpha(Tx_{2n},gx_{2n}) &= \alpha(Sy_{2n},fy_{2n})\ \alpha(y_{2n-1},y_{2n}) \\ &\ge \eta(Sy_{2n},fy_{2n})\ \eta(y_{2n-1},y_{2n}),\ \text{from } (2.1.6) \\ &= \eta(SSx_{2n+1},fSx_{2n+1})\ \eta(Tx_{2n},gx_{2n}). \end{aligned}$$

From (12), we have

$$\begin{aligned} p(Sz,z) &= \lim_{n\to\infty} p(fSx_{2n+1}, gx_{2n}) \\ &\leq \lim_{n\to\infty} \psi(M(Sx_{2n+1}, x_{2n})), from (2.1.2) \end{aligned}$$

where

$$\begin{aligned} M(Sx_{2n+1}, x_{2n}) &= \max \left\{ \begin{array}{c} p(SSx_{2n+1}, Tx_{2n}), p(SSx_{2n+1}, fSx_{2n+1}), \\ p(Tx_{2n}, gx_{2n}), \\ \frac{1}{2}\left[p(SSx_{2n+1}, gx_{2n}) + p(Tx_{2n}, fSx_{2n+1})\right] \end{array} \right\} \\ &\to p(Sz,z),\ from\ (13),(11),(3),(14)\ and\ (15). \end{aligned}$$

Thus

$$p(Sz,z) \leq \phi(p(Sz,z))$$

which in turn yields that $Sz = z$.
Similarly using the continuity of T and partial$^{(*)}$ compatibility of (g,T) and $\alpha(Ty_{2n+1}, gy_{2n+1}) \geq \eta(Ty_{2n+1}, gy_{2n+1})$ from (2.1.6), we can show that $Tz = z$.
We have

$$\begin{aligned} \alpha(Sz,fz)\ \alpha(Tx_{2n}, gx_{2n}) &= \alpha(z,fz)\ \alpha(y_{2n-1}, y_{2n}) \\ &\geq \eta(z,fz)\ \eta(y_{2n-1}, y_{2n}), from\ (1)and(2.1.6) \\ &= \eta(Sz,fz)\ \eta(Tx_{2n}, gx_{2n}). \end{aligned}$$

Now from (2.1.2),

$$\begin{aligned} p(fz,z) &= \lim_{n\to\infty} p(fz, gx_{2n}) \\ &\leq \lim_{n\to\infty} \psi(M(z, x_{2n})) \end{aligned}$$

where

$$\begin{aligned} M(z, x_{2n}) &= \max \left\{ \begin{array}{c} p(z, Tx_{2n}), p(z,fz), p(Tx_{2n}, gx_{2n}), \\ \frac{1}{2}\left[p(z, gx_{2n}) + p(Tx_{2n}, fz)\right] \end{array} \right\} \\ &\to p(z,fz) \text{ from (6),(3), Lemma 6 .} \end{aligned}$$

Hence $p(fz,z) \leq \psi(p(fz,z))$which in turn yields that $fz = z$.
Similarly we can show that $gz = z$ by using $\alpha(z,gz) \geq \eta(z,gz)$ from (2.1.6). Thus z is a common fixed point of f, g, S and T.
Suppose z' is another common fixed point of f, g, S and T. We have

$$\begin{aligned} \alpha(Sz,fz)\ \alpha(Tz', gz') &= \alpha(z,z)\ \alpha(z',z') \\ &\geq \eta(z,z)\ \eta(z',z'), from\ (2.1.7) \\ &= \eta(Sz,fz)\ \eta(Tz', gz'). \end{aligned}$$

Hence from (2.1.2), we have $p(z,z') = p(fz, gz') \leq \psi(M(z,z'))$, where

$$\begin{aligned} M(z,z') &= \max \left\{ p(z,z'), p(z,z), p(z',z'), \tfrac{1}{2}\left[p(z,z') + p(z,z')\right] \right\} \\ &= p(z,z'),\ from\ (p_2). \end{aligned}$$

Thus

$$p(z,z') \leq \psi(p(z,z')) < p(z,z')$$

which is a contradiction. Hence $z = z'$. Thus f, g, S and T have a unique common fixed point.

Now we give an example to support Theorem 2.1.

Example 2.2 Let $X = [0,1]$ be endowed with metric $p(x,y) = \max\{x,y\}$ for all $x, y \in X$ and $f, g, S, T : X \to X$ be defined by $fx = (\frac{x}{2})^8$, $gx = (\frac{x}{2})^4$, $Sx = (\frac{x}{2})^4$ and $Tx = (\frac{x}{2})^2$ for all $x \in X$.
Define $\alpha, \eta : X \times X \to \mathcal{R}^+$ by

$\alpha(x,y) = \begin{cases} 2, \; if \; x,y \in [0,\frac{1}{4}] \\ 0, \; otherwise \end{cases}$ and $\eta(x,y) = \begin{cases} 1, \; if \; x,y \in [0,\frac{1}{4}] \\ 0, \; otherwise \end{cases}$.

Define $\psi : \mathcal{R}^+ \to \mathcal{R}^+$ by $\psi(t) = \frac{1}{4}t$ for all $t \in \mathcal{R}^+$.

One can easily verify all conditions of Theorem 2.1. Here "0" is the unique common fixed point of f, g, S and T.

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